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A REVIEW OF THE SUPRA-CONVERGENCES OF SHORTLEY-WELLER METHOD FOR POISSON EQUATION

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ABSTRACT. The Shortley-Weller method is a basic finite difference method for solving the Poisson equation with Dirichlet boundary condition. In this article, we review the analysis for supra-convergence of the Shortley-Weller method. Though consistency error is first order accurate at some locations, the convergence order is globally second order. We call this increase of the order of accuracy, supra-convergence. Our review is not a simple copy but serves a basic foundation to go toward yet undiscovered analysis for another supra-convergence: we present a partial result for supra-convergence for the gradient of solution.

1. INTRODUCTION

The Poisson equation $-\Delta u = f$ is of primal importance in many physical problems, especially in fluid flows with incompressible condition. The Navier-Stokes equations with divergence free condition are the governing equations for incompressible fluid flows.

$$\begin{aligned} U_t + U \cdot \nabla U + \frac{1}{\rho}\nabla p &= \mu \Delta U \\ \nabla \cdot U &= 0 \end{aligned}$$

The equations misses a time evolution for pressure variable; a state equation $\Delta p = -\rho \nabla \cdot (U \cdot \nabla U)$, which is obtained from the divergence operation on the momentum equation, complements the missed time evolution. Thus solving the Poisson equation is essential in the simulation of incompressible fluid flows. The elliptic equation is coupled either with Dirichlet boundary condition or with Neumann boundary condition. At free surface, pressure value is the same as the default air pressure, so Dirichlet boundary condition $p = p_{air}$ holds. At the surface

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of container, the non-penetration condition $U \cdot n = 0$ results in Neumann boundary condition $\frac{\partial p}{\partial n} = -\rho\left(U \cdot \nabla U\right) \cdot n + \rho\mu\left(\Delta U\right) \cdot n$. There are two important aspects in the application of Poisson solver to incompressible fluid flows. One is the ability to deal with both boundary conditions, and the other is the accuracy of the gradient of the solution.

Shortley-Weller method [1] solves the Poisson equation with Dirichlet boundary condition in irregular domains. It is a dimension-by-dimension approach and hence works in any dimensions. It suffices to state one dimensional case; consider the stencil where grid nodes for u_i and u_{i-1} are located inside the domain and that for u_{i+1} is outside, then the method reads

$$\left(\frac{u_i - u_{i+1}}{h} + \frac{u_i - u_\Gamma}{h_\Gamma}\right)\frac{2}{h + h_\Gamma} = f_i.$$

Here u_Γ is given Dirichlet condition at the interface point between the grid nodes for u_i and u_{i+1}. The equations for each i constitute a non-symmetric linear system whose matrix is an M-matrix. It was proved in [1] that the numerical solution is second order accurate, and it was numerically observed in [4] that the gradient of the solution is also second order accurate.

The work of Gibou et al. [2] also solves the Poisson equation with Dirichlet boundary condition. It is a dimension-by-dimension approach and hence works in any dimensions. It's enough to state one dimensional case; in the same stencil, the method reads

$$\left(\frac{u_i - u_{i-1}}{h} + \frac{u_i - u_\Gamma}{h_\Gamma}\right)\frac{1}{h} = f_i.$$

The equations for each i constitute a symmetric linear system whose matrix is an M-matrix. It was numerically observed in [4] that the numerical solution is second order accurate and the gradient of the solution is only first order accurate.

The work of Purvis [5, 3] solves the Poisson equation with Neumann boundary condition. It is a finite volume approach that works in any dimensions. In the stencil, the method reads

$$\begin{array}{c} \frac{L_{i+\frac{1}{2},j}}{h}\frac{u_{ij}-u_{i+1,j}}{h} + \frac{L_{i-\frac{1}{2},j}}{h}\frac{u_{ij}-u_{i-1,j}}{h} \\ + \frac{L_{i,j+\frac{1}{2}}}{h}\frac{u_{ij}-u_{i,j+1}}{h} + \frac{L_{i,j-\frac{1}{2}}}{h}\frac{u_{ij}-u_{i,j-1}}{h} \end{array} = f_{ij},$$

where $L_{i+\frac{1}{2},j}$ denotes the length of the right edge of the cell and similarly defined are the other lengths. The equations for each i constitute a symmetric linear system whose matrix is an M-matrix. The numerical solution was numerically observed to be second order accurate in [3]. The accuracy of the numerical gradient was observed to be second order with zero Neumann boundary condition but drops to first order with non-zero Neumann boundary condition.

