

The Line n -sigraph of a Symmetric n -sigraph-V

P. SIVA KOTA REDDY*

Department of Mathematics, Siddaganga Institute of Technology, Tumkur-572 103, India.

e-mail: reddy_math@yahoo.com; pskreddy@sit.ac.in

K. M. NAGARAJA

Department of Mathematics, JSS Academy of Technical Education, Bangalore-560 060, India.

e-mail: nagkmn@gmail.com

M. C. GEETHA

Department of Mathematics, East West Institute of Technology, Bangalore-560 091, India.

e-mail: geethalingarajub@gmail.com

ABSTRACT. An n -tuple (a_1, a_2, \dots, a_n) is symmetric, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \dots, a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n -tuples. A symmetric n -sigraph (symmetric n -marked graph) is an ordered pair $S_n = (G, \sigma)$ ($S_n = (G, \mu)$), where $G = (V, E)$ is a graph called the underlying graph of S_n and $\sigma : E \rightarrow H_n$ ($\mu : V \rightarrow H_n$) is a function. The restricted super line graph of index r of a graph G , denoted by $\mathcal{RL}_r(G)$. The vertices of $\mathcal{RL}_r(G)$ are the r -subsets of $E(G)$ and two vertices $P = \{p_1, p_2, \dots, p_r\}$ and $Q = \{q_1, q_2, \dots, q_r\}$ are adjacent if there exists exactly one pair of edges, say p_i and q_j , where $1 \leq i, j \leq r$, that are adjacent edges in G . Analogously, one can define the restricted super line symmetric n -sigraph of index r of a symmetric n -sigraph $S_n = (G, \sigma)$ as a symmetric n -sigraph $\mathcal{RL}_r(S_n) = (\mathcal{RL}_r(G), \sigma')$, where $\mathcal{RL}_r(G)$ is the underlying graph of $\mathcal{RL}_r(S_n)$, where for any edge PQ in $\mathcal{RL}_r(S_n)$, $\sigma'(PQ) = \sigma(P)\sigma(Q)$. It is shown that for any symmetric n -sigraph S_n , its $\mathcal{RL}_r(S_n)$ is i -balanced and we offer a structural characterization of super line symmetric n -sigraphs of index r . Further, we characterize symmetric n -sigraphs S_n for which $\mathcal{RL}_r(S_n) \sim \mathcal{L}_r(S_n)$ and $\mathcal{RL}_r(S_n) \cong \mathcal{L}_r(S_n)$, where \sim and \cong denotes switching equivalence and isomorphism and $\mathcal{RL}_r(S_n)$ and $\mathcal{L}_r(S_n)$ are denotes the restricted super line symmetric n -sigraph of index r and super line symmetric n -sigraph of index r of S_n respectively.

* Corresponding Author.

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1. Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [2]. We consider only finite, simple graphs free from self-loops.

Let $n \geq 1$ be an integer. An n -tuple (a_1, a_2, \dots, a_n) is *symmetric*, if $a_k = a_{n-k+1}$, $1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \dots, a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n -tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m , where $m = \lceil \frac{n}{2} \rceil$.

A *symmetric n -sigraph* (*symmetric n -marked graph*) is an ordered pair $S_n = (G, \sigma)$ ($S_n = (G, \mu)$), where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \rightarrow H_n$ ($\mu : V \rightarrow H_n$) is a function.

In this paper by an *n -tuple/ n -sigraph/ n -marked graph* we always mean a symmetric n -tuple/symmetric n -sigraph/symmetric n -marked graph.

An n -tuple (a_1, a_2, \dots, a_n) is the *identity n -tuple*, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n -tuple*. In an n -sigraph $S_n = (G, \sigma)$ an edge labelled with the identity n -tuple is called an *identity edge*, otherwise it is a *non-identity edge*.

Further, in an n -sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the *n -tuple $\sigma(A)$* is the product of the n -tuples on the edges of A .

In [17], the authors defined two notions of balance in n -sigraph $S_n = (G, \sigma)$ as follows (See also R. Rangarajan and P.S.K.Reddy [6]):

Definition 1.1. Let $S_n = (G, \sigma)$ be an n -sigraph. Then,

- (i) S_n is *identity balanced* (or *i -balanced*), if product of n -tuples on each cycle of S_n is the identity n -tuple, and
- (ii) S_n is *balanced*, if every cycle in S_n contains an even number of non-identity edges.

Note: An i -balanced n -sigraph need not be balanced and conversely.

The following characterization of i -balanced n -sigraphs is obtained in [17].

