KYUNGPOOK Math. J. 54(2014), 31-41
http://dx.doi.org/10.5666/KMJ.2014.54.1.31

Integrability and L_1-convergence of Certain Cosine Sums with Quasi Hyper Convex Coefficients

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Abstract. In this paper criterion for L_1-convergence of a certain cosine sums with quasi hyper-convex coefficients is obtained.

1. Introduction

It is well known that if a trigonometric series converges in L_1-metric to a function $f \in L_1$, then it is the Fourier series of the function f. Riesz [2] gave a counter example showing that in a metric space L_1 we cannot expect the converse of the above said result to hold true. This motivated the various authors to study L_1-convergence of the trigonometric series. During their investigations some authors introduced modified trigonometric sums as these sums approximate their limits better than the classical trigonometric series in the sense that they converge in L_1-metric to the sum of the trigonometric series whereas the classical series itself may not.

In what follows we will denote by

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx, \tag{1.1}$$

with partial sums defined by

$$g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx, \tag{1.2}$$

and

$$g(x) = \lim_{n\to\infty} g_n(x). \tag{1.3}$$

* Corresponding Author.
Received April 6, 2011; accepted September 28, 2011.
2010 Mathematics Subject Classification: 42A16, 40G05.
Key words and phrases: cosine sums, L_1-convergence, quasi hyper-convex null sequences.

In the sequel we will mention some results which are useful for the further work. Dirichlet's kernels are denoted by

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

$$\widetilde{D}_n(t) = \sum_{k=1}^{n} \cos kt$$

$$\overline{\overline{D}}_n(t) = \sum_{k=1}^{n} \sin kt = \frac{\cos\frac{t}{2} - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

$$\overline{D}_n(t) = -\frac{1}{2}\cot\frac{t}{2} + \overline{\overline{D}}_n(t) = -\frac{\cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

In what follows we will briefly describe some known facts which will be very useful for us (see [14]):

$$\begin{aligned} & S_n^0 = S_n = a_0 + a_1 + \cdots + a_n \\ & S_n^k = S_0^{k-1} + S_1^{k-1} + \cdots + S_n^{k-1}, k = 1, 2, \cdots ; n = 1, 2, \cdots ; \\ & A_n^0 = 1, A_n^k = A_0^{k-1} + A_1^{k-1} + \cdots + A_n^{k-1}, k = 1, 2, \cdots ; n = 1, 2, \cdots ; \end{aligned} \tag{1.4}$$

The A_n's are called the binomial coefficients and are given by the following relation:

$$\sum_{k=0}^{\infty} A_k^{\alpha} x^{\alpha} = (1-x)^{(-\alpha-1)}, \tag{1.5}$$

whereas S_n's are given by

$$\sum_{k=0}^{\infty} S_k^{\alpha} x^{\alpha} = (1-x)^{-\alpha} \sum_{k=0}^{\infty} S_k x^{\alpha}, \tag{1.6}$$

and

$$\begin{aligned} & A_n^{\alpha} = \sum_{k=0}^{n} A_k^{\alpha-1}, A_n^{\alpha} - A_{n-1}^{\alpha} = A_n^{\alpha-1}, \\ & A_n^{\alpha} = \binom{n+\alpha}{n} \cong \frac{n^{\alpha}}{\Gamma(\alpha+1)} (\alpha \neq -1, -2, \cdots). \end{aligned} \tag{1.7}$$

In what follows we will consider that $\alpha > 0$.

The Cesaro means T_k^{α} of order α is denoted by $T_k^{\alpha} = \frac{S_k^{\alpha}}{A_k^{\alpha}}.$ Also for $0 < x \leq \pi$, let

$$\begin{aligned} & S_n^0(x) = \widetilde{D}_n(x) = \cos x + \cos 2x + \cdots + \cos nx \\ & S_n^1(x) = S_0(x) + S_1(x) + \cdots + S_n(x), \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & S_n^k(x) = S_0^{k-1}(x) + S_1^{k-1}(x) + \cdots + S_n^{k-1}(x). \end{aligned} \tag{1.8}$$

The Cesaro means $T_k^\alpha(x)$ of order α is denoted by $T_k^\alpha(x) = \frac{S_k^\alpha(x)}{A_k^\alpha}.$

Lemma 1.1. (see [3]) If $\alpha \geq 0, p \geq 0$, $\epsilon_n = o(n^{-p})$, and $\sum_{n=0}^{\infty} A_n^{\alpha+p}|\Delta^{\alpha+1}\epsilon_n| < \infty$, then

$$\sum_{n=0}^{\infty} A_n^{\lambda+p}|\Delta^{\lambda+1}\epsilon_n| < \infty,$$

for $-1 \leq \lambda \leq \alpha$, $A_n^{\lambda+p}\Delta^{\lambda}\epsilon_n$ is of bounded variation for $0 \leq \lambda \leq \alpha$ and tends to zero as $n \to \infty$.