We have listed three finite difference methods for Poisson equations either with Dirichlet or with Neumann boundary condition. All of them work in any dimensions and in arbitrary shaped domains. If the grid node for u_{ij} and its four neighboring nodes are inside the domain, the three finite differences are the same as the standard five point finite difference method whose consistency is second order accurate. Near the boundary, all the methods lose the second order accuracy: the consistency of Shortley-Weller method is only first order accurate, that of Gibou et al. is not convergent, and that of Purvis is not convergent either. See the details in [1, 2, 3, 4] for each.

A unique property of finite difference methods for Elliptic problems is **supra-convergence**. Although consistency order varies from zero to the second, convergence order maintains two. The second order convergence for the solution would result in first order convergence for its gradient, however both the solution and its gradient are second order accurate in the cases of Shortley-Weller and Purvis, but not in the case of Gibou. The phenomena were numerically observed and most of their supporting mechanisms of the supra-convergences are still unknown to the best knowledge of the authors. It was shown in Ciarlet [1] how the supra-convergence for the solution occurs in Shortly-Weller method. In this article, we review the analysis with details and introduce new findings toward the analysis for the supra-convergence of the gradient in Shortley-Weller method.

2. Supra-convergence of Solution

In this section, we review the analysis for the supra-convergence of the solution in Ciarlet [1]. We try it not to be a simple copy but a basic foundation to start from and go toward the yet undiscovered analysis for other supra-convergences.

Consider a uniform grid with step size h. Let the four neighboring nodes of a grid note P inside the domain be named as P_1, P_2, P_3, P_4 and the distances to the neighbors as h_1, h_2, h_3, h_4. Some of the neighboring nodes may be the points on the boundary, thus the distances may be different to each other near the interface while they are all equal to h away from the boundary. In the stencil, Shortley-Weller method reads

$$\begin{aligned} -\Delta_h u(P) = \left(\frac{2}{h_1 h_3} + \frac{2}{h_2 h_4}\right) u(P) - \frac{2}{h_1(h_1+h_3)} u(P_1) - \frac{2}{h_3(h_1+h_3)} u(P_3) \\ - \frac{2}{h_2(h_2+h_4)} u(P_2) - \frac{2}{h_4(h_2+h_4)} u(P_4). \end{aligned}$$

Let u_h be the solution of the following discrete equation.

$$\begin{cases} -\Delta_h u_h(P) = f(P), & P \in \Omega_h \\ u_h(P) = g(P), & P \in \Gamma_h. \end{cases}$$

Let $u(x)$ be the solution of the Poisson problem.

$$\begin{cases} -\Delta u(x) = f(x), & x \in \Omega \\ u(x) = g(x), & x \in \Gamma. \end{cases}$$

- Denote by Ω_h^* the subset of points of Ω_h that are adjacent to Γ_h, so that $h_1 \neq h_3$ and/or $h_2 \neq h_4$.
- $-\Delta_h(u(P) - u_h(P))$ is $O(h^2)$ when $P \in \Omega_h - \Omega_h^*$ but drops to $O(h)$ when $P \in \Omega_h^*$.
- (Discrete maximum principle) If $-\Delta_h u \geq 0$ then minimum should be achieved at Γ_h. Similarly, if $-\Delta_h u(P) \leq 0$ then maximum should be achieved at Γ_h.
- (Discrete Green's function) For each $Q \in \Omega_h$, define the function $G_h(P, Q)$, $P \in \Omega_h \cup \Gamma_h$ as the solution of the discrete problem

$$\begin{cases} -\Delta_h u_h(P) = \begin{cases} 0, & P \neq Q \\ \frac{1}{h^2}, & P = Q \end{cases}, & P \in \Omega_h \\ u_h(P) = 0, & P \in \Gamma_h. \end{cases}$$

Since $-\Delta_h u_h \geq 0$, the minimum should be achieved at Γ_h, and therefore $G_h(P,Q) \geq 0$ for any $P \in \Omega_h \cup \Gamma_h$.

