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Sequence Spaces of Fuzzy Real Numbers Using Fuzzy Metric

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Abstract. The sequence spaces $c^F(M)$, $c_0^F(M)$ and $\ell^F(M)$ of fuzzy real numbers with fuzzy metric are introduced. Some properties of these sequence spaces like solidness, symmetricity, convergence-free etc. are studied. We obtain some inclusion relations involving these sequence spaces.

1. Introduction

The concept of fuzzy set theory was introduced by L.A. Zadeh in the year 1965. Later on different classes of sequences of fuzzy numbers have been investigated by Yu-ru [15], Tripathy and Baruah ([5], [6]), Tripathy and Borgohain [4], Tripathy and Dutta ([8], [9]), Tripathy and Sarma ([16],[17],[18]) and many others.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

If the convexity of the Orlicz function is replaced by $M(x+y) \leq M(x)+M(y)$, then this function is called as modulus function.

Remark 1.1. An Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Throughout the article $w^F, \ell^F, \ell_\infty^F$, represent the classes of *all*, *absolutely summable* and *bounded* sequences fuzzy real numbers respectively.

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2. Definitions and Background

A fuzzy real number X is a fuzzy set on R i.e. a mapping $X : R \to I(= [0,1])$ associating each real number t with its grade of membership $X(t)$.

A fuzzy real number X is called *convex* if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$. If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called *normal*. A fuzzy real number X is said to be *upper semi-continuous* if for each $\varepsilon > 0$, $X^{-1}([0, a+\varepsilon))$, for all $a \in I$ is open in the usual topology of R.

The class of all upper semi-continuous, normal, convex fuzzy real numbers is denoted by $R(I)$. For $X \in R(I)$, the α-level set X^α, for $0 < \alpha \leq 1$ is defined by, $X^\alpha = \{t \in R : X(t) \geq \alpha\}$. The 0-level i.e. X^0 is the closure of strong 0-cut, i.e. $\{t \in R : X(t) > 0\}$.

The *absolute value* of $X \in R(I)$ is defined by,

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{for } t \geq 0 \\ 0 \quad \text{otherwise} \end{cases}$$

For $r \in R$ and $\overline{r} \in R(I)$ is defined as,

$$\overline{r}(t) = \begin{cases} 1 & \text{if } t = r \\ 0 & \text{if } t \neq r \end{cases}$$

The additive and multiplicative identities of $R(I)$ are denoted by $\overline{0}$ and $\overline{1}$.

Let D be the set of all closed bounded intervals $X = [X^L, X^R]$.

Define $d : D \times D \to R$ by $d(X,Y) = \max\{|X^L - Y^L|, |X^R - Y^R|\}$. Then clearly (D, d) is a complete metric space.

Define $\overline{d} : R(I) \times R(I) \to R$ by $\overline{d}(X,Y) = \sup\limits_{0<\alpha\leq 1} d(X^\alpha, Y^\alpha)$, for $X, Y \in R(I)$.

Then it is well known that $(R(I), \overline{d})$ is a complete metric space.

A sequence $X = (X_k)$ of fuzzy real numbers is said to *converge* to the fuzzy number X_0, if for every $\varepsilon > 0$, there exists $k_0 \in N$ such that, $\overline{d}(X_k, X_0) < \varepsilon$ for all $k \geq k_0$.

A sequence space E is said to be *solid* if $(Y_n) \in E$, whenever $(X_n) \in E$ and $|Y_n| \leq |X_n|$, for all $n \in N$.

Let $X = (X_n)$ be a sequence, then $S(X)$ denotes the set of all permutations of the elements of (X_n) i.e. $S(X) = \{(X_{\pi(n)}) : \pi \text{ is a permutation of } N\}$.

A sequence space E is said to be *symmetric* if $S(X) \subset E$ for all $X \in E$.

A sequence space E is said to be *convergence-free* if $(Y_n) \in E$ whenever $(X_n) \in E$ and $X_n = \overline{0}$ implies $Y_n = \overline{0}$.

A sequence space E is said to be *monotone* if E contains the canonical pre-images of all its step spaces.

Lemma 2.1. *A sequence space E is solid implies that E is monotone.*

Lindenstrauss and Tzafriri [13] used the notion of Orlicz function and introduced the sequence space:

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right), \text{ for some } \rho > 0 \right\}$$

The space ℓ_M with the norm,

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space, which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p, which is an Orlicz sequence space with $M(x) = x^p$, for $1 \leq p \leq \infty$.