Definition 1.2. A sequence of scalars (a_n) is said to be semi-convex if $a_n \to 0$ as $n \to \infty$, and

$$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1} + \Delta^2 a_n| < \infty, (a_0 = 0), \tag{1.9}$$

where $\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}$ and $\Delta^\alpha a_n = \Delta^{\alpha-1}a_n - \Delta^{\alpha-1}a_{n+1}$, for $\alpha \geq 2$.

Definition 1.3. A sequence of scalars (a_n) is said to be quasi semi-convex if $a_n \to 0$ as $n \to \infty$, and

$$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1} - \Delta^2 a_n| < \infty, (a_0 = 0). \tag{1.10}$$

The L_1-convergence of cosine and sine sums was studied by several authors. Kolmogorov in [7], proved the following theorem:

Theorem 1.4. If (a_n) is a quasi-convex null sequence, then for the L_1-convergence of the cosine series (1.1), it is necessary and sufficient that $\lim_{n\to\infty} a_n \cdot \log n = 0$.

The case in which sequence (a_n) is convex, of this theorem was established by Young (see [13]). That is why, sometimes, this theorem is known as Young-Kolmogorov Theorem.

Definition 1.5. A sequence of scalars (a_n) is said to be quasi-convex if $a_n \to 0$ as $n \to \infty$, and

$$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1}| < \infty, (a_0 = 0), \tag{1.11}$$

Lemma 1.6. If $a = (a_n)$ is quasi semi-convex than it follows that it is twice quasi semi-convex, but converse is not true.

Proof. Let us suppose that $a = (a_n)$ is quasi semi-convex, which mean that

$$\sum_{n=1}^{\infty} n|\Delta^2 a_n - \Delta^2 a_{n-1}| < \infty.$$

Then from definition of the twice quasi semi-convexity we get:

$$\sum_{n=1}^{\infty} n|\Delta^4 a_n - \Delta^4 a_{n-1}| = \sum_{n=1}^{\infty} n|\Delta^2 a_n - 3\Delta^2 a_{n-1} + 3\Delta^2 a_{n-2} - \Delta^2 a_{n-3}| \leq$$

$$\sum_{n=1}^{\infty} n|\Delta^2 a_n - \Delta^2 a_{n-1}| + 2\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1} - \Delta^2 a_{n-2}| + \sum_{n=1}^{\infty} n|\Delta^2 a_{n-2} - \Delta^2 a_{n-3}| < \infty.$$

Conversely is not true which is given by this example.
Let $\Delta^2 a_n = \frac{1}{n}$, then $\Delta^2 a_{n-1} = \frac{1}{n-1}$ and from this we obtain following estimation:

$$\sum_{n=1}^{\infty} n|\Delta^2 a_n - \Delta^2 a_{n-1}| = \sum_{n=1}^{\infty} n\left|\frac{1}{n(n-1)}\right| = \infty.$$

But

$$\sum_{n=1}^{\infty} n|\Delta^4 a_n - \Delta^4 a_{n-1}| = \sum_{n=1}^{\infty} n\left|\frac{-6}{n(n-1)(n-2)(n-3)}\right| < \infty,$$

with which is proved lemma. □

Definition 1.7. A sequence of scalars (a_n) is said to be quasi hyper-convex if $a_n \to 0$ as $n \to \infty$, and

$$\sum_{n=1}^{\infty} n^{\alpha}|\Delta^{\alpha+1} a_{n-1}| < \infty, (a_0 = 0), \tag{1.12}$$

for $\alpha > 0$.

Remark 1.8. For $\alpha = 1$, from this class of coefficients follows the class defined in definition 1.6 (see Lemma 2.2 bellow).

In paper [9], is given definition of generalized semi-convex coefficients as follows:

Definition 1.9. A sequence of scalars (a_n) is said to be generalized semi-convex if $a_n \to 0$ as $n \to \infty$, and

$$\sum_{n=1}^{\infty} n^{\alpha}|\Delta^{\alpha+1} a_{n-1} + \Delta^{\alpha+1} a_n| < \infty, (a_0 = 0). \tag{1.13}$$

for $\alpha > 0$. For $\alpha = 1$, this class reduces to the class of semi-convex coefficients (definition 1.3).

