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Some Difference Paranormed Sequence Spaces over n-normed Spaces Defined by a Musielak-Orlicz Function

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Abstract. In the present paper we introduce difference paranormed sequence spaces $c_0(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$, $c(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$ and $l_\infty(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$ defined by a Musielak-Orlicz function $\mathcal{M} = (M_k)$ over n-normed spaces. We also study some topological properties and some inclusion relations between these spaces.

1. Introduction and Preliminaries

Let w, l_∞, c and c_0 denote the spaces of all, bounded, convergent and null sequences $x = (x_k)$ with real or complex entries respectively. The zero sequence is denoted by $\theta = (0, 0, ...)$. The notion of difference sequence spaces was introduced by Kızmaz [9], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [4] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_o(\Delta^n)$. Let m, n be non-negative integers, then for $Z = l_\infty$, c and c_0 we have sequence spaces,

$$Z(\Delta_m^n) = \{x = (x_k) \in w : (\Delta_m^n x_k) \in Z\}$$

where $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ and $\Delta_m^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_m^n x_k = \sum_{v=0}^{n} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} x_{k+mv}.$$

Taking $m = n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_o(\Delta)$ introduced and studied by Kızmaz [6].

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

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1. $p(x) \geq 0$, for all $x \in X$;
2. $p(-x) = p(x)$, for all $x \in X$;
3. $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$;
4. if (σ_n) is a sequence of scalars with $\sigma_n \to \sigma \; as \; n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\sigma_n x_n - \sigma x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [18], Theorem 10.4.2, P-183). For more details about sequence spaces (see [1], [2], [3], [14], [15], [16], [17]) and references therein.

An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.
Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \Big\{(x_k) \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \;\text{ for some }\; \rho > 0\Big\}$$

which is called as an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$||(x_k)|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

Also, it was shown in [10] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. An Orlicz function M satisfies Δ_2-condition if and only if for any constant $L > 1$ there exists a constant $K(L)$ such that $M(Lu) \leq K(L)M(u)$ for all values of $u \geq 0$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function (see [11], [13]). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\}, \; k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \;\text{ for some }\; c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{ x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for all } \ c > 0 \Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf \Big\{ k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1 \Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf \Big\{ \frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0 \Big\}.$$

The concept of 2-normed spaces was initially developed by Gähler [5] in the mid of 1960's, while that of n-normed spaces one can see in Misiak[9]. Since then, many others have studied this concept and obtained various results, see Gunawan ([6], [7]) and Gunawan and Mashadi [8]. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n which satisfies the following four conditions:

1. $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
2. $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;
3. $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and
4. $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called a n-norm on X and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{K}$.
For example, we may take $X = \mathbb{R}^n$ being equipped with
the n-norm $||x_1, x_2, \cdots, x_n||_E =$ the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$ and script E denotes the Euclidean norm. Let $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of positive reals such that $u_k \neq 0$ for all k, then we define the following classes of sequences in the present paper:

$c_0(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||) =$

$$\Big\{x = (x_k) \in w : \lim_{k\to\infty} u_k \Big[M_k\Big(||\frac{\Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0, \quad \text{for some } \rho > 0\Big\},$$

$$c(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||) = \Big\{x = (x_k) \in w : \lim_{k\to\infty} u_k \Big[M_k\Big(||\frac{\Delta_m^n x_k - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0, \text{ for some } \rho > 0 \text{ and } L \in X\Big\},$$

and

$l_\infty(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||) =$

$$\Big\{x = (x_k) \in w : \sup_{k\geq 1} u_k \Big[M_k\Big(||\frac{\Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} < \infty, \quad \text{for some } \rho > 0\Big\}.$$

For $p_k = 1$, for all k

$c_0(\mathcal{M}, \Delta_m^n, u, ||\cdot, \cdots, \cdot||) =$

$$\Big\{x = (x_k) \in w : \lim_{k\to\infty} u_k \Big[M_k\Big(||\frac{\Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] = 0, \text{ for some } \rho > 0\Big\},$$

$$c(\mathcal{M}, \Delta_m^n, u, ||\cdot, \cdots, \cdot||) = \Big\{x = (x_k) \in w : \lim_{k\to\infty} u_k \Big[M_k\Big(||\frac{\Delta_m^n x_k - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] = 0, \text{ for some } \rho > 0 \text{ and } L \in X\Big\},$$

and

$l_\infty(\mathcal{M}, \Delta_m^n, u, ||\cdot, \cdots, \cdot||) =$

$$\Big\{x = (x_k) \in w : \sup_{k\geq 1} u_k \Big[M_k\Big(||\frac{\Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] < \infty, \text{ for some } \rho > 0\Big\}.$$