In the later stage different classes of Orlicz sequence spaces were introduced and studied by Altin, Et and Tripathy [2], Tripathy, Altin and Et [3], Tripathy and Mahanta [14], Tripathy and Sarma ([16], [17],[18]) and many others.

Let $d_F : R(I) \times R(I) \to R(I)$ be the fuzzy metric. Let the mappings $L, T : [0,1] \times [0,1] \to [0,1]$ be symmetric, non-decreasing in both arguments and satisfy, $L[0,0] = 0$ and $M[1,1] = 1$. We consider $L = \min\{p, q\}$ and $T = \max\{p, q\}$, where $p, q \in [0,1]$, for our investigations in this article.

Let $\lambda : R(I) \times R(I) \to R$ be such that $\lambda(X, Y) = \sup\limits_{0<\alpha\leq 1} \lambda_\alpha(X^\alpha, Y^\alpha)$, where $\lambda_\alpha : R \times R \to R$ and $\lambda_\alpha(X^\alpha, Y^\alpha) = \min\{|X_1^\alpha - Y_1^\alpha|, |X_2^\alpha - Y_2^\alpha|\}$.

Similarly, let $\rho : R(I) \times R(I) \to R$ be such that $\rho(X, Y) = \sup\limits_{0<\alpha\leq 1} \rho_\alpha(X^\alpha, Y^\alpha)$, where $\rho_\alpha : R \times R \to R$ and $\rho_\alpha(X^\alpha, Y^\alpha) = \max\{|X_1^\alpha - Y_1^\alpha|, |X_2^\alpha - Y_2^\alpha|\}$.

Since the distance between two fuzzy numbers is again a fuzzy number, so the α- level set of this distance d_F between the fuzzy real numbers X and Y is denoted by,

$$[d(X, Y)]_\alpha = [\lambda_\alpha(X^\alpha, Y^\alpha), \rho_\alpha(X^\alpha, Y^\alpha)], 0 < \alpha \leq 1.$$

The quadruple $(R(I), d_F, L, T)$ is called a fuzzy metric space and d_F is a fuzzy metric, if,

1. $d_F(X, Y) = \bar{0}$ if and only if $X = Y$.
2. $d_F(X, Y) = d_F(Y, X)$, for all $X, Y \in R(I)$.
3. For all $X, Y, Z \in R(I)$,
   - $d_F(X, Y)(s+t) \geq L(d_F(X, Z)(s), d_F(Z, Y)(t))$, whenever $s \leq \lambda_1(X, Z)$, $t \leq \lambda_1(Z, Y)$ and $s + t \leq \lambda_1(X, Y)$.
   - $d_F(X, Y)(s+t) \leq T(d_F(X, Z)(s), d_F(Z, Y)(t))$, whenever $s \geq \lambda_1(X, Z)$, $t \geq \lambda_1(Z, Y)$ and $s + t \geq \lambda_1(X, Y)$.

Using the concept of Orlicz function and fuzzy metric, we introduce the following sequence spaces,

$$\ell_\infty^F(M) = \left\{(X_k) \in w^F : \sup_k M\left(\frac{\lambda(X_k,\overline{0})}{r}\right) < \infty \text{ and } \sup_k M\left(\frac{\rho(X_k,\overline{0})}{r}\right) < \infty, \text{ for some } r > 0\right\}$$

$$c^F(M) = \left\{(X_k) \in w^F : M\left(\frac{\lambda(X_k,L)}{r}\right) \to 0 \text{ and } M\left(\frac{\rho(X_k,L)}{r}\right) \to 0, \text{ as } k \to \infty, \text{ for some } r > 0, L \in R(I)\right\}$$

$$c_0^F(M) = \left\{(X_k) \in w^F : M\left(\frac{\lambda(X_k,\overline{0})}{r}\right) \to 0 \text{ and } M\left(\frac{\rho(X_k,\overline{0})}{r}\right) \to 0, \text{ as } k \to \infty, \text{ for some } r > 0\right\}$$

3. Main Results

Theorem 3.1. *The classes of sequences $Z(M)$, where $Z = \ell_\infty^F, c^F, c_0^F$, are metric spaces by the metric defined by,*

$$\overline{d}(X,Y)_M = inf\left\{r > 0 : \sup_k M\left(\frac{\lambda(X_k,Y_k)}{r}\right) \le 1 \text{ and } \sup_k M\left(\frac{\rho(X_k,Y_k)}{r}\right) \le 1\right\}$$

for $X, Y \in Z(M)$, where $Z = \ell_\infty^F, c^F, c_0^F$.

