논리연구 17-3(2014) pp. 441-460

Algebraic Kripke-style Semantics for Three-valued Paraconsistent Logic*

Eunsuk Yang

【Abstract】This paper deals with one sort of Kripke-style semantics for three-valued paraconsistent logic: algebraic Kripke-style semantics. We first introduce two three-valued systems, define their corresponding algebraic structures, and give algebraic completeness results for them. Next, we introduce algebraic Kripke-style semantics for them, and then connect them with algebraic semantics.

【Key Words】(Algebraic) Kripke-style semantics, Algebraic semantics, Three-valued logic, Paraconsistent logic.

접수일자: 2014.08.04 심사 및 수정완료일: 2014.09.24 게재확정일: 2014.09.30

* This paper was supported by research funds of Chonbuk National University in 2014. I must thank the referees for their helpful comments.

1. Introduction

The aim of this paper is to introduce one type of (binary) Kripke-style semantics, i.e., algebraic Kripke-style semantics, for three-valued paraconsistent logic. We have two reasons why we consider three-valued paraconsistent logics and binary Kripke-style semantics. Before introducing these reasons, we first recall one reason to introduce such semantics for three-valued logic in Yang (201+).

> First, the logic and semantics are very simple. Namely, three-valued logic is the most simple among fuzzy logics, and binary Kripke-style semantics are also simple Kripke-style semantics. Thus, for ease and clarity we consider three-valued logic and binary semantics (Yang (201+)).

The present author investigated algebraic and non-algebraic binary Kripke-style semantics for three-valued logic in it. He introduced the well-known two systems $\mathbf{Ł}_3$ (Łukasiewicz three-valued logic), $\mathbf{G}_3$ (Dummett-Gödel three-valued logic), and the system $\mathbf{IUML}_3$ (the three-valued extension of Involutive uninorm mingle logic **IUML**.[1]) Although introducing several important three-valued systems, he did not considered paraconsistent systems distinguished from relevant systems. Note that, while paraconsistent logics have in general weak-Boolean (briefly, wB) negations, relevant system have de Morgan (briefly,

---

1) As he mentioned, the system $\mathbf{IUML}_3$ also can be regarded as a version of $\mathbf{RM}_3$ (Three-valued **R** of relevant implication with mingle), $\mathbf{RM}^{\mathrm{T}}_3$ (see Yang (2013; 201+) for more details).

dM) negations. Note also that wB negations are dual of pseudo-Boolean (briefly, pB) negations such as the intuitionistic and Dummett-Gödel logics **H** and **G** have. Thus, it is not clear whether such semantics work for three-valued *paraconsistent* systems. This is a very natural question because the systems introduced in Yang (201+) have only pB or dM negations. So we have decided to consider such semantics for paraconsistent logic. This is the first and main reason to consider such semantics for three-valued paraconsistent logic.

Algebraic Kripke-style semantics have been recently provided for fuzzy logics based on t-norms and uninorms (see e.g. Montagna & Ono (2002), Montagna and Sacchetti (2003; 2004), Diaconescu & and Georgescu (2007), and Yang (2012b; 2012c; 2014). This kind of semantics is very interesting in the sense that it is closely connected with algebraic semantics. It is obvious that three-valued logics are also fuzzy logics. Thus, we have decided to introduce algebraic Kripke-style semantics for fuzzy paraconsistent logic. This is the very and second reason to consider algebraic Kripke-style semantics for three-valued paraconsistent logic.

This paper is organized as follows. First, in Section 2, as examples we introduce the systems $\mathbf{IUML}^{-}_{3}$ (the $\mathbf{IUML}_{3}$ with a wB negation) and $\mathbf{G}^{wB}_{3}$ (the $\mathbf{G}_{3}$ with a wB negation in place of its pB negation), their corresponding algebraic structures, and their algebraic completeness results. Next, in Section 3, we introduce one kind of binary relational Kripke-style semantics, algebraic Kripke-style semantics, for the above mentioned three-valued systems. We then connect them with algebraic semantics.

For ease, let us denote wB negation by - and dM negation by ~. Moreover, for convenience, we adopt the notations and terminology similar to those in Dunn (2000), Metcalfe & Montagna (2007), Montagna & Sacchetti (2003; 2004), Yang (2012a; 2012b; 2012c; 2014) and assume reader familiarity with them (together with the results found therein).

2. Three-valued paraconsistent systems and their algebraic semantics

We base three-valued paraconsistent logics on a countable propositional language with formulas *Fm* built inductively as usual from a set of propositional variables *VAR*, binary connectives →, &, ∧, ∨, and constants **F**, **f**, **t**, with a defined connective:

df1. A ↔ B := (A → B) ∧ (B → A).

We further define **T** and A_t as **F** → **F** and A ∧ **t**, respectively. We use the axiom systems to provide a consequence relation.