For $\mathcal{M}(x) = x$, we have

$$c_0(\Delta_m^n, p, u, ||\cdot, \cdots, \cdot||) = \Big\{x = (x_k) \in w : \lim_{k\to\infty} u_k\Big(||\frac{\Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} = 0, \text{ for some } \rho > 0\Big\},$$

$$c(\Delta_m^n, p, u, ||\cdot, \cdots, \cdot||) = \Big\{x = (x_k) \in w : \lim_{k\to\infty} u_k\Big(||\frac{\Delta_m^n x_k - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} = 0, \text{ for some } \rho > 0 \text{ and } L \in X\Big\},$$

and

$$l_\infty(\Delta_m^n, p, u, ||\cdot, \cdots, \cdot||) = \Big\{x = (x_k) \in w : \sup_{k\geq 1} u_k\Big(||\frac{\Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} < \infty, \text{ for some } \rho > 0\Big\}.$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = G$, $K = \max(1, 2^{G-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1.1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^G)$ for all $a \in \mathbb{C}$.

The aim of this paper is to study some difference sequence spaces in more general setting i.e. over n- normed spaces defined by a Musielak-Orlicz function.

2. Main Results

In this section, we study some topological properties and inclusion relation between the spaces $c_0(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$, $c(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$ and $l_\infty(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$.

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers, then the classes of sequences $c_0(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$, $c(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$ and $l_\infty(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$ are linear spaces.*

Proof. Let $x = (x_k)$, $y = (y_k) \in c_0(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_{k\to\infty} u_k\Big[M_k\Big(||\frac{\Delta_m^n x_k}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0, \text{ and}$$

$$\lim_{k\to\infty} u_k\Big[M_k\Big(||\frac{\Delta_m^n y_k}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_k)$ is non-decreasing convex function and so by using inequality (1.1), we have

$$\lim_{k\to\infty} u_k\Big[M_k\Big(||\frac{\Delta_m^n(\alpha x_k + \beta y_k)|}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$$

$$\begin{aligned}
&\leq \lim_{k\to\infty} u_k\Big[M_k\Big(||\frac{\alpha\Delta_m^n x_k}{\rho_3}, z_1, \cdots, z_{n-1}|| + ||\frac{\beta\Delta_m^n y_k}{\rho_3}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\\
&\leq K\lim_{k\to\infty} \frac{1}{2^{p_k}} u_k\Big[M_k\Big(||\frac{\Delta_m^n x_k}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\\
&+ K\lim_{k\to\infty} \frac{1}{2^{p_k}} u_k\Big[M_k\Big(||\frac{\Delta_m^n y_k}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\\
&\leq K\lim_{k\to\infty} u_k\Big[M_k\Big(||\frac{\Delta_m^n x_k}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\\
&+ K\lim_{k\to\infty} u_k\Big[M_k\Big(||\frac{\Delta_m^n y_k}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\\
&= 0.
\end{aligned}$$

So, $\alpha x+\beta y \in c_0(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$. Hence $c_0(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$ is a linear space. Similarly, we can prove that $c(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$ and $l_\infty(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$ are linear spaces. □

Theorem 2.2. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. For $Z = l_\infty, c$ and c_0, the spaces $Z(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$ are paranormed spaces, paranormed by*

$$g(x) = \sum_{k=1}^{mn} ||x_k, z_1, \cdots, z_{n-1}|| + \inf\Big\{\rho^{\frac{p_k}{H}} : \sup_k u_k M_k\Big(||\frac{\Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big) \leq 1\Big\}$$

where $H = \max(1, \sup_k p_k)$.

