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RELATIONSHIP BETWEEN THE WIENER INTEGRAL AND THE ANALYTIC FEYNMAN INTEGRAL OF CYLINDER FUNCTION

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Abstract. Cameron and Storvick discovered a change of scale formula for Wiener integral of functionals in a Banach algebra $\mathcal{S}$ on classical Wiener space. We express the analytic Feynman integral of cylinder function as a limit of Wiener integrals. Moreover we obtain the same change of scale formula as Cameron and Storvick's result for Wiener integral of cylinder function. Our result cover a restricted version of the change of scale formula by Kim.

1. Introduction

Let $C_0[0,T]$ denote the Wiener space, that is, the space of real valued continuous functions x on $[0,T]$ with $x(0)=0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let m denote Wiener measure. Then $(C_0[0,T],\mathcal{M},m)$ is a complete measure space and we denote the Wiener integral of a functional F by

$$\int_{C_0[0,T]} F(x)\,dm(x). \tag{1.1}$$

A subset E of $C_0[0,T]$ is said to be scale-invariant measurable [7] provided ρE is measurable for each $\rho>0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m(\rho N)=0$ for each $\rho>0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.).

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Let $\mathbb{C}_+$ denote the set of complex numbers with positive real part. Let F be a complex valued measurable functional on $C_0[0,T]$ such that the Wiener integral

$$J_F(\lambda) = \int_{C_0[0,T]} F(\lambda^{-1/2}x)\, dm(x) \tag{1.2}$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J_F^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J_F^*(\lambda) = J_F(\lambda)$ for all $\lambda > 0$, then $J_F^*(\lambda)$ is defined to be the analytic Wiener integral of F over $C_0[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$\int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x)\, dm(x) = J_F^*(\lambda). \tag{1.3}$$

If the following limit exists for nonzero real number q, then we call it the analytic Feynman integral of F over $C_0[0,T]$ with parameter q and we write

$$\int_{C_0[0,T]}^{\mathrm{anf}_q} F(x)\, dm(x) = \lim_{\lambda \to -iq} \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x)\, dm(x) \tag{1.4}$$

where λ approaches $-iq$ through $\mathbb{C}_+$.

It has long been known that Wiener measure and Wiener measurability behave badly under the change of scale transformation [2] and under translations [1]. Cameron and Storvick [5] expressed the analytic Feynman integral on classical Wiener space as a limit of Wiener integrals. In doing so, they discovered nice change of scale formula for Wiener integral on classical Wiener space $(C_0[0,T], m)$ [4] as follows.

$$\begin{aligned} &\int_{C_0[0,T]} F(\rho x)\, dm(x) \\ &= \lim_{n\to\infty} \rho^{-n} \int_{C_0[0,T]} \exp\Big\{ \frac{\rho^2 - 1}{2\rho^2} \sum_{k=1}^{n} \langle \phi_k, x\rangle^2 \Big\} F(x)\, dm(x), \end{aligned} \tag{1.5}$$

where $\{\phi_1, \phi_2, \ldots\}$ is a complete orthonormal set of functionals in $L_2[0,T]$, $\rho > 0$ and F is a functional in a Banach algebra $\mathcal{S}$ introduced by Cameron and Storvick [3].

In [10, 11], Yoo and Skoug extended these results to an abstract Wiener space (H, B, ν). Moreover Yoo, Song, Kim and Chang [12, 13] established a change of scale formula for Wiener integrals of some unbounded functionals on abstract Wiener space and on a product of abstract Wiener space. Recently Yoo, Kim and Kim [9] obtained a change of scale formula for a function space integral on a generalized Wiener space $C_{a,b}[0,T]$.

On the other hand, Kim [8] established a change of scale formula for cylinder functions on abstract Wiener space B. That is, for $F(x) = f((h_1,x)^\sim,\dots,(h_n,x)^\sim)$, he proved that

$$\int_B F(\rho x)\,d\nu(x) = \rho^{-n}\int_B \exp\Big\{\frac{\rho^2-1}{2\rho^2}\sum_{k=1}^n[(h_k,x)^\sim]^2\Big\}F(x)\,d\nu(x), \tag{1.6}$$

where $\{h_1,\dots,h_n\}$ is an orthonormal set in H and $\rho>0$.