- (Summation property of Green's function) If $u_h \equiv 0$ on Γ_h,

$$u_h(P) = \sum_{Q \in \Omega_h} (-\Delta_h u_h(Q))\, G_h(P,Q)\, h^2, \forall P \in \Omega_h \cup \Gamma_h.$$

Theorem 2.1. *(Bound on Green's function)* $\sum_{Q \in \Omega_h} G_h(P,Q)\, h^2 \leq C$.

Proof. Let $U_h = \sum_{Q \in \Omega_h} G_h(P,Q)\, h^2$, then $-\Delta_h U_h = 1$ in Ω_h and $U_h = 0$ on Γ_h. Consider the analytic solution $U(x)$ such that $-\Delta U = 1$ in Ω and $U = 0$ on Γ. Then we have $0 < 1 - c < -\Delta_h(U)$ for some constant c, since $-\Delta_h(U) = 1 + O(h)$. Using the summation property, noting that $\frac{1}{1-c}U = U_h = 0$ on Γ_h,

$$\begin{aligned} U_h(P) &= \sum_{Q \in \Omega_h} G_h(P,Q)\, h^2 \\ &\leq \sum_{Q \in \Omega_h} \left(-\frac{1}{1-c} \Delta_h U(Q) \right) G_h(P,Q)\, h^2 \\ &= \frac{1}{1-c} U(P) \leq \frac{1}{1-c} |U|_\infty . \end{aligned}$$

$\square$

Theorem 2.2. *(Bound on Green's function)* $\sum_{Q \in \Omega_h^*} G_h(P,Q) \leq 1$.

Proof. Let $U_h = \sum_{Q \in \Omega_h^*} G_h(P,Q)$. Since $-\Delta_h U_h = 0$ in $\Omega_h - \Omega_h^*$, the maximum of U_h should be attained at $\Omega_h^* \cup \Gamma_h$. Since $U_h = 0$ on Γ_h and $U_h \geq 0$ in $\Omega_h \cup \Gamma_h$, the maximum should be attained at Ω_h^*. Let the maximum point be $P^* \in \Omega_h^*$. At that point,

$$\begin{aligned} \left(\frac{2}{h_1 h_3} + \frac{2}{h_2 h_4} \right) u(P^*) \quad & - \frac{2}{h_1(h_1+h_3)} u(P_1) - \frac{2}{h_3(h_1+h_3)} u(P_3) \\ & - \frac{2}{h_2(h_2+h_4)} u(P_2) - \frac{2}{h_4(h_2+h_4)} u(P_4) = \frac{1}{h^2}. \end{aligned}$$

Since $P^* \in \Omega_h^*$, one of its neighborhood should be Γ_h, let us say P_1. Then

$$
\begin{aligned}
\left(\frac{2}{h_1h_3}+\frac{2}{h_2h_4}\right)u\left(P^*\right) &= \frac{2}{h_3\left(h_1+h_3\right)}u\left(P_3\right)+\frac{2}{h_2\left(h_2+h_4\right)}u\left(P_2\right)\\
&\quad +\frac{2}{h_4\left(h_2+h_4\right)}u\left(P_4\right)+\frac{1}{h^2}\\
\left(\frac{2}{h_1h_3}+\frac{2}{h_2h_4}\right)u\left(P^*\right) &\leq \left(\frac{2}{h_3\left(h_1+h_3\right)}+\frac{2}{h_2\left(h_2+h_4\right)}+\frac{2}{h_4\left(h_2+h_4\right)}\right)u\left(P^*\right)\\
&\quad +\frac{1}{h^2}\\
\frac{2}{h_1\left(h_1+h_3\right)}u\left(P^*\right) &\leq \frac{1}{h^2}\\
u\left(P^*\right) &\leq \frac{h_1\left(h_1+h_3\right)}{2h^2}\leq 1.
\end{aligned}
$$

□

Theorem 2.3. *(Convergence of the solution)* $u\left(P\right)-u_h\left(P\right)=O\left(h^2\right)$, $\forall P\in\Omega_h\cup\Gamma_h$.

