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ON CHARACTERIZATIONS OF PARETO AND WEIBULL DISTRIBUTIONS BY CONSIDERING CONDITIONAL EXPECTATIONS OF UPPER RECORD VALUES

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Abstract. Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with absolutely continuous cumulative distribution function(cdf) $F(x)$ and the corresponding probability density function(pdf) $f(x)$. In this paper, we give characterizations of Pareto and Weibull distribution by considering conditional expectations of record values.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed (i.i.d.) random variables with cumulative distribution function(cdf) $F(x)$ and probability density function (pdf) $f(x)$. Let $Y_n = max\{X_1, X_2, \cdots, X_n\}$ for $n \geq 1$. We say X_j is an upper record value of this sequence, if $Y_j > (<) Y_{j-1}$ for $j > 1$. The indices at which the upper record values occur are given by the record times $\{U(n), n \geq 1\}$, where $U(n) = min\{j \mid j > U(n-1), X_j > X_{U(n-1)}, n \geq 2\}$ with $U(1) = 1$. We assume that all upper record values $X_{U(i)}$ for $i \geq 1$ occur at a sequence $\{X_n, n \geq 1\}$ of i.i.d. random variables.

Lee(2003) showed that $X \in PAR(\theta)$ if and only if $(\theta+1)^i E[X_{U(n+i)} \mid X_{U(m)} = y] = \theta^i E[X_{U(n)} \mid X_{U(m)} = y]$ for $i = 1, 2, 3,\ n \geq m+1$. Also, Dembiǹska & Wesolowski(2000) obtained characterizations of the exponential, Pareto, and power distributions by considering this problem with $E(R_{m+k} \mid R_m) = aR_m + b$. Further, Ahsanullah and Shakil(2012)

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proved that $X \in PAR(\beta)$ if and only if $(\delta - k)E[X^k_{U(n+1)} \mid X_{U(m)} = y] = \delta E[X^k_{U(n)} \mid X_{U(m)} = y], m \geq n+1$.

In this paper, we give the characterizations of Pareto and Weibull distributions by considering conditional expectations of record values.

2. Main results

THEOREM 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables with cumulative distribution function $F(x)$ which is absolutely continuous with pdf $f(x)$ and $E(X^k) < \infty$. For positive integers k, l and a positive real number $\theta < k$,*

(2.1)
$$(\theta - k)^l E[X^k_{U(n+l)} \mid X_{U(m)} = y] = \theta^l E[X^k_{U(n)} \mid X_{U(m)} = y]\ ,\ n \geq m+1$$

if and only if

$$F(x) = 1 - cx^{-\theta}, \quad 0 < c \leq x, \quad \theta > 0. \tag{2.2}$$

Proof. For necessary part, it is easy to see that (2.2) implies (2.1).

For the sufficiency part, using Ahsanullah formula(1995), we get the following equation

$$\begin{aligned} &\frac{(\theta - k)^l}{(n+l-m-1)!} \int_y^\infty \Big(- ln \frac{\bar{F}(x)}{\bar{F}(y)} \Big)^{n+l-m-1} x^k f(x) dx \\ &= \frac{\theta^l}{(n-m-1)!} \int_y^\infty \Big(- ln \frac{\bar{F}(x)}{\bar{F}(y)} \Big)^{n-m-1} x^k f(x) dx. \end{aligned} \tag{2.3}$$

Since $F(x)$ is absolutely continuous, we can differentiate $(n-m)$ times both sides of (2.3) with respect to y and simplify to obtain the following equation

$$\frac{1}{\bar{F}(y)\Gamma(l)} \int_y^\infty \Big(- ln \frac{\bar{F}(x)}{\bar{F}(y)} \Big)^{l-1} x^k f(x) dx = \frac{\theta^l}{(\theta - k)^l}\ y^k. \tag{2.4}$$

By setting $s = \frac{\bar{F}(x)}{\bar{F}(y)}$ in (2.4) we get

$$\frac{1}{\Gamma(l)} \int_0^1 \big[\bar{F}^{-1}(s\bar{F}(y))\big]^k \big(- ln(s)\big)^{l-1} ds = \frac{\theta^l}{(\theta - k)^l}\ y^k.$$

Setting $\bar{F}(y) = t$, we find that

$$\frac{1}{\Gamma(l)} \int_0^1 \big[\bar{F}^{-1}(st)\big]^k \big(- ln(s)\big)^{l-1} ds = \frac{\theta^l}{(\theta - k)^l}\ [\bar{F}^{-1}(t)]^k,\ \ 0 < t < 1.$$

Dividing both sides by $\frac{(\theta-k)^l}{\theta^l}$ and substituting once again $s = e^{-v}$ and $t = e^{-w}$, we have

$$\frac{(\theta-k)^l}{\theta^l\Gamma(l)}\int_0^\infty \left[\bar{F}^{-1}(e^{-(v+w)})\right]^k (v)^{l-1}e^{-v}dv = [\bar{F}^{-1}(e^{-w})]^k,\ \ 0 < w < \infty.$$

