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FREQUENTLY CONVERGENT SOLUTIONS OF A DIFFERENCE EQUATION

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Abstract. In this paper, using the definition and properties of frequency measurement, we describe the properties of solutions of a difference equation as the initial value belongs to different intervals of the whole domain. We get the main result that if the initial value belongs to $[-1, 1]$ which is different from $\frac{-1\pm\sqrt{5}}{2}$, then the solution defined by initial value have two frequent limits 0 and 1 of the same degree 0.5.

1. Introduction

Frequent limit is established using the definition of frequency measurement in the early 90s ([1]-[2]), since the classical concept of limit does not capture the fine details of sequences that do not converge. Frequency measurement is a basic tool not only for discussing sequences but also for describing properties of solutions of difference equations.

There have been many results using frequent measurement, such as frequent oscillatory, frequent convergence and frequent stability of difference equations([3]-[15]). Frequency measurement is a basic tool not only for studying sequences, but also for describing properties of solutions of difference equations.

According to the existing literature in the 1930s, frequent measurement was formally known as asymptotic density, natural density and density, etc. In the beginning of the 20th century, it had been applied to study problems of random distribution models ([1]-[2]).

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To any set A and B, denote $A \cup B$, $A \cap B$, $A \setminus B$ the union, the intersection and the difference of A and B respectively. Besides, ϕ means void set. If Z is the set of integers, and $k, l \in Z$, denoting $Z[k, \infty) = \{i \in Z | i \geq k\}$, $Z[k, l] = \{i \in Z | k \leq i \leq l\}$, $Z(-\infty, l] = \{i \in Z | i \leq l\}$. If $\Omega \subseteq Z$, then $|\Omega|$ means the number of elements of set Ω. Denoting $\Omega^{(n)} = \Omega \cap Z(-\infty, n]$.

Let $v = \{v_n\}$ be a sequence, then the set $\{n \in Z[k, \infty) | v_n > c\}$ will be denoted by $(v > c)$. The notations $(v \geq c), (v < c)$ and $(v \leq c)$ will be defined similarly.

Definition 1.1. ([1]-[2]) Let Ω be a subset of Z^+or $(Z[-k, \infty))$, if $\limsup_{n \to \infty} \frac{|\Omega^n|}{n}$ exists, then we call it upper frequency measurement of the set Ω, denoting by $\mu^*(\Omega)$; if $\lim_{n \to \infty} \inf \frac{|\Omega^n|}{n}$ exists, then we call it lower frequency measurement of Ω, denoting by $\mu_*(\Omega)$. Specially, if $\mu^*(\Omega) = \mu_*(\Omega)$, then we call it the frequency measurement of the set Ω, denoting by $\mu(\Omega)$, we also say that Ω is measurable. If Ω can not be measured, we say that Ω is unmeasurable.

The following are some properties of frequency measurement:

Proposition 1.2. ([1]-[2]) *If $\Omega \subseteq Z^+$, $\mu_*(\Omega)$ and $\mu^*(\Omega)$ both exist, then*

$$0 \leq \mu_*(\Omega) \leq \mu^*(\Omega) \leq 1.$$

If Ω is a finite set, then $\mu(\Omega) = 0$, $\mu(Z^+) = 1$. Especially $\mu(\phi) = 0$.

Proposition 1.3. ([1]-[2]) *If Ω and Γ are the subsets of Z^+, $\Omega \subseteq \Gamma$, then $\mu^*(\Omega) \leq \mu^*(\Gamma)$ and $\mu_*(\Omega) \leq \mu_*(\Gamma)$.*

Proposition 1.4. ([1]-[2]) *If Ω and Γ are two subsets of Z^+, then we have*

$$\mu_*(\Omega) + \mu^*(\Gamma) - \mu^*(\Omega \cap \Gamma) \leq \mu^*(\Omega + \Gamma) \leq \mu^*(\Omega) + \mu^*(\Gamma) - \mu_*(\Omega \cap \Gamma)$$

$$\mu_*(\Omega) + \mu_*(\Gamma) - \mu^*(\Omega \cap \Gamma) \leq \mu_*(\Omega + \Gamma) \leq \mu_*(\Omega) + \mu^*(\Gamma) - \mu_*(\Omega \cap \Gamma)$$

