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AN OPTIMAL CONSUMPTION AND INVESTMENT PROBLEM WITH LABOR INCOME AND REGIME SWITCHING

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Abstract. I use the dynamic programming approach to study the optimal consumption and investment problem with regime-switching and constant labor income. I derive the optimal solutions in closed-form with constant absolute risk aversion (CARA) utility and constant disutility.

1. Introduction

Following the seminal works of Merton [3, 4], the field of continuous-time portfolio selection is one of the most important areas in mathematical finance. Also recently regime-switching technique is widely used in mathematical finance (see [1, 7, 5, 6]).

In this work I study the optimal consumption and investment problem with two-state regime-switching and constant labor income under the dynamic programming framework based on Karatzas *et al.* [2]. I use the constant absolute risk aversion (CARA) utility function and constant disutility to derive the optimal solutions in closed-form.

2. The financial market

On a proper probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a standard Brownian motion B_t and a continuous-time two-state Markov chain ϵ_t are defined and it is assumed that they are independent. The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by both the Brownian motion B_t and the Markov chain ϵ_t.

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In the financial market, it is assumed that only two assets are traded: One is a riskless asset with constant interest rate $r > 0$ and the other is a risky asset (or stock). It is also assumed that there are two regime states, 1, 2 in the market and regime i switches into regime j at the first jump time of an independent Poisson process with intensity λ_i, for $i, j \in \{1, 2\}$. In regime $i \in \{1, 2\}$, the risky asset price process follows $dS_t/S_t = \mu_i dt + \sigma_i dB_t$. The market price of risk is defined by $\theta_i := (\mu_i - r)/\sigma_i,\ i = 1,\ 2$. Let π_t be the $\mathcal{F}_t$-progressively measurable portfolio process at time t and c_t be the nonnegative $\mathcal{F}_t$-progressively measurable consumption rate process at time t. I assume that the portfolio process π_t and the consumption rate process c_t satisfy the following conditions:

$$\int_0^t \pi_s^2 ds < \infty \quad \text{and} \quad \int_0^t c_s ds < \infty, \text{ for all } t \geq 0, \text{ almost surely (a.s.)}.$$

In regime $i \in \{1, 2\}$, the agent receives constant labor income $y_i > 0$. The agent's wealth process X_t at time t follows

$$dX_t = [rX_t + \pi_t(\mu_i - r) - c_t + y_i]\, dt + \sigma_i \pi_t dB_t, \quad X_0 = x > -\frac{y_i}{r},\ i = 1,\ 2.$$

3. The optimization problem

The agent's expected utility maximization problem with CARA utility $u(c) := -e^{-\gamma c}/\gamma$ is given by

$$V_i(x) = \sup_{(c,\pi)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^{\tau_i} e^{-\rho t}\left(-\frac{e^{-\gamma c_t}}{\gamma} - l\right) dt + e^{-\rho\tau_i} V_j(X_{\tau_i})\right], \tag{3.1}$$

where τ_i is the first jump time from i-th state to j-th state, $\rho > 0$ is a subjective discount factor, $\gamma > 0$ is the coefficient of absolute risk aversion, $l > 0$ is constant disutility because of labor, and $\mathcal{A}(x)$ is an admissible class of pairs (c, π) at x, where $i, j \in \{1, 2\}$ and $i \neq j$. It is assumed that the following inequality always holds without further comments:

Assumption 3.1.

$$\rho - r + \frac{\theta_i^2}{2} > 0,\ i \in \{1, 2\}.$$

Assumption 3.2. It is assumed that the value function $V_i(x)$ for this optimization problem (3.1) is an increasing function, that is,

$$V_i'(x) > 0, \text{ for } i = 1, 2.$$

In fact, $V_i'(x) > 0$ (see (3.11)).

My main results are given in the next theorem.