Proof. Clearly $g(-x) = g(x)$, $g(\theta) = 0$. Let (x_k) and (y_k) be any two sequences belong to any one of the space $Z(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$, for $Z = c_0,\ c$ and l_∞. Then, we get $\rho_1,\ \rho_2 > 0$ such that

$$\sup_k u_k M_k\Big(||\frac{\Delta_m^n x_k}{\rho_1}, z_1, \cdots, z_{n-1}||\Big) \leq 1$$

and

$$\sup_k u_k M_k\Big(||\frac{\Delta_m^n y_k}{\rho_2}, z_1, \cdots, z_{n-1}||\Big) \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by convexity of $\mathcal{M} = (M_k)$, we have

$$\sup_k u_k M_k\Big(||\frac{\Delta_m^n(x_k+y_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)$$
$$\begin{aligned} &\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\sup_k u_k M_k\Big(||\frac{\Delta_m^n x_k}{\rho_1}, z_1, \cdots, z_{n-1}||\Big) \\ &+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\sup_k u_k M_k\Big(||\frac{\Delta_m^n y_k}{\rho_2}, z_1, \cdots, z_{n-1}||\Big) \\ &\leq 1. \end{aligned}$$

Hence we have,
$g(x+y)$

$$\begin{aligned} &= \sum_{k=1}^{mn} ||(x_k+y_k), z_1, \cdots, z_{n-1}|| \\ &+ \inf\Big\{\rho^{\frac{p_k}{H}} : \sup_k u_k M_k\Big(||\frac{\Delta_m^n(x_k+y_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big) \leq 1\Big\} \\ &\leq \sum_{k=1}^{mn} ||x_k, z_1, \cdots, z_{n-1}|| + \inf\Big\{\rho_1^{\frac{p_k}{H}} : \sup_k u_k M_k\Big(||\frac{\Delta_m^n x_k}{\rho_1}, z_1, \cdots, z_{n-1}||\Big) \leq 1\Big\} \\ &+ \sum_{k=1}^{mn} ||y_k, z_1, \cdots, z_{n-1}|| + \inf\Big\{\rho_2^{\frac{p_k}{H}} : \sup_k u_k M_k\Big(||\frac{\Delta_m^n y_k}{\rho_2}, z_1, \cdots, z_{n-1}||\Big) \leq 1\Big\}. \end{aligned}$$

This implies that
$$g(x+y) \leq g(x) + g(y).$$

The continuity of the scalar multiplication follows from the following inequality

$g(\mu x)$

$$\begin{aligned} &= \sum_{k=1}^{mn} ||\mu x_k, z_1, \cdots, z_{n-1}|| + \inf\Big\{\rho^{\frac{p_k}{H}} : \sup_k u_k M_k\Big(||\frac{\Delta_m^n \mu x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big) \leq 1\Big\} \\ &= |\mu|\sum_{k=1}^{mn} ||x_k, z_1, \cdots, z_{n-1}|| \\ &\quad + \inf\Big\{(t|\mu|)^{\frac{p_k}{H}} : \sup_k u_k M_k\Big(||\frac{\Delta_m^n x_k}{t}, z_1, \cdots, z_{n-1}||\Big) \leq 1\Big\}, \end{aligned}$$

where $t = \frac{\rho}{|\mu|}$. Hence the space $Z(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$, for $Z = c_0, c$ and l_∞ is a paranormed space, paranormed by g. □

Theorem 2.3. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. For $Z = l_\infty, c$ and c_0, the spaces $Z(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$ are complete* *paranormed spaces, paranormed by*

$$g(x) = \sum_{k=1}^{mn} ||x_k, z_1, \cdots, z_{n-1}|| + \inf\Big\{\rho^{\frac{p_k}{H}} : \sup_k u_k M_k\Big(||\frac{\Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big) \leq 1\Big\},$$

where $H = \max(1, \sup_k p_k)$.