Definition 2.1

(i) $\mathbf{IUML}^{-}_{3}$ consists of the following axiom schemes and rules:

df2. -A := (**T** → A) → **F**

A1. A → A (self-implication, SI)

A2. (A ∧ B) → A, (A ∧ B) → B (∧-elimination, ∧-E)

A3. $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$ ($\wedge$-introduction, $\wedge$-I)

A4. $A \to (A \vee B)$, $B \to (A \vee B)$ ($\vee$-introduction, $\vee$-I)

A5. $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$ ($\vee$-elimination, $\vee$-E)

A6. $(A \& B) \to (B \& A)$ (&-commutativity, &-C)

A7. $(A \& \mathbf{t}) \leftrightarrow A$ (push and pop, PP)

A8. $\mathbf{F} \to A$ (ex falsum quodlibet, EF)

A9. $A \to \mathbf{T}$ (verum ex quolibet, VE)

A10. $(A \to (B \to C)) \leftrightarrow ((A \& B) \to C)$ (residuation, RE)

A11. $(A \to B) \to ((B \to C) \to (A \to C))$ (suffixing, SF)

A12. $(A \to B)_{\mathbf{t}} \vee (B \to A)_{\mathbf{t}}$ ($\mathbf{t}$-prelinearity, $PL_{\mathbf{t}}$)

A13. $\sim\sim A \to A$ (double negation elimination, DNE)

A14. $(A \& A) \leftrightarrow A$ (idempotence, ID)

A15. $\mathbf{t} \leftrightarrow \mathbf{f}$ (fixed-point, FP)

A16. $A \to (\sim A \to A)$ (RM3(1))

A17. $A \vee (A \to B)$ (RM3(2))

A18. $--A \to A$ (classical double negation, ClDN)

A19. $A \to (B \vee -B)$ (triviality, TRI)

A20. $(A \to B) \to (-B \to -A)$ (contraposition, CP^-)

A21. $(A \wedge -B) \to -(A \to B)$ (-1)

A22. $\sim A \to -A$ (-2)

A23. $-(A \& B) \to ((A \wedge B) \to (-A \wedge -B))$ (-3)

A24. $--(A \& B) \to (-A \to B)$ (-4)

A25. $((A \to B) \wedge -(A \to B)) \to (A \wedge -B)$ ($IUML^-3$)

$A \to B, A \vdash B$ (modus ponens, mp)

$A, B \vdash A \wedge B$ (adjunction, adj)

(ii) (Yang (2012a)) $\mathbf{G}^{wB}{}_3$ is A1 - A12, A14, A18, A19, (mp), (adj) plus

A26. $A \rightarrow (B \rightarrow A)$ (weakening, W)
A27. $-(A \wedge B) \leftrightarrow (-A \vee -B)$ ($DM1^{-}$)
A28. $-(A \vee B) \leftrightarrow (-A \wedge -B)$ ($DM2^{-}$)
A29. $((A \rightarrow -(C \vee -C)) \rightarrow B) \rightarrow (((B \rightarrow A) \rightarrow B) \rightarrow B)$ (G3)
A30. $((A \rightarrow B) \wedge -(A \rightarrow B)) \rightarrow (--A \wedge -B)$ ($G^{-}3(1)$)
A31. $(--A \wedge -B) \rightarrow -(A \rightarrow B)$ ($G^{-}3(2)$)

Remark 2.2 (1) The Involutive uninorm mingle logic **IUML**, i.e., $\mathbf{RM}_{\perp}$, is the $\mathbf{IUML}^{-}_{3}$ dropping A16 to A25; the system $\mathbf{IUML}^{-}$, i.e., the **IUML** with the negation -, is the **IUML** having A18 to A24; the system $\mathbf{G}^{wB}$ is the $\mathbf{G}^{wB}_{3}$ dropping A29 to A31.

(2) $\mathbf{G}^{wB}$ is equivalent to the system $\mathbf{G}^{\Delta}$ (the **G** with the delta Δ) and so $\mathbf{G}^{wB}_{3}$ to $\mathbf{G}^{\Delta}_{3}$ (see Yang (2012a)). Here we introduced $\mathbf{G}^{wB}_{3}$ instead of $\mathbf{G}^{\Delta}_{3}$, which is a three-valued extension of $\mathbf{G}^{\Delta}$. But since $\mathbf{G}^{wB}_{3}$ is equivalent to $\mathbf{G}^{\Delta}_{3}$, the former can be also regarded as a three-valued extension of $\mathbf{G}^{\Delta}$.

For easy reference, we let Ls_3 be the set of the three-valued systems introduced in Definition 2.1.

Definition 2.3 $Ls_3 = \{\mathbf{IUML}^{-}_{3}, \mathbf{G}^{wB}_{3}\}$.