Note that in the change of scale formula (1.5) by Cameron and Storvick, $\{\phi_1,\phi_2,\dots\}$ may be any complete orthonormal set of functionals in $L_2[0,T]$ and it requires the limiting procedure. While in the change of scale formula (1.6) by Kim, it does not require the limiting procedure but $\{h_1,\dots,h_n\}$ in the exponential of the integrand must be the same as the elements used to define the cylinder function F.

In this paper, we express the analytic Feynman integral of cylinder function as a limit of Wiener integrals on $C_0[0,T]$. And we obtain the original version (1.5) of a change of scale formula for Wiener integral of cylinder function. Of course the changle of scale formula (1.6) can be obtained as a corollary of our result.

Now we will introduce the class of functionals that we work with in this paper. Let α be a nonzero function with $\|\alpha\|=1$ in $L_2[0,T]$. For $1\le p<\infty$ let $\mathcal{A}^{(p)}$ be the space of all functionals F on $C_0[0,T]$ of the form

$$F(x) = f(\langle\alpha,x\rangle) \tag{1.7}$$

for s-a.e. x in $C_0[0,T]$, where $f:\mathbb{R}\to\mathbb{R}$ is in $L_p(\mathbb{R})$ and $\langle\alpha,x\rangle$ denote the Paley-Wiener-Zygmund stochastic integral $\int_0^T\alpha(t)\,dx(t)$. Let $\mathcal{A}^{(\infty)}$ be the space of all functionals of the form (1.7) with $f\in C_0(\mathbb{R})$, the space of bounded continuous functions on $\mathbb{R}$ that vanish at infinity. For simplicity, we restrict our attention to the cylinder function (1.7) which depends on a single function α.

Let $g:\mathbb{R}^n\to\mathbb{R}$ be a Lebesgue measurable function and let $\{\phi_1,\dots,\phi_n\}$ be an orthonormal set in $L_2[0,T]$. Then the following Wiener integration formula is a very fundamental integration formula to study Wiener and Feynman integration theories.

$$\begin{aligned}&\int_{C_0[0,T]} g(\langle\phi_1,x\rangle,\dots,\langle\phi_n,x\rangle)\,dm(x)\\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} g(u_1,\dots,u_n)\exp\Big\{-\frac{1}{2}\sum_{k=1}^n u_k^2\Big\}\,du_1\cdots du_n\end{aligned} \tag{1.8}$$

in the sense that if either side exists, then both sides exist and equality holds.

We close this section by introducing the analytic Wiener and the analytic Feynman integrals of the cylinder function. One can find the results in, for example, [6].

THEOREM 1.1. *Let $1 \le p \le \infty$ and let $F \in \mathcal{A}^{(p)}$ be given by* (1.7). *Then for all $\lambda \in \mathbb{C}_+$, F is analytic Wiener integrable and*

$$\int_{C_0[0,T]}^{\text{anw}_\lambda} F(x)\,dm(x) = \Big(\frac{\lambda}{2\pi}\Big)^{1/2} \int_{\mathbb{R}} f(u) \exp\Big\{-\frac{\lambda}{2}u^2\Big\}\,du. \tag{1.9}$$

Furthermore, if $F \in \mathcal{A}^{(1)}$, then F is analytic Feynman integrable and

$$\int_{C_0[0,T]}^{\text{anf}_q} F(x)\,dm(x) = \Big(-\frac{iq}{2\pi}\Big)^{1/2} \int_{\mathbb{R}} f(u) \exp\Big\{\frac{iq}{2}u^2\Big\}\,du \tag{1.10}$$

for all nonzero real number q.

2. Relationship between the Wiener integral and the analytic Feynman integral of cylinder function

In this section we give a relationship between the Wiener integral and the analytic Feynman integral on $C_0[0,T]$ for cylinder function, that is, we express the analytic Feynman integral of functional of the form (1.7) as a limit of Wiener integrals on $C_0[0,T]$. We begin this section by introducing a generalized Chapman-Kolmogorov equation.