Proof. We prove the result for the space $l_\infty(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$. Let (x^i) be any Cauchy sequence in $l_\infty(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$. Let $x_0 > 0$ be fixed and $t > 0$ be such that for a given $0 < \epsilon < 1$, $\frac{\epsilon}{x_0 t} > 0$ and $x_0 t \geq 1$. Then there exists a positive integer n_0 such that $g(x^i - x^j) < \frac{\epsilon}{x_0 t}$, for all $i, j \geq n_0$. Using the definition of paranorm, we get

$$\begin{aligned} (2.1) \qquad & \sum_{k=1}^{mn} ||(x_k^i - x_k^j, z_1, \cdots, z_{n-1}|| \\ & + \inf\Big\{\rho^{\frac{p_k}{H}} : \sup_k u_k M_k\Big(||\frac{\Delta_m^n (x_k^i - x_k^j)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big\} \\ & < \frac{\epsilon}{x_0 t}, \text{ for all } i, j \geq n_0. \end{aligned}$$

Hence we have,

$$\sum_{k=1}^{mn} ||(x_k^i - x_k^j), z_1, \cdots, z_{n-1}|| < \epsilon, \text{ for all } i, j \geq n_0.$$

This implies that

$$||(x_k^i - x_k^j), z_1, \cdots, z_{n-1}|| < \epsilon, \text{ for all } i, j \geq n_0 \text{ and } 1 \leq k \leq mn.$$

Thus (x_k^i) is a Cauchy sequence for $k = 1, 2, ..., mn$. Hence (x_k^i) is convergent for $k = 1, 2, ..., mn$. Let

$$(2.2) \qquad \lim_{i\to\infty} x_k^i = x_k, \quad \text{say for } k = 1, 2, \cdots, mn.$$

Again from equation (2.1) we have,

$$\inf\Big\{\rho^{\frac{p_k}{H}} : \sup_k u_k M_k\Big(||\frac{\Delta_m^n (x_k^i - x_k^j)}{\rho}, z_1, \cdots, z_{n-1}||\Big) \leq 1\Big\} < \epsilon, \text{ for all } i, j \geq n_0.$$

Hence we get

$$\sup_k u_k M_k\Big(||\frac{\Delta_m^n (x_k^i - x_k^j)}{g(x^i - x^j)}, z_1, \cdots, z_{n-1}||\Big) \leq 1, \quad \text{for all } i, j \geq n_0.$$

It follows that $u_kM_k\Big(||\frac{\Delta_m^n(x_k^i-x_k^j)}{g(x^i-x^j)}, z_1,\cdots,z_{n-1}||\Big) \leq 1$, for each $k \geq 1$ and for all $i,j \geq n_0$. For $t>0$ with $u_kM_k(\frac{tx_0}{2}) \geq 1$, we have

$$u_kM_k\Big(||\frac{\Delta_m^n(x_k^i-x_k^j)}{g(x^i-x^j)}, z_1,\cdots,z_{n-1}||\Big) \leq u_kM_k\Big(\frac{tx_0}{2}\Big).$$

This implies that

$$||\Delta_m^n x_k^i - \Delta_m^n x_k^j, z_1,\cdots,z_{n-1}|| < \frac{tx_0}{2}\frac{\epsilon}{tx_0} = \frac{\epsilon}{2}.$$

Hence $(\Delta_m^n x_k^i)$ is a Cauchy sequence for all $k \in \mathbb{N}$. This implies that $(\Delta_m^n x_k^i)$ is convergent for all $k \in \mathbb{N}$. Let $\lim\limits_{i\to\infty} \Delta_m^n x_k^i = y_k$ for each $k \in \mathbb{N}$. Let $k=1$, then we have

$$\lim_{i\to\infty} \Delta_m^n x_1^i = \lim_{i\to\infty} \sum_{v=0}^{n} (-1)^v \left(\begin{array}{c} n \\ v \end{array}\right) x_{1+mv}^i = y_1. \tag{2.3}$$

We have by equation (2.2) and equation (2.3) $\lim\limits_{i\to\infty} x_{mn+1}^i = x_{mn+1}$, exists. Proceeding in this way inductively, we have $\lim\limits_{i\to\infty} x_k^i = x_k$ exists for each $k \in \mathbb{N}$. Now we have for all $i,j \geq n_0$,

$$\sum_{k=1}^{mn} ||(x_k^i - x_k^j), z_1,\cdots,z_{n-1}|| \\ + \inf\Big\{\rho^{\frac{p_k}{H}} : \sup_k u_kM_k\Big(||\frac{\Delta_m^n(x_k^i-x_k^j)}{\rho}, z_1,\cdots,z_{n-1}||\Big) \leq 1\Big\} < \epsilon.$$