Remark 1.10. If (a_n) is a quasi hyper-convex null scalar sequence, then it is generalized semi-convex scalars sequence too.

Bala and Ram in [1] have proved that Theorem 1.4 holds true for cosine series with semi-convex null sequences in the following form:

Theorem 1.11. If (a_n) is a semi-convex null sequence, then for the convergence of the cosine series (1.1) in the metric space L_1, it is necessary and sufficient that $a_{k-1}\log k = 0(1), k \to \infty$.

Garret and Stanojevic in [5], have introduced modified cosine sums

$$G_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} (\Delta a_j)\cos kx. \tag{1.14}$$

The same authors (see [6]), Ram in [11] and Singh and Sharma in [12] studied the L_1-convergence of this cosine sum under different sets of conditions on the coefficients (a_n). Kumari and Ram in [10], introduced new modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right)\cos kx \tag{1.15}$$

and

$$G'_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right)\sin kx, \tag{1.16}$$

and have studied their L_1-convergence under the condition that the coefficients (a_n) belong to different classes of sequences. Later one, Kulwinder in [8], introduced new modified sine sums as

$$K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n} (\Delta a_{j-1} - \Delta a_{j+1})\sin kx, \tag{1.17}$$

and have studied their L_1-convergence under the condition that the coefficients (a_n) are semi-convex null. In [9], was proved the L_1 convergence of some modified cosine series using in consideration generalized semi-convex coefficients.

2. Results

In this paper we call the modified cosine sums $N_n(x)$, defined in [4] and we will prove that this sums L_1- converges to $g(x)$, under conditions that coefficients

(a_n) are quasi hyper-convex and $\alpha \in \mathbb{N}$. In paper [4], was proved that the above modified cosine sums L_1-converges to $g(x)$ under condition that coefficients (a_n) are semi-convex. First we will prove this trivial fact:

Lemma 2.1. If (a_n) is a quasi hyper-convex null sequence of scalars, then it follows that the following relation

$$\sum_{k=1}^{\infty} k^{\alpha}|(\Delta^{\alpha+1}a_{k-1} - \Delta^{\alpha+1}a_k)| < \infty$$

holds.

Proof. The proof of the lemma follows directly from the following inequality

$$\sum_{k=1}^{\infty} k^{\alpha}|(\Delta^{\alpha+1}a_{k-1} - \Delta^{\alpha+1}a_k)| \leq \sum_{k=1}^{\infty} k^{\alpha}|\Delta^{\alpha+1}a_{k-1}| + \sum_{k=1}^{\infty} k^{\alpha}|\Delta^{\alpha+1}a_k| < \infty,$$

and definition of quasi hyper-convex sequences. □

Lemma 2.2. If (a_n) is a quasi hyper-convex null sequence of scalars, then it is quasi semi-convex null sequence too.

Proof. Because (a_n) is a quasi hyper-convex, then it follows that relation (1.12) holds for every $\alpha > 0$, and for $\alpha = 1$. From this we get the following relation

$$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1} - \Delta^2 a_n| \leq \sum_{n=1}^{\infty} n|\Delta^2 a_{n-1}| + \sum_{n=1}^{\infty} n|\Delta^2 a_n| < \infty.$$

□

The next Theorem, which will be of use to us, also appears in [4]. For the convenience of reader we give its proof.

Theorem 2.3. Let (a_n) a the quasi semi-convex null sequence, then $N_n(x)$ converges to $g(x)$ in L_1 norm.

Proof. We have

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cdot \cos kx = \frac{1}{\left(2\sin\frac{x}{2}\right)^2} \cdot \sum_{k=1}^{n} a_k \cdot \cos kx \cdot \left(2\sin\frac{x}{2}\right)^2$$

$$= -\frac{1}{\left(2\sin\frac{x}{2}\right)^2} \cdot \sum_{k=1}^{n} a_k[\cdot \cos(k+1)x - 2\cos kx + cos(k-1)x]$$

$$= -\frac{1}{\left(2\sin\frac{x}{2}\right)^2} \cdot \sum_{k=1}^{n} (a_{k-1} - 2a_k + a_{k+1}) \cdot \cos kx - \frac{a_0 \cos x}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_n \cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^2} +$$

$$\frac{a_1}{\left(2\sin\frac{x}{2}\right)^2} - \frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^2} \Rightarrow$$

$$S_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\cdot\sum_{k=1}^{n}\Delta^2 a_{k-1}\cos kx - \frac{a_0\cos x}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2} - \frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^2}.$$