Proposition 1.1. (E. Sampathkumar et al. [17])

An n -sigraph $S_n = (G, \sigma)$ is i -balanced if, and only if, it is possible to assign n -tuples to its vertices such that the n -tuple of each edge uv is equal to the product of the n -tuples of u and v .

Let $S_n = (G, \sigma)$ be an n -sigraph. Consider the n -marking μ on vertices of S_n defined as follows: each vertex $v \in V$, $\mu(v)$ is the n -tuple which is the product of the n -tuples on the edges incident with v . *Complement* of S_n is an n -sigraph

$\overline{S}_n = (\overline{G}, \sigma^c)$, where for any edge $e = uv \in \overline{G}$, $\sigma^c(uv) = \mu(u)\mu(v)$. Clearly, \overline{S}_n as defined here is an i -balanced n -sigraph due to Proposition 1.1 [9].

In [17], the authors also have defined switching and cycle isomorphism of an n -sigraph $S_n = (G, \sigma)$ as follows: (See also [3, 7, 8] & [9]-[16])

Let $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$, be two n -sigraphs. Then S_n and S'_n are said to be *isomorphic*, if there exists an isomorphism $\phi : G \rightarrow G'$ such that if uv is an edge in S_n with label (a_1, a_2, \dots, a_n) then $\phi(u)\phi(v)$ is an edge in S'_n with label (a_1, a_2, \dots, a_n) .

Given an n -marking μ of an n -sigraph $S_n = (G, \sigma)$, *switching* S_n with respect to μ is the operation of changing the n -tuple of every edge uv of S_n by $\mu(u)\sigma(uv)\mu(v)$. The n -sigraph obtained in this way is denoted by $\mathcal{S}_\mu(S_n)$ and is called the μ -switched n -sigraph or just *switched n -sigraph*.

Further, an n -sigraph S_n *switches* to n -sigraph S'_n (or that they are *switching equivalent* to each other), written as $S_n \sim S'_n$, whenever there exists an n -marking of S_n such that $\mathcal{S}_\mu(S_n) \cong S'_n$.

Two n -sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : G \rightarrow G'$ such that the n -tuple $\sigma(C)$ of every cycle C in S_n equals to the n -tuple $\sigma(\phi(C))$ in S'_n . We make use of the following known result (see [17]).

Proposition 1.2.(E. Sampathkumar et al. [17])

Given a graph G , any two n -sigraphs with G as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.

In this paper, we introduced the notion called restricted super line n -sigraph of index r and we obtained some interesting results in the following sections. The restricted super line n -sigraph of index r is the generalization of line n -sigraph.

2. Restricted Super Line n -sigraph $\mathcal{L}_r(S_n)$

In [4], K. Manjula introduced the concept of the *restricted super line graph*, which generalizes the notion of line graph. For a given G , its restricted super line graph $\mathcal{RL}_r(G)$ of index r is the graph whose vertices are the r -subsets of $E(G)$, and two vertices $P = \{p_1, p_2, \dots, p_r\}$ and $Q = \{q_1, q_2, \dots, q_r\}$ are adjacent if there exists exactly one pair of edges, say p_i and q_j , where $1 \leq i, j \leq r$, that are adjacent edges in G . In [1], the authors introduced the concept of the *super line graph* as follows: For a given G , its super line graph $\mathcal{L}_r(G)$ of index r is the graph whose vertices are the r -subsets of $E(G)$, and two vertices P and Q are adjacent if there exist $p \in P$

and $q \in Q$ such that p and q are adjacent edges in G . Clearly $\mathcal{RL}_r(G)$ is a spanning subgraph of $\mathcal{L}_r(G)$. From the definitions of $\mathcal{RL}_r(G)$ and $\mathcal{L}_r(G)$, it turns out that $\mathcal{RL}_1(G)$ and $\mathcal{L}_1(G)$ coincides with the line graph $L(G)$.

In this paper, we extend the notion of $\mathcal{RL}_r(G)$ to realm of n -sigraphs as follows: The *restricted super line n -sigraph of index r* of an n -sigraph $S_n = (G, \sigma)$ as an n -sigraph $\mathcal{RL}_r(S_n) = (\mathcal{RL}_r(G), \sigma')$, where $\mathcal{RL}_r(G)$ is the underlying graph of $\mathcal{RL}_r(S_n)$, where for any edge PQ in $\mathcal{RL}_r(S_n)$, $\sigma'(PQ) = \sigma(P)\sigma(Q)$.

Hence, we shall call a given n -sigraph S_n is a *restricted super line n -sigraph of index r* if it is isomorphic to the restricted super line n -sigraph of index r , $\mathcal{RL}_r(S'_n)$ of some n -sigraph S'_n . In the following subsection, we shall present a characterization of restricted super line n -sigraph of index r .