THEOREM 2.1 (Generalized Chapman-Kolmogorov equation). *Let a and b be positive real numbers. Then we have*

$$\begin{aligned}&\int_{\mathbb{R}} \exp\{-a(w-v)^2 - b(v-u)^2\}\,dv \\ &= \Big(\frac{\pi}{a+b}\Big)^{1/2} \exp\Big\{-\frac{ab}{a+b}(w-u)^2\Big\}.\end{aligned} \tag{2.1}$$

Proof. We begin by considering the exponential of the integrand in (2.1), but without the minus sign. This is a quadratic function of v and, by completing the square, we obtain

$$\begin{aligned}a(w-v)^2 + b(v-u)^2 &= (a+b)\Big(v^2 - \frac{2(aw+bu)}{a+b}v\Big) + aw^2 + bu^2 \\ &= (a+b)\Big(v - \frac{aw+bu}{a+b}\Big)^2 + \frac{ab}{a+b}(w-u)^2.\end{aligned}$$

Using the above equation and the translation invariance of the Lebesgue integral, we see that the left hand side of (2.1) equals

$$\begin{aligned}&\exp\Big\{-\frac{ab}{a+b}(w-v)^2\Big\}\int_{\mathbb{R}}\exp\Big\{-(a+b)\Big(v-\frac{aw+bu}{a+b}\Big)^2\Big\}\,dv\\&=\Big(\frac{\pi}{a+b}\Big)^{1/2}\exp\Big\{-\frac{ab}{a+b}(w-u)^2\Big\},\end{aligned}$$

where the last equality follows from the well known integration formula $\int_{\mathbb{R}}\exp\{-k^2v^2\}\,dv=\sqrt{\pi}/k$. □

We next introduce an integration formula which is useful in the proof of Theorem 2.4. Although equation (2.3) below holds even if $\{\phi_1,\dots,\phi_n,\alpha\}$ is linearly dependent (see Remark 2.3 below), we assume for a moment that $\{\phi_1,\dots,\phi_n,\alpha\}$ is linearly independent in Lemma 2.2.

LEMMA 2.2. *Let $1\le p\le\infty$ and let $F\in\mathcal{A}^{(p)}$ be given by (1.7), where $\|\alpha\|=1$. Let $\{\phi_1,\dots,\phi_n\}$ be an orthonormal set in $L_2[0,T]$ and let $\lambda\in\mathbb{C}_+$. Then the functional*

$$\exp\Big\{\frac{1-\lambda}{2}\sum_{k=1}^{n}\langle\phi_k,x\rangle^2\Big\}F(x)\tag{2.2}$$

is Wiener integrable on $C_0[0,T]$ and

$$\begin{aligned}&\int_{C_0[0,T]}\exp\Big\{\frac{1-\lambda}{2}\sum_{k=1}^{n}\langle\phi_k,x\rangle^2\Big\}F(x)\,dm(x)\\&=(2\pi)^{-1/2}\lambda^{-(n-1)/2}C_{n,\lambda}^{-1/2}\int_{\mathbb{R}}\exp\Big\{-\frac{\lambda}{2C_{n,\lambda}}v^2\Big\}f(v)\,dv,\end{aligned}\tag{2.3}$$

where

$$c_k=\langle\phi_k,\alpha\rangle\tag{2.4}$$

for $k=1,2,\dots,n$ and $C_{n,\lambda}=(1-\lambda)(c_1^2+\dots+c_n^2)+\lambda$.