Besides, if Ω and Γ are mutually disjoint, then

$$\begin{aligned} &\mu_*(\Omega) + \mu_*(\Gamma) \\ &\leq \mu_*(\Omega + \Gamma) \leq \mu_*(\Omega) + \mu^*(\Gamma) \leq \mu^*(\Omega + \Gamma) \leq \mu^*(\Omega) + \mu^*(\Gamma). \end{aligned}$$

Proposition 1.5. ([1]-[2]) *For any set $\Omega \subseteq Z^+$, we have*

$$\mu_*(\Omega) + \mu^*(Z^+ \setminus \Omega) = 1.$$

Proposition 1.6. ([1]-[2]) *If* Ω *and* Γ *are two subsets of* Z^+, *and* $\Omega \subseteq \Gamma$, *then we have*

$$\mu^*(\Gamma) - \mu^*(\Omega) \le \mu^*(\Gamma \setminus \Omega) \le \mu^*(\Gamma) - \mu_*(\Omega),$$
$$\mu_*(\Gamma) - \mu^*(\Omega) \le \mu_*(\Gamma \setminus \Omega) \le \mu_*(\Gamma) - \mu_*(\Omega).$$

Proposition 1.7. ([1]-[2]) *If* Ω *and* Γ *are two subsets of* Z^+, *and* $\mu^*(\Omega) + \mu_*(\Gamma) \ge 1$, *then the set* $\Omega \cap \Gamma$ *must be an infinite set.*

2. Definitions and properties about frequent convergence

Definition 2.1. ([1]) Let $X = \{x_n\}_{n=k}^{\infty}$ be a real sequence and $I \subseteq R$. If there exists a constant $\omega \in [0,1]$ such that $\mu^*(X \notin I) \le \omega$ (or equivalently, $\mu_*(X \in I) \ge 1 - \omega$), then X is said to be frequently inside I of upper degree ω. If $\mu_*(X \notin I) \le \omega$ (or equivalently, $\mu^*(X \in I) \ge 1 - \omega$), then X is said to be frequently inside I of lower degree ω.

In particular, if $\mu^*(X \notin I) = 0$, then X is said to be frequently inside I.

Definition 2.2. ([1]-[2]) Let $X = \{x_n\}_{n=k}^{\infty}$ be a real sequence and c a constant. If for any given number $\varepsilon > 0$, there is a constant $\omega \in [0,1)$ such that $\mu^*(|X-c| \ge \varepsilon) \le \omega$ (or $(\mu_*(|X-c| \ge \varepsilon) \le \omega)$), then c is called a frequent limit of upper (respectively lower) degree ω of the sequence X, and X is said to be frequently convergent to c of upper (respectively lower) degree ω.

If there exists a constant ε_0 such that $\mu\{|X - c| \ge \varepsilon\} = \omega$ for any number $\varepsilon \in (0, \varepsilon_0)$ then the sequence X is said to be frequently convergent to c of degree ω and c is said to be a frequent limit of degree ω of X. In particular, if $\omega = 0$, we say that X frequently converges to c, and c is the frequent limit of X.

The following are properties of frequent limit, where $X = \{x_n\}$, $Y = \{y_n\}$, $Z = \{z_n\}$ are all real sequences.

Proposition 2.3. ([1]-[2]) *If* $f\lim_{n\to\infty} x_n = f\lim_{n\to\infty} y_n = a$ *and* $\mu(X \le Z \le Y) = 1$, *then* $f\lim_{n\to\infty} z_n = a$.

Proposition 2.4. ([1]-[2]) *If* $f\lim_{n\to\infty} x_n = a$ *and* $f\lim_{n\to\infty} y_n = b$, *then* $f\lim_{n\to\infty}(x_n \pm y_n) = a \pm b$ *and* $f\lim_{n\to\infty}(x_n y_n) = ab$.

Proposition 2.5. ([1]-[2]) *If* $f\lim_{n\to\infty} x_n = a$ *and* $f\lim_{n\to\infty} y_n = b \ne 0$, *then* $f\lim_{n\to\infty}(x_n/y_n) = a/b$.

Proposition 2.6. ([1]-[2]) *If* $f\lim_{n\to\infty} x_n = a$ *and function* $g(t)$ *is continuous near point* a, *then* $f\lim_{n\to\infty} g(x_n) = g(a)$.