The following result indicates the limitations of the notion $\mathcal{RL}_r(S_n)$ as introduced above, since the entire class of i -unbalanced n -sigraphs is forbidden to be restricted super line n -sigraphs of index r .

Proposition 2.1. *For any n -sigraph $S_n = (G, \sigma)$, its $\mathcal{RL}_r(S_n)$ is i -balanced.*

Proof. Let σ' denote the n -tuple of $\mathcal{RL}_r(S_n)$ and let the n -tuple σ of S_n be treated as an n -marking of the vertices of $\mathcal{RL}_r(S_n)$. Then by definition of $\mathcal{RL}_r(S_n)$ we see that $\sigma'(P, Q) = \sigma(P)\sigma(Q)$, for every edge PQ of $\mathcal{RL}_r(S_n)$ and hence, by Proposition 1.1, the result follows. \square

For any positive integer k , the k^{th} iterated restricted super line n -sigraph of index r , $\mathcal{RL}_r(S_n)$ of S_n is defined as follows:

$$\mathcal{RL}_r^0(S_n) = S_n, \mathcal{RL}_r^k(S_n) = \mathcal{RL}_r(\mathcal{RL}_r^{k-1}(S_n))$$

Corollary 2.2. *For any n -sigraph $S_n = (G, \sigma)$ and any positive integer k , $\mathcal{RL}_r^k(S_n)$ is i -balanced.*

In [16], the authors introduced the notion of the *super line n -sigraph*, which generalizes the notion of line n -sigraph [18]. The *super line n -sigraph of index r* of an n -sigraph $S_n = (G, \sigma)$ as an n -sigraph $\mathcal{L}_r(S_n) = (\mathcal{L}_r(G), \sigma')$, where $\mathcal{L}_r(G)$ is the underlying graph of $\mathcal{L}_r(S_n)$, where for any edge PQ in $\mathcal{L}_r(S_n)$, $\sigma'(PQ) = \sigma(P)\sigma(Q)$. The above notion restricted super line n -sigraph is another generalization of line n -sigraphs.

Proposition 2.3. (P.S.K.Reddy et al. [16])

For any n -sigraph $S_n = (G, \sigma)$, its $\mathcal{L}_r(S_n)$ is i -balanced.

In [4], the author characterized whose restricted super line graphs of index r that are isomorphic to $\mathcal{L}_r(G)$.

Proposition 2.4. (K. Manjula [4])

For a graph $G = (V, E)$, $\mathcal{RL}_r(G) \cong \mathcal{L}_r(G)$ if, and only if, G is either $K_{1,2} \cup nK_2$ or nK_2 .

We now characterize n -sigraphs those $\mathcal{RL}_r(S_n)$ are switching equivalent to their $\mathcal{L}_r(S_n)$.

Proposition 2.5. For any n -sigraph $S_n = (G, \sigma)$, $\mathcal{RL}_r(S_n) \sim \mathcal{L}_r(S_n)$ if, and only if, G is either $K_{1,2} \cup nK_2$ or nK_2 .

Proof. Suppose $\mathcal{RL}_r(S_n) \sim \mathcal{L}_r(S_n)$. This implies, $\mathcal{RL}_r(G) \cong \mathcal{L}_r(G)$ and hence by Proposition 2.4, we see that the graph G must be isomorphic to either $K_{1,2} \cup nK_2$ or nK_2 .

Conversely, suppose that G is either $K_{1,2} \cup nK_2$ or nK_2 . Then $\mathcal{RL}_r(G) \cong \mathcal{L}_r(G)$ by Proposition 2.4. Now, if S_n any n -sigraph on any of these graphs, by Proposition 2.1 and Proposition 2.3, $\mathcal{RL}_r(S_n)$ and $\mathcal{L}_r(S_n)$ are i -balanced and hence, the result follows from Proposition 1.2. \square

We now characterize n -sigraphs those $\mathcal{RL}_r(S_n)$ are isomorphic to their $\mathcal{L}_r(S_n)$. The following result is a stronger form of the above result.

Proposition 2.6. For any n -sigraph $S_n = (G, \sigma)$, $\mathcal{RL}_r(S_n) \cong \mathcal{L}_r(S_n)$ if, and only if, G is either $K_{1,2} \cup nK_2$ or nK_2 .