Applying Abel's transformation, we have

$$S_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\cdot\sum_{k=1}^{n-1}(\Delta^2 a_{k-1} - \Delta^2 a_k)\widetilde{D}_k(x) + \frac{\Delta^2 a_{n-1}\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}$$

$$-\frac{a_0\cos x}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_n\cos(n+1)x}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2} - \frac{a_{n+1}\cos nx}{\left(2\sin\frac{x}{2}\right)^2}.$$

Since $\widetilde{D}_n(x)$ is uniformly bounded on every segment $[\epsilon, \pi-\epsilon]$, for every $\epsilon > 0$,

$$g(x) = \lim_{n\to\infty} S_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\cdot\sum_{k=1}^{\infty}(\Delta^2 a_{k-1} - \Delta^2 a_k)\widetilde{D}_k(x) + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}.$$

Also

$$N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta^2 a_{j-1} - \Delta^2 a_j)\cos kx + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2},$$

respectively

$$N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{n}\Delta^2 a_{k-1}\cos kx + \frac{\Delta^2 a_n\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2} + \frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}.$$

Now applying Abel's transformation we get the following relation:

$$N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{n-1}(\Delta^2 a_{k-1} - \Delta^2 a_k)\widetilde{D}_k(x) + \frac{\Delta^2 a_{n-1}\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2} + \frac{\Delta^2 a_n\cdot\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}$$

$$+\frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}$$

From above relation we will have:

$$g(x) - N_n(x) = -\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=n+1}^{\infty}(\Delta^2 a_{k-1} - \Delta^2 a_k)\widetilde{D}_k(x)$$

$$-\frac{\Delta^2 a_{n-1}\cdot \widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}-\frac{\Delta^2 a_n\cdot \widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}\Rightarrow$$

$$g(x)-N_n(x)=-\lim_{m\to\infty}\left(\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=n+1}^{m}(\Delta^2 a_{k-1}-\Delta^2 a_k)\widetilde{D}_k(x)\right)$$

$$-\frac{\Delta^2 a_{n-1}\cdot \widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}-\frac{\Delta^2 a_n\cdot \widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}.$$

Thus, we have

$$\int_0^\pi |g(x)-N_n(x)|dx\to 0,$$

for $n\to\infty$ and definition . □

Theorem 2.4. Let (a_n) be a quasi hyper-convex null sequence, then $N_n(x)$ converges to $g(x)$ in L_1 norm.

Proof. Let us start from the modified cosine sums:

$$N_n(x)=-\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta^2 a_{j-1}-\Delta^2 a_j)\cos kx+\frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}$$

From Theorem 2.3, it follows that

$$||g(x)-N_n(x)||_{L_1}\to 0, n\to\infty,$$

if (a_n), are quasi semi-convex null coefficients(in our case sequence (a_n), is quasi semi-convex(see Lemma 2.2). In what follows we will prove that

$$||g(x)-N_n(x)||_{L_1}\to 0, n\to\infty,$$

if (a_n), are quasi hyper-convex null coefficients, using in consideration Cesaro's mean of integral order.

Applying Abel's transformation, we have

$$N_n(x)=-\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\cdot\sum_{k=1}^{n-1}(\Delta^2 a_{k-1}-\Delta^2 a_k)\widetilde{D}_k(x)+\frac{\Delta^2 a_{n-1}\cdot \widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}+\frac{\Delta^2 a_n\cdot \widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}$$

$$+\frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}.$$

If we use Abel's transformation α times, we get this relation:

$$N_n(x)=-\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{n-\alpha}(\Delta^{\alpha+1}a_{k-1}-\Delta^{\alpha+1}a_k)S_k^{\alpha-1}(x)$$

$$+\sum_{k=1}^{\alpha-1}\frac{(\Delta^{k+1}a_{n-k-1}-\Delta^{k+1}a_{n-k})S_{n-k}^{k}(x)}{\left(2\sin\frac{x}{2}\right)^2}$$

$$+\frac{\Delta^2 a_{n-1}\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}+\frac{\Delta^2 a_n\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}+\frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}. \tag{2.1}$$