Now we use applications of integrated Cauchy functional equation in C.R. Rao & D.N. Shanbhag(1994) and let $H(w) = [\bar{F}^{-1}(e^{-w})]^k$. Consequently,

$$\int_{R^+} H(v+w)\mu(dv) = H(w),\ \ 0 < w < \infty, \tag{2.5}$$

where μ is a finite measure on R^+, which is absolutely continuous with respect to the $[L]$ measure and is defined by

$$\mu(dv) = \frac{(\theta-k)^l}{\theta^l\Gamma(l)}(v)^{l-1}e^{-v}dv. \tag{2.6}$$

Observe that H is strictly increasing on $[0,\infty)$ since it is a superposition of two strictly decreasing functions. Hence, since H is continuous it follows that

$$H(w) = \begin{cases}\gamma + \alpha(1-exp(\eta w)) \ \ if \ \ \eta \neq 0, \\ \gamma + \beta w \ \ if \ \ \eta = 0, w > 0,\end{cases} \tag{2.7}$$

where α, β, γ are all constants and η is determined such that

$$\int_{R^+} exp(\eta x)\mu(dx) = 1. \tag{2.8}$$

From (2.8) we get

$$1 = \frac{(\theta-k)^l}{\theta^l\Gamma(l)}\int_0^\infty (x)^{l-1}e^{(\eta-1)x}dx. \tag{2.9}$$

which means that $\eta < 1$, since the integral at the right-hand side has to converge. Consequently,

$$\frac{(\theta-k)^l}{\theta^l} = (1-\eta)^l. \tag{2.10}$$

From equation (2.10) we obtain $\eta = \frac{k}{\theta}$ and it follows that $0 < \eta < 1$. Using equation (2.7) we get

$$[\bar{F}^{-1}(e^{-x})]^k = \gamma + \alpha(1 - e^{\eta x}).$$

By applying Theorem 1 in Anna Dembiǹska & Jacek Wesolowski(2000) for $a = \frac{\theta^l}{(\theta-k)^l} > 1$ and $\alpha + \gamma = 0$, we have

$$\bar{F}(z) = cz^{-\theta}, \quad 0 < c \leq z, \quad \theta > 0, \quad c = \gamma^{\frac{\theta}{k}}.$$

Hence, the theorem is proved. □

THEOREM 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables with cumulative distribution function $F(x)$ which is absolutely continuous with pdf $f(x)$. For positive integers k, l and a positive real number θ,*

(2.11)
$$E[X^k_{U(n+l)} \mid X_{U(m)} = y] = E[X^k_{U(n)} \mid X_{U(m)} = y] + l\theta^k \ , \ n \geq m+1.$$

if and only if

$$F(x) = 1 - e^{-(\frac{x}{\theta})^k} \ , \ 0 \leq x \ , \ \theta > 0. \tag{2.12}$$

Proof. For necessary part, it is easy to see that (2.12) implies (2.11).
For the sufficiency part, we get

$$\begin{aligned} &\frac{1}{(n+l-m-1)!}\int_{-\infty}^{y}\left(-ln\frac{\bar{F}(x)}{\bar{F}(y)}\right)^{n+l-m-1} x^k f(x)dx \\ &= \frac{1}{(n-m-1)!}\int_{-\infty}^{y}\left(-ln\frac{\bar{F}(x)}{\bar{F}(y)}\right)^{n-m-1} x^k f(x)dx + l\theta^k. \end{aligned} \tag{2.13}$$

Since $F(x)$ is absolutely continuous, we can differentiate $(n-m)$ times both sides of (2.13) with respect to y and simplify to obtain the following equation

$$\frac{1}{\bar{F}(y)\Gamma(l)}\int_{y}^{\infty}\left(-ln\frac{\bar{F}(x)}{\bar{F}(y)}\right)^{l-1} x^k f(x)dx = y^k + l\theta^k. \tag{2.14}$$

By setting $s = \frac{\bar{F}(x)}{\bar{F}(y)}$ in (2.14) we get

$$\frac{1}{\Gamma(l)}\int_0^1 \left[\bar{F}^{-1}(s\bar{F}(y))\right]^k \left(-ln(s)\right)^{l-1} ds = y^k + l\theta^k.$$

Setting $\bar{F}(y) = t$, we find that

$$\frac{1}{\Gamma(l)}\int_0^1 \left[\bar{F}^{-1}(st)\right]^k \left(-ln(s)\right)^{l-1} ds = [\bar{F}^{-1}(t)]^k + l\theta^k, \ \ 0 < t < 1.$$

By setting $s = e^{-v}$ and $t = e^{-w}$, we obtain

$$\frac{1}{\Gamma(l)}\int_0^\infty \left[\bar{F}^{-1}(e^{-(v+w)})\right]^k (v)^{l-1} e^{-v} dv = [\bar{F}^{-1}(e^{-w})]^k + l\theta^k, \ \ 0 < w < \infty.$$

In the same way as the proof of Theorem 1.1, we get

$$1 = \frac{1}{\Gamma(l)} \int_0^\infty (x)^{l-1} e^{(\eta-1)x} dx, \tag{2.15}$$

which means that $\eta < 1$, since the integral at the right-hand side has to converge. Consequently,

$$1 = (1-\eta)^l. \tag{2.16}$$

From equation (2.16) we obtain $\eta = 0$.

Using equation (2.7) we get

$$[\bar{F}^{-1}(e^{-x})]^k = \gamma + \beta x.$$

By applying Theorem 1 in Anna Dembiǹska & Jacek Wesolowski(2000) for $\gamma = 0$, $a = 1$ and $\beta = \theta^k$, we have

$$\bar{F}(z) = e^{-(\frac{z}{\theta})^k}, \quad z \geq 0, \quad \theta > 0.$$

Hence, the theorem is proved. □

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