Proof. Consider the sequence space $\ell_\infty^F(M)$. We have to show that $\ell_\infty^F(M)$ is a metric space.

For $X, Y \in \ell_\infty^F(M)$, we have,

(i) $\overline{d}(X,Y)_M = 0$. This implies that,

$$\lambda(X_k,Y_k) = 0 \text{ and } \rho(X_k,Y_k) = 0, \text{ for all } k \in N.(\text{ Since } M(0) = 0).$$

Which implies that, for all $\alpha \in (0,1]$,

$$\lambda(X_k,Y_k) = \sup_{0<\alpha\le 1} \lambda_\alpha(X_k^\alpha,Y_k^\alpha) = 0 \Rightarrow \lambda_\alpha(X_k^\alpha,Y_k^\alpha) = 0, \text{ for all } \alpha \in (0,1].$$

$$\Rightarrow \min\{|X_{k1}^\alpha - Y_{k1}^\alpha|, |X_{k2}^\alpha - Y_{k2}^\alpha|\} = 0, \text{ for all } \alpha \in (0,1]. \tag{3.1}$$

Similarly, we get that, for all $\alpha \in (0,1]$,

$$\rho(X_k,Y_k) = \sup_{0<\alpha\le 1} \rho_\alpha(X_k^\alpha,Y_k^\alpha) = 0 \Rightarrow \rho_\alpha(X_k^\alpha,Y_k^\alpha) = 0, \text{ for all } \alpha \in (0,1].$$

$$\Rightarrow \max\{|X_{k1}^{\alpha} - Y_{k1}^{\alpha}|, |X_{k2}^{\alpha} - Y_{k2}^{\alpha}|\} = 0, \text{ for all } \alpha \in (0,1]. \tag{3.2}$$

From (3.1) and (3.2), it follows that, for all $k \in N, X_k = Y_k \Rightarrow X = Y$.

Conversely, assume that, $X = Y$.

Then, using the defintion of λ and ρ, we get,

$$\lambda_\alpha(X_k^\alpha, Y_k^\alpha) = 0 \text{ and } \rho_\alpha(X_k^\alpha, Y_k^\alpha) = 0, \text{ for all } k \in N, \alpha \in (0,1].$$

Which implies that,

$$\sup_{0<\alpha\leq 1} \lambda_\alpha(X_k^\alpha, Y_k^\alpha) = 0 \text{ and } \sup_{0<\alpha\leq 1} \rho_\alpha(X_k^\alpha, Y_k^\alpha) = 0, \text{for all } k \in N.$$

It follows that, $\lambda(X_k, Y_k) = 0$ and $\rho(X_k, Y_k) = 0$.

Using the continuity of M, we get, $\overline{d}(X,Y)_M = 0$.

Which shows that, $\overline{d}(X,Y)_M = 0$ if and only if $X = Y$.

(ii) $\overline{d}(X,Y)_M$

$$= \inf\left\{r > 0 : \sup_k M\left(\frac{\lambda(X_k, Y_k)}{r}\right) \leq 1; \sup_k M\left(\frac{\rho(X_k, Y_k)}{r}\right) \leq 1\right\}.$$

From the defintion of λ, it follows that,

$$\begin{aligned}
\lambda(X_k, Y_k) &= \sup_{0<\alpha\leq 1} \lambda_\alpha(X_k^\alpha, Y_k^\alpha) \\
&= \sup_{0<\alpha\leq 1} [\min\{|X_{k1}^\alpha, Y_{k1}^\alpha|, |X_{k2}^\alpha, Y_{k2}^\alpha|\}] \\
&= \sup_{0<\alpha\leq 1} [\min\{|Y_{k1}^\alpha, X_{k1}^\alpha|, |Y_{k2}^\alpha, X_{k2}^\alpha|\}] \\
&= \sup_{0<\alpha\leq 1} \lambda_\alpha(Y_k^\alpha, X_k^\alpha) \\
&= \lambda(Y_k, X_k).
\end{aligned}$$

Proceeding in the same way, we get, $\rho(X_k, Y_k) = \rho(Y_k, X_k)$.