Proof. Since $u-u_h=0$ on Γ_h, we can apply the summation formula. $\forall P\in\Omega_h\cup\Gamma_h$

$$
\begin{aligned}
\left(u-u_h\right)\left(P\right) &= \sum_{Q\in\Omega_h}\left(-\Delta_h\left(u-u_h\right)\left(Q\right)\right)G_h\left(P,Q\right)h^2\\
&= \sum_{Q\in\Omega_h^*}\left(-\Delta_h\left(u-u_h\right)\left(Q\right)\right)G_h\left(P,Q\right)h^2\\
&\quad + \sum_{Q\in\Omega_h-\Omega_h^*}\left(-\Delta_h\left(u-u_h\right)\left(Q\right)\right)G_h\left(P,Q\right)h^2\\
&= O\left(h^3\right)+O\left(h^2\right).
\end{aligned}
$$

□

3. Symmetrization Implementation

We listed Shortley-Weller method and Gibou method for Poisson equation with Dirichlet boundary condition. Shortley-Weller method excels Gibou method since it produces second order accurate gradient, but Gibou method is preferred to Shortley-Weller for its implementation is a symmetric linear system while that of Shortley-Weller is non-symmetric. In this section, we introduce a new implementation of Shortley-Weller in one dimension as symmetric linear system which is obviously the winner over the two choices.

The following equations are the discretization of Shortley-Weller method in one dimension.

$$\begin{aligned} &&&\vdots \\ \frac{2}{h_{i-\frac{1}{2}}h_{i+\frac{1}{2}}}u_i - \frac{2}{h_{i-\frac{1}{2}}\left(h_{i-\frac{1}{2}}+h_{i+\frac{1}{2}}\right)}u_{i-1} - \frac{2}{h_{i+\frac{1}{2}}\left(h_{i-\frac{1}{2}}+h_{i+\frac{1}{2}}\right)}u_{i+1} &=& f_i \\ \frac{2}{h_{i+\frac{1}{2}}h_{i+\frac{3}{2}}}u_{i+1} - \frac{2}{h_{i+\frac{1}{2}}\left(h_{i+\frac{1}{2}}+h_{i+\frac{3}{2}}\right)}u_{i} - \frac{2}{h_{i+\frac{3}{2}}\left(h_{i+\frac{1}{2}}+h_{i+\frac{3}{2}}\right)}u_{i+2} &=& f_{i+1} \\ &&&\vdots \end{aligned}$$

Here $h_{i+\frac{1}{2}} = x_{i+1} - x_i$. The scheme results in non-symmetric linear system, because $\frac{2}{h_{i+\frac{1}{2}}\left(h_{i-\frac{1}{2}}+h_{i+\frac{1}{2}}\right)} \neq \frac{2}{h_{i+\frac{1}{2}}\left(h_{i+\frac{1}{2}}+h_{i+\frac{3}{2}}\right)}$. The idea of the new implementation is fairly simple. Multiplying the volume element on each equation, we get

$$\begin{aligned} &&&\vdots \\ \frac{2\left(h_{i-\frac{1}{2}}+h_{i+\frac{1}{2}}\right)}{h_{i-\frac{1}{2}}h_{i+\frac{1}{2}}}u_i - \frac{2}{h_{i-\frac{1}{2}}}u_{i-1} - \frac{2}{h_{i+\frac{1}{2}}}u_{i+1} &=& f_i\left(h_{i-\frac{1}{2}}+h_{i+\frac{1}{2}}\right) \\ \frac{2\left(h_{i+\frac{1}{2}}+h_{i+\frac{3}{2}}\right)}{h_{i+\frac{1}{2}}h_{i+\frac{3}{2}}}u_{i+1} - \frac{2}{h_{i+\frac{1}{2}}}u_{i} - \frac{2}{h_{i+\frac{3}{2}}}u_{i+2} &=& f_{i+1}\left(h_{i+\frac{1}{2}}+h_{i+\frac{3}{2}}\right) \\ &&&\vdots \end{aligned}$$

Now, the scheme results in symmetric linear system, now note the equality $\frac{2}{h_{i+\frac{1}{2}}} = \frac{2}{h_{i+\frac{1}{2}}}$.