3. Main result and its proof

In this section, we discuss the frequent convergence of solutions of the following difference equation:

$$x_{n+1} = 1 - x_n^2 \tag{3.1}$$

Given the initial-value x_0, we can use equation (3.1) to deduce sequence $X = \{x_n\}_{n=0}^{\infty}$, then we call it the solution of the difference equation (3.1).

Obviously, if the initial-value $x_0 = \frac{-1\pm\sqrt{5}}{2}$, then we can deduce that $x_n = \frac{1\pm\sqrt{5}}{2}$, $n = 0, 1, 2\cdots$, which means the solution of the difference equation (3.1) is constant-valued. If the initial-value x_0 doesn't equal to $\frac{-1\pm\sqrt{5}}{2}$, we have the following conclusion.

THEOREM 3.1. *Let x_0 be the initial-value of the difference equation (3.1), $X = \{x_n\}_{n=0}^{\infty}$ be its solution , then we have the following results.*

1) *If $x_0 \in (-\infty, \frac{-1-\sqrt{5}}{2}) \cup (\frac{1+\sqrt{5}}{2}, +\infty)$, then $X = \{x_n\}_{n=0}^{\infty}$ frequently inside $(-\infty, \frac{-1-\sqrt{5}}{2})$;*
2) *If $x_0 \in (\frac{-1-\sqrt{5}}{2}, -1) \cup [1, \frac{1+\sqrt{5}}{2})$, then $X = \{x_n\}_{n=0}^{\infty}$ frequently inside $(\frac{-1-\sqrt{5}}{2}, 1)$;*
3) *If $x_0 \in (-1, \frac{1-\sqrt{5}}{2}) \cup (\frac{1-\sqrt{5}}{2}, 0) \cup [0, \frac{-1+\sqrt{5}}{2}) \cup (\frac{-1+\sqrt{5}}{2}, 1]$, then $X = \{x_n\}_{n=0}^{\infty}$ has two frequent limits 0 and 1 of the same degree 0.5.*

Proof. Let $H(t) = 1 - (1 - t^2)^2$ and $G(t) = H(t) - t = 2t^2 - t^4 - t$. If $G(t) = 0$, then we can get four roots: $t_1 = \frac{-1-\sqrt{5}}{2}$, $t_2 = 0$, $t_3 = \frac{-1+\sqrt{5}}{2}$, $t_4 = 1$.

By elementary analysis, it is easy to see that $G(t) \le 0$ for $t \in (-\infty, \frac{-1-\sqrt{5}}{2}) \cup [0, \frac{-1+\sqrt{5}}{2}) \cup [1, +\infty)$ and $G(t) \ge 0$ for $t \in (\frac{-1-\sqrt{5}}{2}, 0) \cup (\frac{-1+\sqrt{5}}{2}, 1]$, that is,

$$\begin{cases} t \ge 1-(1-t^2)^2, & t \in (-\infty, \frac{-1-\sqrt{5}}{2}) \cup [0, \frac{-1+\sqrt{5}}{2}) \cup [1, +\infty) \\ t \le 1-(1-t^2)^2, & t \in (\frac{-1-\sqrt{5}}{2}, 0) \cup (\frac{-1+\sqrt{5}}{2}, 1] \end{cases} \tag{3.2}$$

According to the initial-value in different intervals, we divide the problem into five cases :

$$I : x_0 \in (-\infty, \frac{-1-\sqrt{5}}{2}); \quad II : x_0 \in (\frac{-1-\sqrt{5}}{2}, 0)$$

$$III : x_0 \in [0, \frac{-1+\sqrt{5}}{2}); \quad IV : x_0 \in (\frac{-1+\sqrt{5}}{2}, 1]; \quad V : x_0 \in (1, +\infty].$$

We then discuss each case in details:

Case I : $x_0 \in (-\infty, \frac{-1-\sqrt{5}}{2})$.