Thus we get,

$$\begin{aligned}
\overline{d}(X,Y)_M &= \inf\left\{r > 0 : \sup_k M\left(\frac{\lambda(X_k, Y_k)}{r}\right) \leq 1; \sup_k M\left(\frac{\rho(X_k, Y_k)}{r}\right) \leq 1\right\} \\
&= \inf\left\{r > 0 : \sup_k M\left(\frac{\lambda(Y_k, X_k)}{r}\right) \leq 1; \sup_k M\left(\frac{\rho(Y_k, X_k)}{r}\right) \leq 1\right\} \\
&= \overline{d}(Y,X)_M.
\end{aligned}$$

Hence, $\overline{d}(X,Y)_M = \overline{d}(Y,X)_M$.

(iii) Let $r_1 > 0, r_2 > 0$ such that,

$$\sup_k M\left(\frac{\lambda(X_k, Z_k)}{r_1}\right) \leq 1.$$

$$\sup_k M\left(\frac{\lambda(Z_k,Y_k)}{r_2}\right)\leq 1.$$

Let $r=r_1+r_2$. Following the definition of λ, we get,

$$\begin{aligned}\lambda(X_k,Y_k) &= \sup_{0<\alpha\leq 1}\lambda_\alpha(X_k^\alpha,Y_k^\alpha)\quad\text{where}\quad \lambda_\alpha(X^\alpha,Y^\alpha)\\ &= \min\{|X_1^\alpha-Y_1^\alpha|,|X_2^\alpha-Y_2^\alpha|\}.\end{aligned}$$

Following the defintion of λ, we get,

$$\lambda_\alpha(X^\alpha,Y^\alpha)\leq\lambda_\alpha(X^\alpha,Z^\alpha)+\lambda_\alpha(Z^\alpha,Y^\alpha),\quad\text{for all }\alpha\in(0,1].$$

Taking the supremum throughout α, we get,

$$\sup_{0<\alpha\leq 1}\lambda_\alpha(X_k^\alpha,Y_k^\alpha)\leq\sup_{0<\alpha\leq 1}\lambda_\alpha(X_k^\alpha,Z_k^\alpha)+\sup_{0<\alpha\leq 1}\lambda_\alpha(Z_k^\alpha,Y_k^\alpha).$$

which implies that, $\lambda(X_k,Y_k)\leq\lambda(X_k,Z_k)+\lambda(Z_k,Y_k)$
Using the continuity of M, we get,

$$\begin{aligned}&\sup_k M\left(\frac{\lambda(X_k,Y_k)}{r}\right)\\ &\leq\sup_k M\left(\frac{\lambda(X_k,Z_k)}{r_1+r_2}+\frac{\lambda(Z_k,Y_k)}{r_1+r_2}\right)\\ &\leq\sup_k M\left(\frac{r_1}{r_1+r_2}\left(\frac{\lambda(X_k,Z_k)}{r_1}\right)+\frac{r_2}{r_1+r_2}\left(\frac{\lambda(Z_k,Y_k)}{r_2}\right)\right)\\ &\leq\sup_k\left(\frac{r_1}{r_1+r_2}\right)M\left(\frac{\lambda(X_k,Z_k)}{r_1}\right)+\sup_k\left(\frac{r_1}{r_1+r_2}\right)M\left(\frac{\lambda(Z_k,Y_k)}{r_2}\right)\\ &\leq 1.\end{aligned}$$

Since r's are non-negative, so taking the infimum of such r's, we get,

$$\begin{aligned}&\inf\left\{r>0:\sup_k M\left(\frac{\lambda(X_k,Y_k)}{r}\right)\leq 1\right\}\\ &\leq\inf\left\{r_1>0:\sup_k M\left(\frac{\lambda(X_k,Z_k)}{r_1}\right)\leq 1\right\}+\inf\left\{r_2>0:\sup_k M\left(\frac{\lambda(Z_k,Y_k)}{r_2}\right)\leq 1\right\}\end{aligned}$$