Proof. Clearly $\mathcal{RL}_r(S_n) \cong \mathcal{L}_r(S_n)$, where G is either $K_{1,2} \cup nK_2$ or nK_2 . Consider the map $f : V(\mathcal{RL}_r(G)) \rightarrow V(\mathcal{L}_r(S))$ defined by $f(e_1e_2, e_2e_3) = (e'_1e'_2, e'_2e'_3)$ is an isomorphism. Let σ be any n -tuple on $K_{1,2} \cup nK_2$ or nK_2 . Let $e = (e_1e_2, e_2e_3)$ be an edge in $\mathcal{RL}_r(G)$, where G is $K_{1,2} \cup nK_2$ or nK_2 . Then the n -tuple of the edge e in $\mathcal{RL}_r(G)$ is the $\sigma(e_1e_2)\sigma(e_2e_3)$ which is the n -tuple of the edge $(e'_1e'_2, e'_2e'_3)$ in $\mathcal{L}_r(G)$, where G is $K_{1,2} \cup nK_2$ or nK_2 . Hence the map f is also an n -sigraph isomorphism between $\mathcal{RL}_r(S_n)$ and $\mathcal{L}_r(S_n)$. \square

3. Characterization of Restricted Super Line n -sigraphs $\mathcal{RL}_r(S_n)$

The following result characterize n -sigraphs which are restricted super line n -sigraphs of index r .

Proposition 3.1. An n -sigraph $S_n = (G, \sigma)$ is a restricted super line n -sigraph of index r if and only if S_n is i -balanced n -sigraph and its underlying graph G is a restricted super line graph of index r .

Proof. Suppose that S_n is i -balanced and G is a $\mathcal{RL}_r(G)$. Then there exists a graph H such that $\mathcal{L}_r(H) \cong G$. Since S_n is i -balanced, by Proposition 1.1, there exists an n -marking μ of G such that each edge uv in S_n satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the n -sigraph $S'_n = (H, \sigma')$, where for any edge e in H , $\sigma'(e)$ is the

n -marking of the corresponding vertex in G . Then clearly, $\mathcal{RL}_r(S'_n) \cong S_n$. Hence S_n is a restricted super line n -sigraph of index r .

Conversely, suppose that $S_n = (G, \sigma)$ is a restricted super line n -sigraph of index r . Then there exists an n -sigraph $S'_n = (H, \sigma')$ such that $\mathcal{RL}_r(S'_n) \cong S_n$. Hence G is the $\mathcal{RL}_r(G)$ of H and by Proposition 2.1, S_n is i -balanced. \square

If we take $r = 1$ in $\mathcal{RL}_r(S_n)$, then this is the ordinary line n -sigraph. In [18], the authors obtained structural characterization of line n -sigraphs and clearly Proposition 3.1 is the generalization of line signed graphs.

Proposition 3.2. *An n -sigraph $S_n = (G, \sigma)$ is a line n -sigraph if, and only if, S_n is i -balanced n -sigraph and its underlying graph G is a line graph.*

4. Complementation

In this section, we investigate the notion of complementation of a graph whose edges have signs (a *sigraph*) in the more general context of graphs with multiple signs on their edges. We look at two kinds of complementation: complementing some or all of the signs, and reversing the order of the signs on each edge.

For any $m \in H_n$, the m -complement of $a = (a_1, a_2, \dots, a_n)$ is: $a^m = am$. For any $M \subseteq H_n$, and $m \in H_n$, the m -complement of M is $M^m = \{a^m : a \in M\}$.

For any $m \in H_n$, the m -complement of an n -sigraph $S_n = (G, \sigma)$, written (S_n^m) , is the same graph but with each edge label $a = (a_1, a_2, \dots, a_n)$ replaced by a^m .

For an n -sigraph $S_n = (G, \sigma)$, the $\mathcal{RL}_r(S_n)$ is i -balanced (Proposition 2.1). We now examine, the condition under which m -complement of $\mathcal{RL}_r(S_n)$ is i -balanced, where for any $m \in H_n$.

Proposition 4.1. *Let $S_n = (G, \sigma)$ be an n -sigraph. Then, for any $m \in H_n$, if $\mathcal{RL}_r(G)$ is bipartite then $(\mathcal{RL}_r(S_n))^m$ is i -balanced.*

Proof. Since, by Proposition 2.1, $\mathcal{RL}_r(S_n)$ is i -balanced, for each k , $1 \leq k \leq n$, the number of n -tuples on any cycle C in $\mathcal{RL}_r(S_n)$ whose k^{th} co-ordinate are $-$ is even. Also, since $\mathcal{RL}_r(G)$ is bipartite, all cycles have even length; thus, for each k , $1 \leq k \leq n$, the number of n -tuples on any cycle C in $\mathcal{RL}_r(S_n)$ whose k^{th} co-ordinate are $+$ is also even. This implies that the same thing is true in any m -complement, where for any $m \in H_n$. Hence $(\mathcal{RL}_r(S_n))^t$ is i -balanced. \square

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