Since $x_0^2 > \frac{3+\sqrt{5}}{2}$, we have $x_1 = 1 - x_0^2 < \frac{-1-\sqrt{5}}{2}$ and $x_1^2 > \frac{3+\sqrt{5}}{2}$, then $x_2 = 1 - x_1^2 < \frac{-1-\sqrt{5}}{2}$, thus we can easily deduce that

$$\{x_n\}_{n=0}^{\infty} \subset (-\infty, \frac{-1-\sqrt{5}}{2}).$$

Thus in view of the inequality $t \geq 1 - t^2$ on $t \in (-\infty, \frac{-1-\sqrt{5}}{2})$, we have

$$\frac{-1-\sqrt{5}}{2} > x_0 > x_1 > x_2 > \cdots > x_n > \cdots,$$

that is, $\{x_n\}_{n=0}^{\infty}$ is a decreasing sequence which belongs to $(-\infty, \frac{-1-\sqrt{5}}{2})$. It also means $\mu(X \notin (-\infty, \frac{-1-\sqrt{5}}{2})) = 0$, hence $\{x_n\}$ is frequently inside $(-\infty, \frac{-1-\sqrt{5}}{2})$.

Case II : $x_0 \in (\frac{-1-\sqrt{5}}{2}, 0)$.

In this case, we will divide the interval $(\frac{-1-\sqrt{5}}{2}, 0)$ into three subintervals:

(I) $x_0 \in (\frac{-1-\sqrt{5}}{2}, -1)$.

Since $1 < x_0^2 < \frac{3+\sqrt{5}}{2}$, we have $\frac{-1-\sqrt{5}}{2} < x_1 = 1 - x_0^2 < 0$ and $0 < x_1^2 < \frac{3+\sqrt{5}}{2}$, then from $\frac{-1-\sqrt{5}}{2} < x_2 = 1 - x_1^2 < 1$ and $0 < x_2^2 < \frac{3+\sqrt{5}}{2}$ we have $\frac{-1-\sqrt{5}}{2} < x_3 = 1 - x_2^2 < 1$, thus we can deduce that $\{x_n\}_{n=2}^{\infty} \subset (\frac{-1-\sqrt{5}}{2}, 1)$ except for $x_0 \in (\frac{-1-\sqrt{5}}{2}, -1)$ and $x_1 \in (\frac{-1-\sqrt{5}}{2}, 0)$, that is to say, $\mu(X \notin (\frac{-1-\sqrt{5}}{2}, 1)) = 0$, hence X is frequently inside $(\frac{-1-\sqrt{5}}{2}, 1)$.

(II) $x_0 \in (-1, \frac{1-\sqrt{5}}{2})$.

Since $\frac{3-\sqrt{5}}{2} < x_0^2 < 1$, we have $0 < x_1 = 1 - x_0^2 < \frac{-1+\sqrt{5}}{2}$ and $0 < x_1^2 < \frac{3-\sqrt{5}}{2}$, then from $\frac{-1+\sqrt{5}}{2} < x_2 = 1 - x_1^2 < 1$ and $\frac{3-\sqrt{5}}{2} < x_2^2 < 1$ we have $0 < x_3 = 1 - x_2^2 < \frac{-1+\sqrt{5}}{2}$ and $0 < x_3^2 < \frac{3-\sqrt{5}}{2}$, then $\frac{-1+\sqrt{5}}{2} < x_4 = 1 - x_3^2 < 1$. Thus we can deduce that $\{x_{2n+1}\}_{n=0}^{\infty} \subset (0, \frac{-1+\sqrt{5}}{2})$ and $\{x_{2n}\}_{n=1}^{\infty} \subset (\frac{-1+\sqrt{5}}{2}, 1)$.

From (3.1), we obtain that

$$x_{n+2} = 1 - x_{n+1}^2 = 1 - (1 - x_n^2)^2, n = 0, 1, 2, \ldots. \tag{3.3}$$

In view of the inequality (3.2), we have

$$\frac{-1+\sqrt{5}}{2} > x_1 \geq 1-(1-x_1^2)^2$$
$$= x_3 \geq \ldots \geq x_{2n+1} \geq 1-(1-x_{2n+1}^2)^2 = x_{2n+3} \geq \ldots \geq 0$$
$$\frac{-1+\sqrt{5}}{2} < x_2 \leq 1-(1-x_2^2)^2$$
$$= x_4 \leq \ldots \leq x_{2n-2} \leq 1-(1-x_{2n}^2)^2 = x_{2n+2} \leq \ldots \leq 1$$