Proof. Since F is measurable, it is only necessary to prove that the Wiener integral of functional in (2.2) is finite. But this is obvious if we show (2.3). By the Gram-Schmidt process, we obtain $\phi_{n+1}\in L_2[0,T]$ such that $\{\phi_1,\dots,\phi_{n+1}\}$ is an orthonormal set in $L_2[0,T]$ and

$$\alpha=\sum_{k=1}^{n+1}c_k\phi_k,$$

where c_k for $k=1,\dots,n$ are given as in (2.4) and

$$c_{n+1} = \Big(1 - \sum_{j=1}^{n} \langle \phi_j, \alpha \rangle^2\Big)^{1/2}.$$

Assume that $\lambda > 0$ and let K be the Wiener integral on the left hand side of (2.3). Then by (1.7),

$$K = \int_{C_0[0,T]} \exp\Big\{\frac{1-\lambda}{2} \sum_{k=1}^{n} \langle \phi_k, x \rangle^2\Big\} f\Big(\sum_{k=1}^{n+1} c_k \langle \phi_k, x \rangle\Big)\, dm(x).$$

By the Wiener integration formula (1.8),

$$K = (2\pi)^{-(n+1)/2} \int_{\mathbb{R}^{n+1}} \exp\Big\{\frac{1-\lambda}{2} \sum_{k=1}^{n} u_k^2 - \frac{1}{2} \sum_{k=1}^{n+1} u_k^2\Big\} f\Big(\sum_{k=1}^{n+1} c_k u_k\Big)\, du_1 \cdots du_{n+1}.$$

Now we evaluate the last integral by changing variables, that is, let

$$v_k = \sum_{j=1}^{k} c_j u_j, \quad k = 1, 2, \ldots, n+1.$$

Then

$$u_k = \frac{1}{c_k}(v_k - v_{k-1}), \quad k = 1, 2, \ldots, n+1,$$

where $v_0 = 0$, and the Jacobian is given by

$$\frac{\partial(u_1, \ldots, u_{n+1})}{\partial(v_1, \ldots, v_{n+1})} = \frac{1}{c_1 \cdots c_{n+1}}.$$

Hence

$$K = (2\pi)^{-(n+1)/2} \frac{1}{c_1 \cdots c_{n+1}} \int_{\mathbb{R}^{n+1}} \exp\Big\{-\frac{\lambda}{2} \sum_{k=1}^{n} \frac{(v_k - v_{k-1})^2}{c_k^2} - \frac{1}{2} \frac{(v_{n+1} - v_n)^2}{c_{n+1}^2}\Big\} f(v_{n+1})\, dv_1 \cdots dv_{n+1}.$$

Using the generalized Chapman Kolmogorov equation (2.1), we evaluate the last integral with respect to $v_1, \ldots, v_{n-1}$ to obtain

$$K = (2\pi)^{-2/2} \lambda^{-(n-1)/2} [(c_1^2 + \cdots + c_n^2) c_{n+1}^2]^{-1/2} \int_{\mathbb{R}^2} \exp\Big\{-\frac{\lambda v_n^2}{2(c_1^2 + \cdots + c_n^2)} - \frac{(v_{n+1} - v_n)^2}{2c_{n+1}^2}\Big\} f(v_{n+1})\, dv_n\, dv_{n+1}.$$

Applying the generalized Chapman Kolmogorov equation once more, we have

$$K = (2\pi)^{-1/2}\lambda^{-(n-1)/2}(c_1^2+\cdots+c_n^2+\lambda c_{n+1}^2)^{-1/2}$$
$$\int_{\mathbb{R}} \exp\Big\{-\frac{\lambda}{2(c_1^2+\cdots+c_n^2+\lambda c_{n+1}^2)}v_{n+1}^2\Big\} f(v_{n+1})\,dv_{n+1}.$$

Finally by the definition of c_k for $k=1,\dots,n+1$, we have $c_1^2+\cdots+c_n^2+\lambda c_{n+1}^2 = C_{n,\lambda}$ and so we know that (2.3) is valid for all $\lambda>0$. But each side of (2.3) is an analytic function of $\lambda \in \mathbb{C}_+$. By the uniqueness of the analytic extension, we conclude that (2.3) holds for all $\lambda \in \mathbb{C}_+$ and this completes the proof. □