In higher dimensions, the simple trick does not work. For example, in two dimensions, the original discretizations are

$$\begin{aligned} &\vdots \\ \left(\frac{2}{h_{i-\frac{1}{2},j}h_{i+\frac{1}{2},j}} + \frac{2}{h_{i,j-\frac{1}{2}}h_{i,j+\frac{1}{2}}}\right)u_{i,j} \quad & -\frac{2}{h_{i-\frac{1}{2},j}\left(h_{i-\frac{1}{2},j}+h_{i+\frac{1}{2},j}\right)}u_{i-1,j} \\ -\frac{2}{h_{i+\frac{1}{2},j}\left(h_{i-\frac{1}{2},j}+h_{i+\frac{1}{2},j}\right)}u_{i+1,j} \quad & -\frac{2}{h_{i,j-\frac{1}{2}}\left(h_{i,j-\frac{1}{2}}+h_{i,j+\frac{1}{2}}\right)}u_{i,j-1} \\ -\frac{2}{h_{i,j+\frac{1}{2}}\left(h_{i,j-\frac{1}{2}}+h_{i,j+\frac{1}{2}}\right)}u_{i,j+1} \quad & = \quad f_{i,j} \end{aligned}$$

$$
\begin{aligned}
\left(\frac{2}{h_{i+\frac{1}{2},j}h_{i+\frac{3}{2},j}}+\frac{2}{h_{i+1,j-\frac{1}{2}}h_{i+1,j+\frac{1}{2}}}\right)u_{i+1,j}- &\qquad \frac{2}{h_{i+\frac{1}{2},j}\left(h_{i+\frac{1}{2},j}+h_{i+\frac{3}{2},j}\right)}u_{i,j}\\
-\frac{2}{h_{i+\frac{3}{2},j}\left(h_{i+\frac{1}{2},j}+h_{i+\frac{3}{2},j}\right)}u_{i+2,j}- &\quad \frac{2}{h_{i,+1j-\frac{1}{2}}\left(h_{i+1,j-\frac{1}{2}}+h_{i+1,j+\frac{1}{2}}\right)}u_{i+1,j-1}\\
-\frac{2}{h_{i+1,j+\frac{1}{2}}\left(h_{i+1,j-\frac{1}{2}}+h_{i+1,j+\frac{1}{2}}\right)}u_{i+1,j+1}= &\qquad f_{i+1,j}\\
&\vdots
\end{aligned}
$$

The scheme results in non-symmetric linear system, because

$$
\frac{2}{h_{i+\frac{1}{2},j}\left(h_{i-\frac{1}{2},j}+h_{i+\frac{1}{2},j}\right)}\neq\frac{2}{h_{i+\frac{1}{2},j}\left(h_{i+\frac{1}{2},j}+h_{i+\frac{3}{2},j}\right)}.
$$

Multiplying by the volume element $\frac{h_{i-\frac{1}{2},j}+h_{i+\frac{1}{2},j}}{2}\frac{h_{i,j-\frac{1}{2},j}+h_{i,j+\frac{1}{2}}}{2}$ the first equation and multiplying by $\frac{h_{i+\frac{1}{2},j}+h_{i+\frac{3}{2},j}}{2}\frac{h_{i+1,j-\frac{1}{2}}+h_{i+1,j+\frac{1}{2}}}{2}$ the second equation does not fix the non-equality. Up to now, we know not of any way to implement Shortley-Weller as symmetric linear system, and guess that such simple conversion in one dimension might be impossible in higher dimensions.

4. Supra-Convergence of Gradient

It was shown in [6] how supra-convergence for the gradient of solution occurs in the standard five-point method for Poisson equation with rectangular domains. The analysis is based on integration-by-parts $\int_\Omega u\cdot\Delta u\,d\Omega=-\int_\Omega|\nabla u|^2d\Omega$. Let $\Delta_h u_h=f$ be an approximation for $\Delta u=f$, then $\Delta e=c$, where $e=u-u_h$ is the error of the approximation and $c=\Delta_h u-\Delta u$ is the consistency error. When the following discrete integration-by-parts holds

$$
\int\limits_{\Omega_h} u_h\cdot\Delta_h u_h\,d\Omega_h=-\int\limits_{\Omega_h}|\nabla u_h|^2,
$$

then it implies that $\int_{\Omega_h}|\nabla u_h-\nabla u|^2\,d\Omega_h\leq\int_{\Omega_h}|e\cdot c|\,d\Omega_h\leq\|e\|_{L^2}\|c\|_{L^2}=O\left(h^2\right)O\left(h^2\right)$ and we obtain a proof for the second order convergence of the solution gradient in L^2. In the usual setting, we show that the discrete integration-by-parts does not hold in the following. In this article, we present only our current partial results, but the analysis should go on by calculating the estimate of the error in integration-by-parts.