If we denote $y_n = x_{2n+1}$ for $n = 0, 1, 2, \ldots$, then $\lim_{n\to\infty} y_n = y_* \in [0, \frac{-1+\sqrt{5}}{2})$. We next assert that $y_* = 0$. Note that we can write (3.3) in the following form

$$y_n = H(y_{n-1}), n = 1, 2, \ldots$$

where $H(u) = 1-(1-u^2)^2$. Note that the polynomial $G(t) = H(t)-t = 1-t-(1-t^2)^2$ has only one root 0 in $[0, \frac{-1+\sqrt{5}}{2})$, and from (3.3) we have $y_* = 1-(1-y_*^2)^2$, i.e., $G(y_*) = 0$, hence $y_* = 0$.

Since $G(t)$ also has only one root in $(\frac{-1+\sqrt{5}}{2}, 1]$, by similar arguments, we may show that $\lim_{n\to\infty} x_{2n-1} = 1$, thus the solution X of the difference equation (3.1) have two frequent limits 0 and 1 of the same degree 0.5.

(III) $x_0 \in (\frac{1-\sqrt{5}}{2}, 0)$.

Since $0 < x_0^2 < \frac{3-\sqrt{5}}{2}$, we have $\frac{-1+\sqrt{5}}{2} < x_1 = 1-x_0^2 < 1$ and $\frac{3-\sqrt{5}}{2} < x_1^2 < 1$, then from $0 < x_2 = 1-x_1^2 < \frac{-1+\sqrt{5}}{2}$ and $0 < x_2^2 < \frac{3-\sqrt{5}}{2}$ we have $\frac{-1+\sqrt{5}}{2} < x_3 = 1-x_2^2 < 1$ and $\frac{3-\sqrt{5}}{2} < x_3^2 < 1$, then $0 < x_4 = 1-x_3^2 < \frac{-1+\sqrt{5}}{2}$, thus we can deduce that $\{x_{2n+1}\}_{n=0}^{\infty} \subset (\frac{-1+\sqrt{5}}{2}, 1)$ and $\{x_{2n}\}_{n=1}^{\infty} \subset (0, \frac{-1+\sqrt{5}}{2})$. By (3.2), we get that

$$\frac{-1+\sqrt{5}}{2} < x_1 < x_3 < x_5 < \cdots < x_{2n+1} < \cdots < 1, (n = 0, 1, 2\cdots)$$
$$\frac{-1+\sqrt{5}}{2} > x_2 > x_4 > x_6 > \cdots > x_{2n} > \cdots > 0, (n = 1, 2\cdots)$$

By similar argument with (II), we can conclude that the solution of different equation (3.1) have two frequent limits 0 and 1 of the same degree 0.5.

Case III : $x_0 \in [0, \frac{-1+\sqrt{5}}{2})$.

Since $0 \leq x_0^2 < \frac{3-\sqrt{5}}{2}$, we have $\frac{-1+\sqrt{5}}{2} < x_1 = 1-x_0^2 \leq 1$, then from $\frac{3-\sqrt{5}}{2} < x_1^2 \leq 1$ we have $0 \leq x_2 = 1-x_1^2 < \frac{-1+\sqrt{5}}{2}$; and from

$0 \le x_2^2 < \frac{3-\sqrt{5}}{2}$ we have $\frac{-1+\sqrt{5}}{2} < x_3 = 1 - x_2^2 \le 1$; also from $\frac{3-\sqrt{5}}{2} < x_3^2 \le 1$ we have $0 \le x_4 = 1 - x_3^2 < \frac{-1+\sqrt{5}}{2}$, thus we can deduce that $\{x_{2n+1}\}_{n=0}^{\infty} \subset (\frac{-1+\sqrt{5}}{2}, 1]$ and $\{x_{2n}\}_{n=0}^{\infty} \subset [0, \frac{-1+\sqrt{5}}{2})$, hence by the similar argument with the above, we can get the conclusion that the solution of the difference equation (3.1) have two frequent limits 0 and 1 of the same degree 0.5.

Case IV : $x_0 \in (\frac{-1+\sqrt{5}}{2}, 1]$.