Theorem 3.3. *The value function for this optimization problem (3.1) is given by*

$$V_i(x) = -\frac{1}{r\gamma} M_i e^{-\gamma(rx+y_i)} - \frac{l}{\rho}, \ i = 1,\ 2,$$

where (M_1, M_2) *is the unique pair solution of the system of algebraic equations*

$$\left(\rho - r + \lambda_i + \frac{1}{2}\theta_i^2\right) e^{-\gamma y_i} M_i + r e^{-\gamma y_i} M_i \log M_i - \lambda_i e^{-\gamma y_j} M_j = 0,$$

for $i, j \in \{1, 2\}$ *and* $i \neq j$. *And the optimal policies* (c_i^*, π_i^*) *for this optimization problem (3.1) are given by*

$$c_i^* = rx + y_i - \frac{1}{\gamma}\log M_i \quad \textit{and} \quad \pi_i^* = \frac{\theta_i}{\sigma_i r\gamma}, \ i = 1,\ 2.$$

Proof. From the expected utility optimization problem (3.1), I derive the coupled Bellman equations

$$\max_{(c_i,\pi_i)} \Bigg[\{rx + \pi_i(\mu_i - r) - c_i + y_i\} V_i'(x) + \frac{1}{2}\sigma_i^2 \pi_i^2 V_i''(x) \\ -(\rho + \lambda_i)V_i(x) + \lambda_i V_j(x) - \frac{e^{-\gamma c_i}}{\gamma} - l \Bigg] = 0, \tag{3.2}$$

where $i, j \in \{1, 2\}$ and $i \neq j$. The first-order conditions (FOCs) for the Bellman equations (3.2) give

$$c_i^* = -\frac{1}{\gamma}\log\{V_i'(x)\} \quad \text{and} \quad \pi_i^* = -\frac{\theta_i V_i'(x)}{\sigma_i V_i''(x)}, \ i = 1,\ 2. \tag{3.3}$$

Plugging the FOCs (3.3) into the equations (3.2), then I obtain

$$rxV_i'(x) + y_i V_i'(x) - \frac{1}{2}\theta_i^2 \frac{\{V_i'(x)\}^2}{V_i''(x)} + \frac{1}{\gamma}V_i'(x)\log\{V_i'(x)\} \\ -(\rho + \lambda_i)V_i(x) + \lambda_i V_j(x) - \frac{1}{\gamma}V_i'(x) - l = 0, \tag{3.4}$$

where $i, j \in \{1, 2\}$ and $i \neq j$. Now it is assumed that the optimal consumption $c_i^* = C_i(x)$, $i = 1,\ 2$, is a function of wealth x. And let $X_i(\cdot)$, $i = 1,\ 2$, be the inverse function of $C_i(\cdot)$, $i = 1,\ 2$, that is, $X_i(\cdot) = C_i^{-1}(\cdot)$, $i = 1,\ 2$. Then the FOCs (3.3) give

$$V_i'(x) = e^{-\gamma C_i(x)} \quad \text{and} \quad V_i''(x) = -\frac{\gamma e^{-\gamma C_i(x)}}{X_i'(c_i)}, \ i = 1,\ 2. \tag{3.5}$$

Substituting (3.5) into the equations (3.4), then I have

$$
\begin{aligned}
(3.6) \qquad & rX_i(c_i)e^{-\gamma c_i} + y_i e^{-\gamma c_i} + \frac{1}{2\gamma}\theta_i^2 X_i'(c_i)e^{-\gamma c_i} - c_i e^{-\gamma c_i} \\
& - (\rho+\lambda_i)V_i(X_i(c_i)) + \lambda_i V_j(X_i(c_i)) - \frac{1}{\gamma}e^{-\gamma c_i} - l = 0,
\end{aligned}
$$

where $i, j \in \{1, 2\}$ and $i \neq j$. Taking derivative of the equations (3.6) with respect to c_i yields

$$
\begin{aligned}
& rX_i'(c_i)e^{-\gamma c_i} - r\gamma X_i(c_i)e^{-\gamma c_i} - \gamma y_i e^{-\gamma c_i} + \frac{1}{2\gamma}\theta_i^2 X_i''(c_i)e^{-\gamma c_i} \\
& - \frac{1}{2}\theta_i^2 X_i'(c_i)e^{-\gamma c_i} + \gamma c_i e^{-\gamma c_i} - (\rho+\lambda_i)X_i'(c_i)e^{-\gamma c_i} + \lambda_i X_i'(c_i)e^{-\gamma c_j} = 0,
\end{aligned}
$$

where $i, j \in \{1, 2\}$ and $i \neq j$. Thus I derive the coupled second order ordinary differential equations (ODEs)

$$
\begin{aligned}
(3.7) \qquad & \frac{1}{2\gamma}\theta_i^2 X_i''(c_i) - \left(\rho - r + \lambda_i + \frac{1}{2}\theta_i^2\right) X_i'(c_i) - r\gamma X_i(c_i) \\
& + \gamma c_i - \gamma y_i + \lambda_i X_i'(c_i)e^{-\gamma(c_j - c_i)} = 0,
\end{aligned}
$$

where $i, j \in \{1, 2\}$ and $i \neq j$. If I conjecture the solution $X_i(c_i)$ to the coupled ODEs (3.7) of the form