Since $S_n^k(x)$, $T_n(x)$, $\widetilde{D}_n(x)$ are uniformly bounded in any segment $[\epsilon,\pi-\epsilon]$, for any $\epsilon>0$, and $T_n^k(x)=\frac{S_n^k(x)}{A_n^k}$ we have
(2.2)

$$g(x)=\lim_{n\to\infty}N_n(x)=-\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{\infty}(\Delta^{\alpha+1}a_{k-1}-\Delta^{\alpha+1}a_k)S_k^{\alpha-1}(x)+\frac{a_1}{\left(2\sin\frac{x}{2}\right)^2}.$$

From relations (2.1) and (2.2) we have:

$$g(x)-N_n(x)=-\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=n-(\alpha+1)}^{\infty}(\Delta^{\alpha+1}a_{k-1}-\Delta^{\alpha+1}a_k)S_k^{\alpha-1}(x)$$

$$-\sum_{k=1}^{\alpha-1}\frac{(\Delta^{k+1}a_{n-k-1}-\Delta^{k+1}a_{n-k})S_{n-k}^{k}(x)}{\left(2\sin\frac{x}{2}\right)^2}-\frac{\Delta^2 a_{n-1}\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}-\frac{\Delta^2 a_n\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}.$$

Respectively

$$||g(x)-N_n(x)||\le\left\|\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=n-(\alpha+1)}^{\infty}(\Delta^{\alpha+1}a_{k-1}-\Delta^{\alpha+1}a_k)S_k^{\alpha-1}(x)\right\|$$

$$+\left\|\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{\alpha-1}\Delta^{k+1}a_{n-k-1}S_{n-k}^{k}(x)\right\|+\left\|\frac{1}{\left(2\sin\frac{x}{2}\right)^2}\sum_{k=1}^{\alpha-1}\Delta^{k+1}a_{n-k}S_{n-k}^{k}(x)\right\|$$

$$+\left\|\frac{\Delta^2 a_{n-1}\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}\right\|+\left\|\frac{\Delta^2 a_n\widetilde{D}_n(x)}{\left(2\sin\frac{x}{2}\right)^2}\right\|$$

$$\le C_1\int_0^{\pi}\left|\sum_{k=n-(\alpha+1)}^{\infty}(\Delta^{\alpha+1}a_{k-1}-\Delta^{\alpha+1}a_k)S_k^{\alpha-1}(x)\right|dx$$

$$+C_1\int_0^{\pi}\left|\sum_{k=1}^{\alpha-1}\Delta^{k+1}a_{n-k-1}S_{n-k}^{k}(x)\right|dx+C_1\int_0^{\pi}\left|\sum_{k=1}^{\alpha-1}\Delta^{k+1}a_{n-k}S_{n-k}^{k}(x)\right|dx$$

$$+C_1\int_0^{\pi}\left|\Delta^2 a_{n-1}\widetilde{D}_n(x)\right|dx+C_1\int_0^{\pi}\left|\Delta^2 a_n\widetilde{D}_n(x)\right|dx$$

$$\leq C_1 \cdot \sum_{k=n-(\alpha+1)}^{\infty} A_k^{\alpha-1} \left|(\Delta^{\alpha+1}a_{k-1} - \Delta^{\alpha+1}a_k)\right| \int_0^{\pi} \left|T_k^{\alpha-1}(x)\right| dx$$

$$+C_1 \cdot \sum_{k=1}^{\alpha-1} A_{n-k}^{k} \left|\Delta^{k+1}a_{n-k-1}\right| \int_0^{\pi} \left|T_{n-k}^{k}(x)\right| dx$$

$$+C_1 \cdot \sum_{k=1}^{\alpha-1} A_{n-k}^{k} \left|\Delta^{k+1}a_{n-k}\right| \int_0^{\pi} \left|T_{n-k}^{k}(x)\right| dx$$

$$+C_1 \cdot A_n^0 \cdot |\Delta^2 a_{n-1}| \int_0^{\pi} |T_n^0(x)| dx + C_1 \cdot A_n^0 \cdot |\Delta^2 a_n| \int_0^{\pi} |T_n^0(x)| dx.$$

The first summand in the above assertion tends to zero from Lemma 2.1, the other summands in the above assertion tends to zero based in Lemma 1.1, where $n \to \infty$. Finally we get $||g(x) - N_n(x)|| \to 0$, for $n \to \infty$. □

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