This implies that

$$\lim_{j\to\infty}\Big\{\sum_{k=1}^{mn} ||(x_k^i - x_k^j), z_1,\cdots,z_{n-1}|| \\ + \inf\Big\{\rho^{\frac{p_k}{H}} : \sup_k u_kM_k\Big(||\frac{\Delta_m^n(x_k^i-x_k^j)}{\rho}, z_1,\cdots,z_{n-1}||\Big) \leq 1\Big\}\Big\} < \epsilon,$$

for all $i \geq n_0$. Using the continuity of (M_k), we have

$$\sum_{k=1}^{mn} ||(x_k^i - x_k), z_1,\cdots,z_{n-1}|| \\ + \inf\Big\{\rho^{\frac{p_k}{H}} : \sup_k u_kM_k\Big(||\frac{\Delta_m^n x_k^i-\Delta_m^n x_k}{\rho}, z_1,\cdots,z_{n-1}||\Big) \leq 1\Big\} < \epsilon,$$

for all $i \geq n_0$. It follows that $(x^i - x) \in l_\infty(\mathcal{M}, \Delta_m^n, p, u, ||\cdot,\cdots,\cdot||)$. Since $x^i \in l_\infty(\mathcal{M}, \Delta_m^n, p, u, ||\cdot,\cdots,\cdot||)$ and $l_\infty(\mathcal{M}, \Delta_m^n, p, u, ||\cdot,\cdots,\cdot||)$ is a linear space, so we have $x = x^i - (x^i - x) \in l_\infty(\mathcal{M}, \Delta^n_m, p, u, ||\cdot, \cdots, \cdot||)$. This completes the proof. Similarly, we can prove that $c(\mathcal{M}, \Delta^n_m, p, u, ||\cdot, \cdots, \cdot||)$ and $c_0(\mathcal{M}, \Delta^n_m, p, u, ||\cdot, \cdots, \cdot||)$ are complete paranormed spaces in view of the above proof. □

Theorem 2.4. *If* $0 < p_k \leq q_k < \infty$ *for each* k, *then* $Z(\mathcal{M}, \Delta^n_m, p, u, ||\cdot, \cdots, \cdot||) \subseteq Z(\mathcal{M}, \Delta^n_m, q, u, ||\cdot, \cdots, \cdot||)$, *for* $Z = c_0$ *and* c.

Proof. Let $x = (x_k) \in c(\mathcal{M}, \Delta^n_m, p, u, ||\cdot, \cdots, \cdot||)$. Then there exists some $\rho > 0$ and $L \in X$ such that

$$\lim_{k\to\infty} u_k\Big(M_k\Big(||\frac{\Delta^n_m x_k - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} = 0.$$

This implies that $u_k M_k\Big(||\frac{\Delta^n_m x_k - L}{\rho}, z_1, \cdots, z_{n-1}||\Big) < \epsilon, \quad (0 < \epsilon < 1)$ for sufficiently large k. Hence we get

$$\begin{aligned}\lim_{k\to\infty} u_k\Big(M_k\Big(||\frac{\Delta^n_m x_k - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{q_k}&\\ \leq \lim_{k\to\infty} u_k\Big(M_k\Big(||\frac{\Delta^n_m x_k - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k}&\\ = 0.&\end{aligned}$$

This implies that $x = (x_k) \in c(\mathcal{M}, \Delta^n_m, q, u, ||\cdot, \cdots, \cdot||)$. This completes the proof. Similarly, we can prove for the case $Z = c_0$. □

Theorem 2.5. *If* $\mathcal{M}' = (M'_k)$ *and* $\mathcal{M}'' = (M''_k)$ *be two Musielak-Orlicz functions. Then*

(i) $Z(\mathcal{M}', \Delta^n_m, p, u, ||\cdot, \cdots, \cdot||) \subseteq Z(\mathcal{M}'' \circ \mathcal{M}', \Delta^n_m, p, u, ||\cdot, \cdots, \cdot||)$,

(ii) $Z(\mathcal{M}', \Delta^n_m, p, u, ||\cdot, \cdots, \cdot||) \cap Z(\mathcal{M}'', \Delta^n_m, p, u, ||\cdot, \cdots, \cdot||)$

$$\subseteq Z(\mathcal{M}' + \mathcal{M}'', \Delta^n_m, p, u, ||\cdot, \cdots, \cdot||),$$

for $Z = l_\infty, c$ *and* c_0.