4.1. **Discrete Integration-by-parts.**

We want to derive a discrete version of $\int_\Omega u_x v=-\int_\Omega uv_x$ when $u|_\Gamma=0$. Assume that u_{ij} is defined on Ω_h, and $v_{i+\frac{1}{2},j}$ is defined whenever $(x_i,y_j)\in\Omega_h$ or $(x_{i+1},y_j)\in\Omega_h$. Based on the conventional control volumes, the two integrals are discretized as

$$\int_\Omega u_x v \quad := \sum_{\substack{(x_i,y_j)\in\Omega_h\\(x_{i+1},y_j)\in\Omega_h}} \frac{u_{i+1,j}-u_{ij}}{h_{i+\frac{1}{2},j}} \cdot v_{i+\frac{1}{2},j} \cdot h_{i+\frac{1}{2},j} \cdot \begin{pmatrix} \frac{h_{i,j+\frac{1}{2}}+h_{i,j-\frac{1}{2}}}{4} \\ +\frac{h_{i+1,j+\frac{1}{2}}+h_{i+1,j-\frac{1}{2}}}{4} \end{pmatrix}$$

$$\int_\Omega u v_x \quad := \sum_{(x_i,y_j)\in\Omega_h} u_{ij} \frac{v_{i+\frac{1}{2},j}-v_{i-\frac{1}{2}j}}{\frac{h_{i+\frac{1}{2},j}+h_{i-\frac{1}{2},j}}{2}} \cdot \frac{h_{i+\frac{1}{2},j}+h_{i-\frac{1}{2},j}}{2} \cdot \frac{h_{i,j+\frac{1}{2}}+h_{i,j-\frac{1}{2}}}{2}.$$

The summation of $\int_\Omega u_x v$ requires the definition of $h_{i,j\pm\frac{1}{2}}$ when $(x_i, y_j) \in \Gamma_h$. Note that $(x_i, y_j) \in \Gamma_h$ implies either $(x_{i-1}, y_j) \in \Omega_h$ or $(x_{i+1}, y_j) \in \Omega_h$. In the case when $(x_i, y_j) \in \Gamma_h$ and $(x_{i+1}, y_j) \in \Omega_h$, we define $h_{i,j+\frac{1}{2}} = h_{i+1,j+\frac{1}{2}}$ and $h_{i,j-\frac{1}{2}} = h_{i+1,j-\frac{1}{2}}$, and the other cases are defined in the same fashion.

Now we compare the difference of the two summations.

$$\begin{aligned}
\int_\Omega u v_x &= \sum_{(x_i,y_j)\in\Omega_h} u_{ij}\left(v_{i+\frac{1}{2},j}-v_{i-\frac{1}{2}j}\right)\frac{h_{i,j+\frac{1}{2}}+h_{i,j-\frac{1}{2}}}{2} \\
&= \sum_{(x_i,y_j)\in\Omega_h} u_{ij}v_{i+\frac{1}{2},j}\frac{h_{i,j+\frac{1}{2}}+h_{i,j-\frac{1}{2}}}{2} - \sum_{(x_i,y_j)\in\Omega_h} u_{ij}v_{i-\frac{1}{2},j}\frac{h_{i,j+\frac{1}{2}}+h_{i,j-\frac{1}{2}}}{2} \\
&= \sum_{(x_i,y_j)\in\Omega_h} u_{ij}v_{i+\frac{1}{2},j}\frac{h_{i,j+\frac{1}{2}}+h_{i,j-\frac{1}{2}}}{2} - \sum_{(x_{i+1},y_j)\in\Omega_h} u_{i+1j}v_{i+\frac{1}{2},j}\frac{h_{i+1,j+\frac{1}{2}}+h_{i+1,j-\frac{1}{2}}}{2} \\
&= \sum_{\substack{(x_i,y_j)\in\Omega_h\\(x_{i+1},y_j)\in\Omega_h}} u_{ij}v_{i+\frac{1}{2},j}\frac{h_{i,j+\frac{1}{2}}+h_{i,j-\frac{1}{2}}}{2} \\
&\quad - \sum_{\substack{(x_i,y_j)\in\Omega_h\\(x_{i+1},y_j)\in\Omega_h}} u_{i+1j}v_{i+\frac{1}{2},j}\frac{h_{i+1,j+\frac{1}{2}}+h_{i+1,j-\frac{1}{2}}}{2} \\
&= \sum_{\substack{(x_i,y_j)\in\Omega_h\\(x_{i+1},y_j)\in\Omega_h}} v_{i+\frac{1}{2},j}\left(u_{ij}\frac{h_{i,j+\frac{1}{2}}+h_{i,j-\frac{1}{2}}}{2} - u_{i+1j}\frac{h_{i+1,j+\frac{1}{2}}+h_{i+1,j-\frac{1}{2}}}{2}\right)
\end{aligned}$$