(3.8)

$$
X_i(c_i) = \frac{c_i - y_i}{r} + \frac{1}{r\gamma}\log M_i \quad \text{and} \quad c_i = rx + y_i - \frac{1}{\gamma}\log M_i, \ i = 1, \ 2,
$$

for some constant $M_i > 0$, then $X_i'(c_i) = 1/r$ and $X_i''(c_i) = 0, \ i = 1, \ 2.$ The equations (3.8) yield

$$
c_j - c_i = y_j - y_i + \frac{1}{\gamma}\log\frac{M_i}{M_j},
$$

where $i, j \in \{1, 2\}$ and $i \neq j$. Thus the coupled ODEs (3.7) can be reduced into the system of algebraic equations

$$
(3.9) \quad \left(\rho - r + \lambda_i + \frac{1}{2}\theta_i^2\right) e^{-\gamma y_i} M_i + r e^{-\gamma y_i} M_i \log M_i - \lambda_i e^{-\gamma y_j} M_j = 0,
$$

where $i, j \in \{1, 2\}$ and $i \neq j$. Let $N_i := e^{-\gamma y_i} M_i > 0$, then I obtain

$$
(3.10) \qquad \left(\rho - r + \lambda_i + \frac{1}{2}\theta_i^2 + r\gamma y_i\right) N_i + rN_i \log N_i - \lambda_i N_j = 0,
$$

where $i, j \in \{1, 2\}$ and $i \neq j$.

Now I want to show that there exists a unique pair solution (M_1, M_2) to the system of algebraic equations (3.9). Thus it is enough to show that there exists a unique pair solution (N_1, N_2) to the system of algebraic equations (3.10). The proof is very similar to the proof of Theorem 3.2 in Shin [6]. Without loss of generality, I may assume that $\theta_i < \theta_j$. If I define

$$N_j = f(N_i) := \frac{1}{\lambda_i}\left(\rho - r + \lambda_i + \frac{1}{2}\theta_i^2 + r\gamma y_i\right) N_i + \frac{r}{\lambda_i} N_i \log N_i > 0,$$

for $N_i > 0$, and

$$f_1(N_i) := \frac{f(N_i)}{N_i} = \frac{1}{\lambda_i}\left(\rho - r + \lambda_i + \frac{1}{2}\theta_i^2 + r\gamma y_i\right) + \frac{r}{\lambda_i}\log N_i > 0,$$

then $f_1'(N_i) = r/(\lambda_i N_i) > 0$, that is, $f_1(\cdot)$ is increasing. Now I define the constants $\bar{x}$ and $\underline{x}$ with $\bar{x} > \underline{x}$ as follows:

$$\bar{x} := e^{-\frac{1}{r}\left(\rho - r + \frac{1}{2}\theta_i^2\right)} < 1 \quad \text{and} \quad \underline{x} := e^{-\frac{1}{r}\left(\rho - r + \lambda_i + \frac{1}{2}\theta_i^2 + r\gamma y_i\right)} < 1,$$

where the inequalities are obtained from Assumption 3.1. Then $f_1(\bar{x}) = 1 + r\gamma y_i/\lambda_i,\ f_1(\underline{x}) = 0$. Thus I have $N_i > \underline{x}$ since $f_1(N_i) > 0$ and $f_1(\cdot)$ is increasing.