REMARK 2.3. Suppose that $\{\phi_1,\dots,\phi_n,\alpha\}$ is linearly dependent in Lemma 2.2. Then $c_{n+1}=0$ and

$$\alpha = \sum_{k=1}^n c_k\phi_k,$$

where $c_k = \langle \phi_k, \alpha\rangle$ for $k=1,\dots,n$. Moreover $c_1^2+\cdots+c_n^2 = 1$ and $C_{n,\lambda}=1$. In this case, the K in the proof of Lemma 2.2 can be simplified as

$$\begin{aligned} K &= \int_{C_0[0,T]} \exp\Big\{\frac{1-\lambda}{2}\sum_{k=1}^n \langle\phi_k,x\rangle^2\Big\} f\Big(\sum_{k=1}^n c_k\langle\phi_k,x\rangle\Big)\,dm(x) \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n}\exp\Big\{-\frac{\lambda}{2}\sum_{k=1}^n u_k^2\Big\} f\Big(\sum_{k=1}^n c_k u_k\Big)\,du_1\cdots du_n. \end{aligned}$$

By changing variables

$$v_k = \sum_{j=1}^k c_j u_j, \quad k=1,2,\dots,n,$$

we have that

$$K = (2\pi)^{-n/2}\frac{1}{c_1\cdots c_n}\int_{\mathbb{R}^n}\exp\Big\{-\frac{\lambda}{2}\sum_{k=1}^n \frac{(v_k-v_{k-1})^2}{c_k^2}\Big\} f(v_n)\,dv_1\cdots dv_n.$$

We evaluate the last integral using the generalized Chapman Kolmogorov equation to obtain

$$K = (2\pi)^{-1/2}\lambda^{-(n-1)/2}\int_{\mathbb{R}}\exp\Big\{-\frac{\lambda}{2}v^2\Big\} f(v)\,dv.$$

But this is the same as (2.3) when $C_{n,\lambda}=1$. Hence we conclude that (2.3) is true whether $\{\phi_1,\dots,\phi_n,\alpha\}$ is linearly independent or not.

Now we give a relationship between the analytic Feynman integral and the Wiener integral on $C_0[0,T]$ for functionals of the form (1.7). In this theorem we express the analytic Feynman integral of functionals of the form (1.7) as a limit of Wiener integrals.

THEOREM 2.4. *Let $F \in \mathcal{A}^{(1)}$ be given by* (1.7), *where* $\|\alpha\| = 1$. *Let $\{\phi_1, \phi_2, \ldots\}$ be a complete orthonormal set of functionals in $L_2[0,T]$. Let q be a nonzero real number and let $\{\lambda_1, \lambda_2, \ldots\}$ be a sequence of complex numbers in $\mathbb{C}_+$ such that $\lambda_n \to -iq$. Then we have*

$$\begin{aligned} &\int_{C_0[0,T]}^{\mathrm{anf}_q} F(x)\,dm(x) \\ &= \lim_{n\to\infty} \lambda_n^{n/2} \int_{C_0[0,T]} \exp\Big\{\frac{1-\lambda_n}{2}\sum_{k=1}^{n}\langle \phi_k, x\rangle^2\Big\} F(x)\,dm(x). \end{aligned} \tag{2.5}$$

Proof. Let $\Gamma(\lambda_n)$ be the Wiener integral on the right hand side of (2.5). Then by Lemma 2.2,

$$\lim_{n\to\infty} \lambda_n^{n/2}\Gamma(\lambda_n) = \lim_{n\to\infty}\Big(\frac{\lambda_n}{2\pi}\Big)^{1/2} C_{n,\lambda_n}^{-1/2} \int_{\mathbb{R}} \exp\Big\{-\frac{\lambda_n}{2C_{n,\lambda_n}}v^2\Big\} f(v)\,dv.$$

As we described in Remark 2.3, if α can be expressed as a finite linear combination of $\{\phi_1, \phi_2, \ldots\}$, then $C_{n,\lambda_n} = 1$ in the last expression. Since $\{\phi_1, \phi_2, \ldots\}$ is a complete orthonormal set of functionals in $L_2[0,T]$, by the Parseval's identity, we have $C_{n,\lambda_n} \to \|\alpha\| = 1$ as $n \to \infty$. Moreover since $f \in L_1(\mathbb{R})$ we apply the dominated convergence theorem to obtain

$$\lim_{n\to\infty} \lambda_n^{n/2}\Gamma(\lambda_n) = \Big(-\frac{iq}{2\pi}\Big)^{1/2} \int_{\mathbb{R}} \exp\Big\{\frac{iq}{2}v^2\Big\} f(v)\,dv.$$