Proceeding in the same way, we get,

$$\begin{aligned}&\inf\left\{r>0:\sup_k M\left(\frac{\rho(X_k,Y_k)}{r}\right)\leq 1\right\}\\ &\leq\inf\left\{r_1>0:\sup_k M\left(\frac{\rho(X_k,Z_k)}{r_1}\right)\leq 1\right\}+\inf\left\{r_2>0:\sup_k M\left(\frac{\rho(Z_k,Y_k)}{r_2}\right)\leq 1\right\}\end{aligned}$$

Thus we have,

$$\begin{aligned}
&\inf\left\{r>0:\sup_k M\left(\frac{\lambda(X_k,Y_k)}{r}\right)\le 1;\sup_k M\left(\frac{\rho(X_k,Y_k)}{r}\right)\le 1\right\}\\
&\le \inf\left\{r_1>0:\sup_k M\left(\frac{\lambda(X_k,Z_k)}{r_1}\right)\le 1;\sup_k M\left(\frac{\rho(X_k,Z_k)}{r_1}\right)\le 1\right\}\\
&+\inf\left\{r_2>0:\sup_k M\left(\frac{\lambda(Z_k,Y_k)}{r_2}\right)\le 1;\sup_k M\left(\frac{\rho(Z_k,Y_k)}{r_2}\right)\le 1\right\}\\
&\Rightarrow \overline{d}(X,Y)_M \le \overline{d}(X,Z)_M+\overline{d}(Z,Y)_M.
\end{aligned}$$

This proves that $\ell_\infty^F(M)$ is a metric space.
This completes the proof. □

Similarly, it can be proved that $Z(M)$, where $Z = c^F$ and c_0^F are metric spaces with the same metric using the above technique.

Theorem 3.2. *The classes of sequences $Z(M)$, where $Z = \ell_\infty^F, c^F, c_0^F$, is a complete metric space with the metric defined by,*

$$\overline{d}(X,Y)_M = \ inf\ \left\{r>0:\sup_k M\left(\frac{\lambda(X_k,Y_k)}{r}\right)\le 1;\sup_k M\left(\frac{\rho(X_k,Y_k)}{r}\right)\le 1\right\},$$

for $X, Y \in Z(M)$, where $Z = \ell_\infty^F, c^F, c_0^F$.

Proof. Consider the sequence space $\ell_\infty^F(M)$.
Let $(X^{(i)})$ be a Cauchy sequence in $\ell_\infty^F(M)$ such that, $X^{(i)} = (X_n^{(i)})_{n=1}^\infty$.
Let $\varepsilon > 0$ be given. For a fixed $x_0 > 0$, choose $p > 0$ such that $M\left(\frac{px_0}{2}\right) \ge 1$.
Then there exists a positive integer $n_0 = n_0(\varepsilon)$ such that,

$$\overline{d}(X^{(i)},X^{(j)})_M < \frac{\varepsilon}{px_0},\ \text{for all } i,j\ge n_0.$$

By the definition of $\overline{d}_M$, we get

$$\inf\left\{r>0:\sup_k M\Big(\frac{\lambda(X_k^{(i)},X_k^{(j)})}{r}\Big)\le 1;\sup_k M\Big(\frac{\rho(X_k^{(i)},X_k^{(j)})}{r}\Big)\le 1\right\}<\varepsilon, \tag{3.3}$$

$$\text{for all } i,j\ge n_0.$$

Which implies that,

$$\sup_k M\left(\frac{\lambda(X_k^{(i)},X_k^{(j)})}{r}\right)\le 1 \tag{3.4}$$

$$\sup_k M\left(\frac{\rho(X_k^{(i)},X_k^{(j)})}{r}\right)\le 1 \tag{3.5}$$

From (3.4) we get,

$$\sup_k M\left(\frac{\lambda(X_k^{(i)}, X_k^{(j)})}{r}\right) \le 1$$

$$\Rightarrow M\left(\frac{\lambda(X_k^{(i)}, X_k^{(j)})}{\overline{d}(X^{(i)}, X^{(j)})}\right) \le 1 \le M\left(\frac{px_0}{2}\right).$$

Using the continuity of M, we get,

$$\lambda(X_k^{(i)}, X_k^{(j)}) \le \frac{px_0}{2}.\frac{\varepsilon}{px_0} = \frac{\varepsilon}{2},$$

i.e $(X_k^{(i)})$ is a Cauchy sequence of $R(I)$. Since $R(I)$ is complete, so it follows that, $(X_k^{(i)})$ is also convergent.