Proof. (i) We prove this part for $Z = l_\infty$ and the rest of the cases will follow similarly. Let $(x_k) \in l_\infty(\mathcal{M}', \Delta^n_m, p, u, ||\cdot, \cdots, \cdot||)$, then there exists $0 < U < \infty$ such that

$$u_k\Big(M'_k\Big(||\frac{\Delta^n_m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} \leq U, \text{ for all } k \in \mathbb{N}.$$

Let $y_k = u_k M'_k\Big(||\frac{\Delta^n_m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)$. Then $y_k \leq U^{\frac{1}{p_k}} \leq V$, say for all $k \in \mathbb{N}$. Hence we have

$$\Big((M''_k \circ M'_k)\Big(||\frac{\Delta^n_m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} = (M''_k(y_k))^{p_k} \leq (M''_k(V))^{p_k} < \infty,$$

for all $k \in \mathbb{N}$. Hence $\sup_k u_k\Big((M_k'' \circ M_k')\Big(||\frac{\Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} < \infty$. Thus $x = (x_k) \in l_\infty(\mathcal{M}'' \circ \mathcal{M}', \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$.

(ii) We prove the result for the case $Z = c$ and the rest of the cases will follow similarly. Let $x = (x_k) \in c(\mathcal{M}', \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||) \cap c(\mathcal{M}'', \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$, then there exist some $\rho_1, \rho_2 > 0$ and $L \in X$ such that

$$\lim_{k\to\infty} u_k\Big(M_k'\Big(||\frac{\Delta_m^n x_k - L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} = 0$$

and

$$\lim_{k\to\infty} u_k\Big(M_k''\Big(||\frac{\Delta_m^n x_k - L}{\rho_2}, z - 1, \cdots, z_{n-1}||\Big)\Big)^{p_k} = 0.$$

Let $\rho = \rho_1 + \rho_2$. Then we have

$$\begin{aligned} & u_k\Big((M_k' + M_k'')\Big(||\frac{\Delta_m^n x_k - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} \\ & \quad \leq K\Big[\Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)u_k M_k'\Big(||\frac{\Delta_m^n x_k - L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\ & \quad + K\Big[\Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)u_k M_k''\Big(||\frac{\Delta_m^n x_k - L}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}. \end{aligned}$$

This implies that

$$\lim_{k\to\infty} u_k\Big((M_k' + M_k'')\Big(||\frac{\Delta_m^n x_k - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} = 0.$$

Thus $x = (x_k) \in c(\mathcal{M}' + \mathcal{M}'', \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$. This completes the proof. □

Theorem 2.6. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers, then $Z(\mathcal{M}, \Delta_m^{n-1}, p, u, ||\cdot, \cdots, \cdot||) \subset Z(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$, for $Z = l_\infty, c$ and c_0.*

Proof. We prove the result for the case $Z = l_\infty$ and the rest of the cases will follow similarly. Let $x = (x_k) \in l_\infty(\mathcal{M}, \Delta_m^{n-1}, p, u, ||\cdot, \cdots, \cdot||)$. Then we can have $\rho > 0$ such that

$$u_k\Big(M_k\Big(||\frac{\Delta_m^{n-1} x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} < \infty, \quad \text{for all } k \in \mathbb{N}. \tag{2.4}$$

On considering 2ρ and using the convexity of (M_k), we have

$$\begin{aligned} u_k M_k\Big(||\frac{\Delta_m^n x_k}{2\rho}, z_1, \cdots, z_{n-1}||\Big) & \leq \frac{1}{2} u_k M_k\Big(||\frac{\Delta_m^{n-1} x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big) \\ & + \frac{1}{2} u_k M_k\Big(||\frac{\Delta_m^{n-1} x_{k+m}}{\rho}, z_1, \cdots, z_{n-1}||\Big). \end{aligned}$$