Since $\frac{3-\sqrt{5}}{2} < x_0^2 \le 1$, we have $0 \le x_1 = 1 - x_0^2 < \frac{-1+\sqrt{5}}{2}$ and $0 \le x_1^2 < \frac{3-\sqrt{5}}{2}$, then from $\frac{-1+\sqrt{5}}{2} < x_2 = 1 - x_1^2 \le 1$ and $\frac{3-\sqrt{5}}{2} < x_2^2 \le 1$ we have $0 \le x_3 = 1 - x_2^2 < \frac{-1+\sqrt{5}}{2}$ and $0 \le x_3^2 < \frac{3-\sqrt{5}}{2}$, then $\frac{-1+\sqrt{5}}{2} < x_4 = 1 - x_3^2 \le 1$, thus we can deduce that $\{x_{2n+1}\}_{n=0}^{\infty} \subset [0, \frac{-1+\sqrt{5}}{2})$ and $\{x_{2n}\}_{n=0}^{\infty} \subset (\frac{-1+\sqrt{5}}{2}, 1]$, so we can get the same conclusion with to *case III*.

Case V : $x_0 \in [1, +\infty)$.

We divide the interval $[1, +\infty)$ into two subintervals:

(I) $x_0 \in (\frac{1+\sqrt{5}}{2}, +\infty)$.

Since $\frac{3+\sqrt{5}}{2} < x_0^2 < +\infty$, we have $-\infty < x_1 = 1 - x_0^2 < \frac{-1-\sqrt{5}}{2}$ and $\frac{3+\sqrt{5}}{2} < x_1^2 < +\infty$, then from $-\infty < x_2 = 1 - x_1^2 < \frac{-1-\sqrt{5}}{2}$ and $\frac{3+\sqrt{5}}{2} < x_2^2 < +\infty$, we have $-\infty < x_3 = 1 - x_2^2 < \frac{-1-\sqrt{5}}{2}$, thus we can deduce that $\{x_n\}_{n=1}^{\infty} \subset (-\infty, \frac{-1-\sqrt{5}}{2})$. In view of the inequality $t > 1 - t^2$ on $t \in (-\infty, \frac{-1-\sqrt{5}}{2})$, we have

$$\frac{-1-\sqrt{5}}{2} > x_1 > x_2 > x_3 > \cdots > x_n > \cdots, (n = 1, 2 \cdots)$$

thus the solution X of the difference equation (3.1) is frequently inside $(-\infty, \frac{-1-\sqrt{5}}{2})$.

(II) $x_0 \in [1, \frac{1+\sqrt{5}}{2})$.

Since $1 < x_0^2 < \frac{3+\sqrt{5}}{2}$, we have $\frac{-1-\sqrt{5}}{2} < x_1 = 1 - x_0^2 < 0$ and $0 < x_1^2 < \frac{3+\sqrt{5}}{2}$, then $\frac{-1-\sqrt{5}}{2} < x_2 = 1 - x_1^2 < 1$, i.e., $x_2 \in (\frac{-1-\sqrt{5}}{2}, 1)$. Similar with the discuss of first situation (I) of *caseII*, we can conclude that the solution X of the difference equation (3.1) is frequently inside $(\frac{-1-\sqrt{5}}{2}, 1)$.

All in all, if $x_0 \in (-\infty, \frac{-1-\sqrt{5}}{2}) \cup (\frac{1+\sqrt{5}}{2}, +\infty)$, then the solution $\{x_n\}_{n=0}^{\infty}$ is frequently inside $(-\infty, \frac{-1-\sqrt{5}}{2})$; if $x_0 \in (\frac{-1-\sqrt{5}}{2}, -1) \cup [1, \frac{1+\sqrt{5}}{2})$, then the solution $\{x_n\}_{n=0}^{\infty}$ is frequently inside $(\frac{-1-\sqrt{5}}{2}, 1)$; if $x_0 \in (-1, \frac{1-\sqrt{5}}{2}) \cup$

$(\frac{1-\sqrt{5}}{2}, 0) \cup [0, \frac{-1+\sqrt{5}}{2}) \cup (\frac{-1+\sqrt{5}}{2}, 1]$, then the solution $\{x_n\}_{n=0}^{\infty}$ have two frequent limits 0 and 1 of the same degree 0.5. $\square$

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