Let, $\lim_i X_k^{(i)} = X_k,$ for each $k \in N$.

We have to establish that,

$$\lim_i X^{(i)} = X \text{ and } X \in \ell_\infty^F(M).$$

Since M is continuous, so on taking $j \to \infty$ and fixing i, we get from (3.4);

$$\sup_k M\left(\frac{\lambda(X_k^{(i)}, X_k)}{r}\right) \le 1, \text{ for some } r > 0 \text{ and } i \ge n_0.$$

Proceeding in the same way, we get from (3.5):

$$\sup_k M\left(\frac{\rho(X_k^{(i)}, X_k)}{r}\right) \le 1, \text{ for some } r > 0 \text{ and } i \ge n_0.$$

Now on taking the infimum of such r's together, we get from (3.3):

$$\inf\left\{r > 0 : \sup_k M\left(\frac{\lambda(X_k^{(i)}, X_k)}{r}\right) \le 1; \sup_k M\left(\frac{\rho(X_k^{(i)}, X_k)}{r}\right) \le 1\right\} < \varepsilon,$$

for some $i \ge n_0$.

Which shows that, $\overline{d}(X^{(i)}, X)_M < \varepsilon$, for all $i \ge n_0$.

i.e. $\lim_i X^{(i)} = X$.

Now, it is to show that $X \in \ell_\infty^F(M)$.

We know that,

$$\begin{aligned}\overline{d}(X, \overline{\theta})_M &\le \overline{d}(X, X^{(i)})_M + \overline{d}(X^{(i)}, \overline{\theta})_M \\ &< \varepsilon + M, \text{ for all } i \ge n_0(\varepsilon).\end{aligned}$$

i.e. $\overline{d}(X, \overline{\theta})_M$ is finite.

Which implies that $X \in \ell_\infty^F(M)$.

Hence $\ell_\infty^F(M)$ is a complete metric space.

Similarly it can be established that the other classes of sequences are complete metric spaces.

This completes the proof of the theorem. □

Theorem 3.3. *The classes of sequences $Z(M)$, where $Z = \ell_\infty^F$ and c_0^F, are solid whereas $c^F(M)$ is not solid.*

Proof. Let $(X_k) \in \ell_\infty^F(M)$. Then we have, for some $r > 0$,

$$\sup_k M\left(\frac{\lambda(X_k,\overline{0})}{r}\right) < \infty; \sup_k M\left(\frac{\rho(X_k,\overline{0})}{r}\right) < \infty.$$

Let (Y_k) be a sequence of fuzzy numbers with,

$$[d(Y_k,\overline{0})]_\alpha = [\lambda_\alpha(Y_k^\alpha,0), \rho_\alpha(Y_k^\alpha,0)], \text{ for } 0 < \alpha \leq 1,$$

Such that,$\lambda(Y_k,\overline{0}) \leq \lambda(X_k,\overline{0})$ and $\rho(Y_k,\overline{0}) \leq \rho(X_k,\overline{0})$.

Since M is non-decreasing continuous function, so we get, for some $r > 0$,

$$M\left(\frac{\lambda(Y_k,\overline{0})}{r}\right) \leq M\left(\frac{\lambda(X_k,\overline{0})}{r}\right) \text{ and } M\left(\frac{\rho(Y_k,\overline{0})}{r}\right) \leq M\left(\frac{\rho(X_k,\overline{0})}{r}\right),$$

which implies that,

$$\sup_k M\left(\frac{\lambda(Y_k,\overline{0})}{r}\right) \leq \sup_k M\left(\frac{\lambda(X_k,\overline{0})}{r}\right) < \infty, \quad \text{for some } r > 0.$$

$$\sup_k M\left(\frac{\rho(Y_k,\overline{0})}{r}\right) \leq \sup_k M\left(\frac{\rho(X_k,\overline{0})}{r}\right) < \infty, \quad \text{for some } r > 0.$$

Which implies that,

$$\sup_k M\left(\frac{\lambda(Y_k,\overline{0})}{r}\right) < \infty, \quad \text{for some } r > 0.$$

$$\sup_k M\left(\frac{\rho(Y_k,\overline{0})}{r}\right) < \infty, \quad \text{for some } r > 0.$$

Which shows that, $(Y_k) \in \ell_\infty^F(M)$.

Hence, $\ell_\infty^F(M)$ is solid.