Now, combining the two summations,

$$
\begin{aligned}
\int_\Omega u_x v \;&+\; \int_\Omega u v_x \\
&= \sum_{\substack{(x_i,y_j)\in\Omega_h \\ (x_{i+1},y_j)\in\Omega_h}} (u_{i+1,j}-u_{ij})\, v_{i+\frac{1}{2},j} \frac{h_{i,j+\frac{1}{2}}+h_{i,j-\frac{1}{2}}+h_{i+1,j+\frac{1}{2}}+h_{i+1,j-\frac{1}{2}}}{4} \\
&+ \sum_{\substack{(x_i,y_j)\in\Omega_h \\ (x_{i+1},y_j)\in\Omega_h}} v_{i+\frac{1}{2},j} \left(u_{ij}\frac{h_{i,j+\frac{1}{2}}+h_{i,j-\frac{1}{2}}}{2} - u_{i+1j}\frac{h_{i+1,j+\frac{1}{2}}+h_{i+1,j-\frac{1}{2}}}{2} \right) \\
&= \sum_{\substack{(x_i,y_j)\in\Omega_h \\ (x_{i+1},y_j)\in\Omega_h}} v_{i+\frac{1}{2},j} \left(\begin{array}{c} u_{i+1,j}\frac{\left(h_{i,j+\frac{1}{2}}+h_{i,j-\frac{1}{2}}\right)-\left(h_{i+1,j+\frac{1}{2}}+h_{i+1,j-\frac{1}{2}}\right)}{4} \\ +u_{i,j}\frac{\left(h_{i,j+\frac{1}{2}}+h_{i,j-\frac{1}{2}}\right)-\left(h_{i+1,j+\frac{1}{2}}+h_{i+1,j-\frac{1}{2}}\right)}{4} \end{array} \right) \\
&= \sum_{\substack{(x_i,y_j)\in\Omega_h \\ (x_{i+1},y_j)\in\Omega_h}} v_{i+\frac{1}{2},j} \frac{u_{i+1,j}+u_{ij}}{2} \frac{\left(h_{i,j+\frac{1}{2}}+h_{i,j-\frac{1}{2}}\right)-\left(h_{i+1,j+\frac{1}{2}}+h_{i+1,j-\frac{1}{2}}\right)}{4} \\
&\neq 0
\end{aligned}
$$

Note that $\left(h_{i,j+\frac{1}{2}}+h_{i,j-\frac{1}{2}}\right)-\left(h_{i+1,j+\frac{1}{2}}+h_{i+1,j-\frac{1}{2}}\right)=0$ for every grid node (x_i, y_j) away from boundary, so the above sum is not zero but should be near zero.

5. CONCLUSION

We reviewed the analysis for supra-convergence of Shortley-Weller method in section 2. It was not a simple copy but serves a basic foundation to go toward the yet undiscovered analysis for other supra-convergences. In section 3, we introduced a new symmetric implementation of Shortley-Weller method in one dimension. In section 4, we presented a partial result for supra-convergence for the gradient of solution.

The Poisson equation is of primal importance in many physical problems, especially in fluid flows with incompressible condition. Both the supra-convergences of solution and its gradient are important. Actually in incompressible fluid flows, that of gradient is more important. Each of Shortley-Weller's, Gibou's, and Purvis' is one of the most popular methods for solving Poisson equation and their supra-convergence properties have been numerically observed and verified through numerous trials. Compared to their importances, mathematical foundation for understating the properties are far short. Though partial, this article serves an introduction and mathematical setting for the analysis.

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