Hence we have

$$u_k\Big(M_k\Big(||\tfrac{\Delta_m^n x_k}{2\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k}$$

$$\leq K\Big\{u_k\Big(\frac{1}{2}M_k\Big(||\frac{\Delta_m^{n-1}x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} + u_k\Big(\frac{1}{2}M_k\Big(||\frac{\Delta_m^{n-1}x_{k+m}}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k}\Big\}.$$

Then using equation (2.4), we have

$$u_k\Big(M_k\Big(||\frac{\Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} < \infty, \quad \text{for all } \ k \in \mathbb{N}.$$

Thus $l_\infty(\mathcal{M}, \Delta_m^{n-1}, p, u, ||\cdot, \cdots, \cdot||) \subset l_\infty(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$. □

Theorem 2.7. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Then*

$$c_0(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||) \subset c(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||) \subset l_\infty(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||).$$

Proof. It is obvious that $c_0(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||) \subset c(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$. We shall prove that $c(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||) \subset l_\infty(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$. Let $x = (x_k) \in c(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$. Then there exists some $\rho > 0$ and $L \in X$ such that

$$\lim_{k\to\infty} u_k\Big(M_k\Big(||\frac{\Delta_m^n x_k - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)^{p_k} = 0.$$

On taking $\rho = 2\rho_1$, we have

$$\begin{aligned}
u_k\Big(M_k\Big(||\frac{\Delta_m^n x_k}{\rho}, z_1, \cdots, z_{n-1}\Big)\Big)^{p_k} &\leq K\Big[\frac{1}{2}u_k\Big(M_k\Big(||\frac{\Delta_m^n x_k - L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \\
&\quad + K\Big[\frac{1}{2}u_k M_k\Big(||\frac{L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&\leq K\Big(\frac{1}{2}\Big)^{p_k} u_k\Big[M_k\Big(||\frac{\Delta_m^n x_k - L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&\quad + K\Big(\frac{1}{2}\Big)^{p_k} \max\Big(1, u_k\Big(M_k\Big(||\frac{L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big)^H\Big),
\end{aligned}$$

where $H = \max(1, \sup p_k)$. Thus we get $x = (x_k) \in l_\infty(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$. Hence $c_0(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||) \subset c(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$ $\subset l_\infty(\mathcal{M}, \Delta_m^n, p, u, ||\cdot, \cdots, \cdot||)$. □

Theorem 2.8. *The sequence space* $l_\infty(\mathcal{M},\Delta_m^n,p,u,||\cdot,\cdots,\cdot||)$ *is solid.*

Proof. Let $x=(x_k)\in l_\infty(\mathcal{M},\Delta_m^n,p,u,||\cdot,\cdots,\cdot||)$, that is

$$\lim_{k\to\infty} u_k\Big[M_k\Big(||\frac{\Delta_m^n x_k}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}<\infty.$$

Let (α_k) be a sequence of scalars such that $|\alpha_k|\le 1$ for all $k\in\mathbb{N}$. Thus we have

$$\begin{aligned}\lim_{k\to\infty} u_k\Big[M_k\Big(||\frac{\alpha_k\Delta_m^n x_k}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}&\\ \le\ & \lim_{k\to\infty} u_k\Big[M_k\Big(||\frac{\Delta_m^n x_k}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\ <\ & \infty.\end{aligned}$$

This shows that $(\alpha_k x_k)\in l_\infty(\mathcal{M},\Delta_m^n,p,u,||\cdot,\cdots,\cdot||)$ for all sequences of scalars (α_k) with $|\alpha_k|\le 1$ for all $k\in\mathbb{N}$, whenever $(x_k)\in l_\infty(\mathcal{M},\Delta_m^n,p,u,||\cdot,\cdots,\cdot||)$. Hence the space $l_\infty(\mathcal{M},\Delta_m^n,p,u,||\cdot,\cdots,\cdot||)$ is a solid sequence space. □

Theorem 2.9. *The sequence space* $l_\infty(\mathcal{M},\Delta_m^n,p,u,||\cdot,\cdots,\cdot||)$ *is monotone.*

Proof. The proof of the theorem is obvious and so we omit it.

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