By (1.10) in Theorem 1.1 it follows that (2.5) holds and the theorem is proved. □

Although relationship (2.5) holds whether $\{\alpha, \phi_1, \phi_2, \ldots\}$ is linearly independent or not, we restate the relationship when $\{\alpha, \phi_1, \phi_2, \ldots\}$ is linearly dependent in the following corollary as it is given in Theorem 3.4 of [8].

COROLLARY 2.5. *Let $F \in \mathcal{A}^{(1)}$ be given by* (1.7), *where* $\|\alpha\| = 1$. *Let l be a positive integer and let $\{\phi_1, \ldots, \phi_l\}$ be an orthonormal set of functionals in $L_2[0,T]$ such that $\{\phi_1, \ldots, \phi_l, \alpha\}$ is linearly dependent. Let q be a nonzero real number and let $\{\lambda_1, \lambda_2, \ldots\}$ be a sequence of* *complex numbers in* $\mathbb{C}_+$ *such that* $\lambda_n \to -iq$ *as* $n \to \infty$*. Then we have*

$$\begin{aligned}(2.6)\qquad &\int_{C_0[0,T]}^{\mathrm{anf}_q} F(x)\,dm(x)\\ &= \lim_{n\to\infty} \lambda_n^{l/2} \int_{C_0[0,T]} \exp\Big\{\frac{1-\lambda_n}{2}\sum_{k=1}^{l}\langle\phi_k, x\rangle^2\Big\} F(x)\,dm(x).\end{aligned}$$

In Theorem 2.4 and Corollary 2.5 above, since λ_n goes to $-iq$, to pass the limit under the integral sign using the dominated convergence theorem, we need to restrict the functional F belongs to $\mathcal{A}^{(1)}$. But if λ_n goes to some value in $\mathbb{C}_+$ or $\lambda_n = \lambda$ for all $n = 1, 2, \ldots$ for some fixed $\lambda \in \mathbb{C}_+$, then the restriction $F \in \mathcal{A}^{(1)}$ is not necessary. Of course in this case we can just say on the analytic Wiener integral but not on the analytic Feynman integral.

In Theorem 2.6 below we give a relationship between the analytic Wiener integral and the Wiener integral on $C_0[0,T]$ for functionals of the form (1.7).

THEOREM 2.6. *Let* $1 \le p \le \infty$ *and let* $F \in \mathcal{A}^{(p)}$ *be given by* (1.7)*, where* $\|\alpha\| = 1$*. Let* $\{\phi_1, \phi_2, \ldots\}$ *be a complete orthonormal set of functionals in* $L_2[0,T]$ *and let* $\lambda \in \mathbb{C}_+$*. Then we have*

$$\begin{aligned}(2.7)\qquad &\int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x)\,dm(x)\\ &= \lim_{n\to\infty} \lambda^{n/2} \int_{C_0[0,T]} \exp\Big\{\frac{1-\lambda}{2}\sum_{k=1}^{n}\langle\phi_k, x\rangle^2\Big\} F(x)\,dm(x).\end{aligned}$$

Proof. To prove this theorem, we modify the first part of the proof of Theorem 2.4 by replacing λ_n by λ whenever it occurs. Then we have

$$\lim_{n\to\infty} \lambda^{n/2}\Gamma(\lambda) = \lim_{n\to\infty} \Big(\frac{\lambda}{2\pi}\Big)^{1/2} C_{n,\lambda}^{-1/2} \int_{\mathbb{R}} \exp\Big\{-\frac{\lambda}{2C_{n,\lambda}}v^2\Big\} f(v)\,dv.$$