Now I define

$$\begin{aligned} g(N_i) :=& \left(\rho - r + \lambda_j + \frac{1}{2}\theta_j^2 + r\gamma y_j\right) N_i f_1(N_i) \\ &+ r N_i f_1(N_i)\log\{N_i f_1(N_i)\} - \lambda_j N_i, \end{aligned}$$

and

$$\begin{aligned} g_1(N_i) :=& \frac{g(N_i)}{N_i} \\ =& \left(\rho - r + \lambda_j + \frac{1}{2}\theta_j^2 + r\gamma y_j\right) f_1(N_i) + r f_1(N_i)\log\{N_i f_1(N_i)\} - \lambda_j. \end{aligned}$$

It can be checked that

$$g_1(\bar{x}) = \left\{\frac{1}{2}(\theta_j^2 - \theta_i^2) + r\gamma y_j + r\log\left(1 + \frac{r\gamma y_i}{\lambda_i}\right)\right\}\left(1 + \frac{r\gamma y_i}{\lambda_i}\right) + \frac{r\gamma\lambda_j y_i}{\lambda_i} > 0.$$

Since, by l'Hospital's rule,

$$\begin{aligned} &\lim_{N_i \to \underline{x}+} f_1(N_i)\log\{N_i f_1(N_i)\} \\ &= \lim_{N_i \to \underline{x}+} \frac{\log\{N_i f_1(N_i)\}}{1/f_1(N_i)} = \lim_{N_i \to \underline{x}+} \frac{f_1(N_i)(f_1(N_i) + N_i f_1'(N_i))}{-N_i f_1'(N_i)} = 0, \end{aligned}$$

$\lim_{N_i \to \underline{x}+} g_1(N_i) = -\lambda_j < 0$. Thus, by intermediate value theorem, there exists $\bar{N} > 0$ such that $g_1(\bar{N}) = 0$ and $\underline{x} < \bar{N} < \bar{x}$. Taking derivative of

$g_1(N_i)$ gives

$$g_1'(N_i) = \left(\rho + \lambda_j + \frac{1}{2}\theta_j^2 + r\gamma y_j\right) f_1'(N_i) + r f_1'(N_i)\log\{N_i f_1(N_i)\} + r\frac{f_1(N_i)}{N_i}$$
$$= \frac{r}{\lambda_i N_i} h(N_i),$$

where

$$h(N_i) := \left(2\rho - r + \lambda_i + \lambda_j + \frac{\theta_i^2 + \theta_j^2}{2} + r\gamma(y_i + y_j)\right) + r\log\left\{N_i^2 f_1(N_i)\right\}.$$

Taking derivative of $h(N_i)$ implies

$$h'(N_i) = \frac{2r}{N_i} + r\frac{f_1'(N_i)}{f_1(N_i)} > 0.$$

Thus $h(\cdot)$ is increasing. Also note that $\lim_{N_i \to \underline{x}+} h(N_i) = -\infty$ and

$$h(\bar{x}) = r + \lambda_i + \lambda_j + \frac{\theta_j^2 - \theta_i^2}{2} + r\gamma(y_i + y_j) + r\log\left(1 + \frac{r\gamma y_i}{\lambda_i}\right) > 0.$$

Again, by intermediate value theorem, there exists a unique $x^* > 0$ such that $h(x^*) = 0$ and $\underline{x} < x^* < \bar{x}$. Thus $h(N_i) < 0$ for $(\underline{x}, x^*)$ and $h(N_i) > 0$ for (x^*, ∞) since $h(\cdot)$ is increasing. This means $g_1'(N_i) < 0$ for $(\underline{x}, x^*)$ and $g_1'(N_i) > 0$ for (x^*, ∞). Thus $g_1(N_i)$ is decreasing and negative for $(\underline{x}, x^*)$ and $g_1(N_i)$ is increasing for (x^*, ∞). Therefore $\bar{N}$ with $x^* < \bar{N} < \bar{x}$ is the unique solution to $g_1(N_i) = 0$, and this implies that I have the unique pair solution (N_1, N_2) to (3.10). Therefore I obtain the unique pair solution (M_1, M_2) to (3.9).

Now substituting c_i in (3.8) into (3.5) yields

(3.11)
$$V_i'(x) = M_i e^{-\gamma(rx + y_i)} > 0 \quad \text{and} \quad V_i''(x) = -r\gamma M_i e^{-\gamma(rx+y_i)} < 0,\ i = 1, 2.$$

Also substituting (3.11) into the FOCs (3.3) implies the optimal policies

$$c_i^* = rx + y_i - \frac{1}{\gamma}\log M_i \quad \text{and} \quad \pi_i^* = \frac{\theta_i}{\sigma_i r\gamma},\ i = 1,\ 2.$$

Therefore the Bellman equations (3.2) gives the value function

$$V_i(x) = -\frac{1}{r\gamma} M_i e^{-\gamma(rx+y_i)} - \frac{l}{\rho}.$$

□

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