Similarly we can establish that the class of sequences $c_0^F(M)$ is solid.

Proof of the second part follows from the following example.

Example 3.1. Let,

$$X_k(t) = \begin{cases} 1+t, & \text{for } t \in [-1,0] \\ 1-t, & \text{for } t \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Then, $(X_k) \in c^F(M)$.

Now, let

$$Y_k(t) = \begin{cases} 1+3t, & \text{for } t \in [-3^{-1}, 0], \text{ all } k \text{ odd} \\ 1-3t, & \text{for } t \in [0, 3^{-1}], \text{ all } k \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Such that, using the continuity of M,

$$\sup_k M\left(\frac{\lambda(Y_k, \bar{0}}{r}\right) \leq \sup_k M\left(\frac{\lambda(X_k, \bar{0}}{r}\right), \text{for some} r > 0.$$

$$\sup_k M\left(\frac{\rho(Y_k, \bar{0}}{r}\right) \leq \sup_k M\left(\frac{\rho(X_k, \bar{0}}{r}\right), \text{for some} r > 0.$$

But, (Y_k) is not convergent.

Hence $c^F(M)$ is not solid. □

Theorem 3.4. *The classes of sequences $Z(M)$, where $Z = \ell_\infty^F, c^F(M)$ and c_0^F are symmetric.*

Proof. Let $(X_k) \in \ell_\infty^F(M)$ and (Y_k) be a rearrangement of (X_k), such that,

$$X_k = Y_{m_k}, \text{ for each } k \in N.$$

Then, we have, $\lambda(X_k, \bar{0}) = \lambda(Y_{m_k}, \bar{0})$ and $\rho(X_k, \bar{0}) = \rho(Y_{m_k}, \bar{0})$.

Using the continuity of M, we get,

$$\sup_k M\left(\frac{\lambda(X_k, \bar{0})}{r}\right) = \sup_k M\left(\frac{\lambda(Y_{m_k}, \bar{0})}{r}\right), \quad \text{for some } r > 0.$$

$$\sup_k M\left(\frac{\rho(X_k, \bar{0})}{r}\right) = \sup_k M\left(\frac{\rho(Y_{m_k}, \bar{0})}{r}\right), \quad \text{for some } r > 0.$$

which implies that,

$$\sup_k M\left(\frac{\lambda(Y_{m_k}, \bar{0})}{r}\right) < \infty \text{ and } \sup_k M\left(\frac{\rho(Y_{m_k}, \bar{0})}{r}\right) < \infty, \quad \text{for some } r > 0.$$

Which shows that, $(Y_k) \in \ell_\infty^F(M)$.

Hence $\ell_\infty^F(M)$ is symmetric.

This completes the proof.

Proof is similar for the cases also. □

Proposition 3.5 *The classes of sequences $Z(M)$, where $Z = \ell_\infty^F, c^F(M)$ and c_0^F are not convergence-free.*

Proof. The result follows from the following example.

Example 3.2. Consider the sequence (X_k) defined as follows:

$$X_k(t) = \begin{cases} 1 + kt, & \text{for } t \in [-k^{-1}, 0] \\ 1 - kt, & \text{for } t \in [0, k^{-1}] \\ 0 & \text{otherwise} \end{cases}$$

Then we have, for some $r > 0$,

$$\sup_k M\left(\frac{\lambda(X_k, \bar{0})}{r}\right) < \infty$$

and

$$\sup_k M\left(\frac{\rho(X_k, \bar{0})}{r}\right) < \infty$$

Which shows that, $(X_k) \in \ell_\infty^F(M)$.

Now, let us consider the sequence (Y_k) such that,

$$Y_k(t) = \begin{cases} 1 + tk^{-2}, & \text{for } t \in [-k^2, 0] \\ 1 - tk^{-2}, & \text{for } t \in [0, k^2] \\ 0 & \text{otherwise} \end{cases}$$

Clearly we have, $\sup_k M\left(\frac{\lambda(X_k, \bar{0})}{r}\right) = \infty$ and $\sup_k M\left(\frac{\rho(X_k, \bar{0})}{r}\right) = \infty$

Thus, $(Y_k) \notin \ell_\infty^F(M)$.

Thus $\ell_\infty^F(M)$ is not convergence-free.

From the above example it follows that the classes of sequences and $c_0^F(M)$ are not convergence free.

This completes the proof. □

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