Since f belongs to $L_p(\mathbb{R}^n)$ and $\mathrm{Re}\lambda > 0$, we apply the dominated convergence theorem to obtain

$$\lim_{n\to\infty} \lambda^{n/2}\Gamma(\lambda) = \Big(\frac{\lambda}{2\pi}\Big)^{1/2} \int_{\mathbb{R}} \exp\Big\{-\frac{\lambda}{2}v^2\Big\} f(v)\,dv.$$

By (1.9) in Theorem 1.1 the proof is completed. □

If $\{\phi_1, \dots, \phi_n, \alpha\}$ is linearly dependent for some $n = 1, 2, \dots$, then $C_{n,\lambda} = 1$ in the proof of Theorem 2.6 and so

$$\lambda^{n/2}\Gamma(\lambda) = \Big(\frac{\lambda}{2\pi}\Big)^{1/2} \int_{\mathbb{R}} \exp\Big\{-\frac{\lambda}{2}v^2\Big\} f(v)\, dv = \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x)\, dm(x),$$

that is, we need not the limit in (2.7). Hence we have the following corollary which is given in Theorem 3.3 of [8].

Corollary 2.7. *Let $1 \le p \le \infty$ and let $F \in \mathcal{A}^{(p)}$ be given by (1.7), where $\|\alpha\| = 1$. Let n be a positive integer and let $\{\phi_1, \dots, \phi_n\}$ be an orthonormal set of functionals in $L_2[0,T]$ such that $\{\phi_1, \dots, \phi_n, \alpha\}$ is linearly dependent. Let $\lambda \in \mathbb{C}_+$. Then we have*

(2.8)
$$\int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x)\, dm(x) = \lambda^{n/2} \int_{C_0[0,T]} \exp\Big\{\frac{1-\lambda}{2} \sum_{k=1}^{n} \langle \phi_k, x\rangle^2\Big\} F(x)\, dm(x).$$

Our main result, namely a change of scale formula for Wiener integral of cylinder function on $C_0[0,T]$ now follows from Theorem 2.6.

Theorem 2.8. *Let $1 \le p \le \infty$ and let $F \in \mathcal{A}^{(p)}$ be given by (1.7), where $\|\alpha\| = 1$. Let $\{\phi_1, \phi_2, \dots\}$ be a complete orthonormal set of functionals in $L_2[0,T]$. Then we have*

$$\begin{aligned} &\int_{C_0[0,T]} F(\rho x)\, dm(x) \\ &= \lim_{n\to\infty} \rho^{-n} \int_{C_0[0,T]} \exp\Big\{\frac{\rho^2-1}{2\rho^2} \sum_{k=1}^{n} \langle \phi_k, x\rangle^2\Big\} F(x)\, dm(x) \end{aligned} \tag{2.9}$$

where $\rho > 0$.

Proof. By letting $\lambda = \rho^{-2}$ in (2.7) we obtain (2.9). □

Note that the change of scale formula (2.9) for cylinder function on Wiener space is the same as the Cameron and Storvick's change of scale formula for Wiener integral of functionals in a Banach algebra $\mathcal{S}$ in [4].

If $\{\phi_1, \dots, \phi_n, \alpha\}$ is linearly dependent for some $n = 1, 2, \dots$, then by letting $\lambda = \rho^{-2}$ in (2.8) we have the following corollary which is given in Theorem 3.5 of [8]. In fact, Kim considered in [4] the case $\phi_1 = \alpha$.

Corollary 2.9. *Let $1 \le p \le \infty$ and let $F \in \mathcal{A}^{(p)}$ be given by (1.7), where $\|\alpha\| = 1$. Let n be a posiitve integer and let $\{\phi_1, \dots, \phi_n\}$ be an* *orthonormal set of functionals in* $L_2[0,T]$ *such that* $\{\phi_1,\ldots,\phi_n,\alpha\}$ *is linearly dependent. Then we have*

(2.10)

$$\int_{C_0[0,T]} F(\rho x)\,dm(x) = \rho^{-n}\int_{C_0[0,T]} \exp\Big\{\frac{\rho^2-1}{2\rho^2}\sum_{k=1}^{n}\langle\phi_k,x\rangle^2\Big\}F(x)\,dm(x)$$

where $\rho > 0$.

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