A *theory* is a set of formulas closed under consequence relation. A *proof* in a theory Γ over L_3 ($\in Ls_3$) is a sequence *s* of formulas such that each element of s is either an axiom of L_3, a member of Γ, or is derivable from previous elements of s by means of a rule of L_3. $\Gamma \vdash A$, more exactly $\Gamma \vdash_{L_3} A$, means that A is *provable* in Γ with respect to (w.r.t.) L_3, i.e., there is an L_3-proof of A in Γ. A theory Γ is *trivial* if Γ ⊢ **F**; otherwise, it is *non-trivial*.

The deduction theorems for L_3 are as follows:

Proposition 2.4 Let Γ be a theory over L_3 and A, B be formulas.

(i) $\Gamma \cup \{A\} \vdash_{\mathbf{IUML}^-_3} B$ iff $\Gamma \vdash_{\mathbf{IUML}^-_3} A_t \to B$.

(ii) $\Gamma \cup \{A\} \vdash_{\mathbf{G}^{wB}_3} B$ iff $\Gamma \vdash_{\mathbf{G}^{wB}_3} A \to B$.

Proof: For (i) and (ii), see Dunn (1986) and Yang (2012b). □

The following formulas can be proved straightforwardly.

Proposition 2.5 (i) L_3 (∈ Ls_3) proves:

(1) (A & (B & C)) → ((A & B) & C) (associativity, AS)

(2) (A → B) ∨ (B → A) (prelinearity, PL)

(3) A ∨ -A (excluded middle, EM)

(ii) $\mathbf{IUML}^-_3$ proves:

(1) ~~ A ↔ A (double negation, DN)

(iii) $\mathbf{G}^{wB}_3$ proves (CP^-) and:

(1) **t** ↔ **T** (INT).

Suitable algebraic structures for L_3 (∈ Ls_3) are obtained as varieties of residuated lattices in the sense of Galatos et al. (2007).

Definition 2.6 (i) A *pointed bounded commutative residuated lattice* is a structure (A, ⊤, ⊥, t, f, ∧, ∨, *, →) such that:
(I) (A, ⊤, ⊥, ∧, ∨) is a bounded lattice with ⊤ element ⊤ and bottom element ⊥.
(II) (A, *, t) is a commutative monoid.
(III) y ≤ x → z iff x * y ≤ z, for all x, y, z in A (residuation).
(IV) f is an element of A.

(ii) (UL-algebra) Let $x_t := x \wedge t$. A *UL-algebra* is a pointed bounded commutative residuated lattice satisfying the condition: for all x, y ∈ A, (PL_t^A) $t \leq (x \to y)_t \vee (y \to x)_t$.
(iii) (MTL-algebra) An *MTL-algebra* is a UL-algebra satisfying the condition: (INT^A) t = ⊤.

A pointed commutative residuated lattice is said to be *linearly ordered* if the ordering of its algebra is linear, i.e., x ≤ y or y ≤ x (equivalently, x ∧ y = x or x ∧ y = y) for each pair x, y.
For convenience, '~', '-', '→', '∧', and '∨' are used ambiguously as propositional connectives and as algebraic operators, but context should make their meaning clear.

Definition 2.7 (i) (L-algebras) We call the following algebras *L-algebras*.
(a) An $IUML^-$-algebra is a UL-algebra satisfying the following conditions:
(DN^A) ~~x = x

(ID^{A}) $x * x = x$
(FP^{A}) $t = f$
$(ClDN^{A})$ $--x = x$
(TRI^{A}) $x \leq y \vee -y$
(CP^{-A}) $x \rightarrow y \leq -y \rightarrow -x$
(-1^{A}) $x \wedge -y \leq -(x \rightarrow y)$
(-2^{A}) $\sim x \leq -x$
(-3^{A}) $-(x * y) \leq (x \wedge y) \rightarrow (-x \wedge -y)$
(-4^{A}) $--(x * y) \leq -x \rightarrow y$.

(b) A G^{wB}-algebra is an MTL-algebra satisfying (ID^{A}), $(ClDN^{A})$, (TRI^{A}), and the following conditions:
$(DM1^{A})$ $-(x \wedge y) = -x \vee -y$
$(DM2^{A})$ $-(x \vee y) = -x \wedge -y$.
(ii) (L_3-algebras) We call the following algebras *L_3-algebras*.
(a) An $IUML^{-}_{3}$-algebra is an $IUML^{-}$-algebra satisfying the following conditions:
$(RM3(1)^{A})$ $x \leq \sim x \rightarrow x$
$(RM3(2)^{A})$ $t \leq x \vee (x \rightarrow y)$
$(IUML^{-}3^{A})$ $(x \rightarrow y) \wedge -(x \rightarrow y) \leq x \wedge -y$.
(b) A $G^{wB}{}_{3}$-algebra is a G^{wB}-algebra satisfying the following conditions:
$(G3^{A})$ $(x \rightarrow -(z \vee -z)) \rightarrow y \leq ((y \rightarrow x) \rightarrow y) \rightarrow y$
$(G^{-}3(1)^{A})$ $(x \rightarrow y) \wedge -(x \rightarrow y) \leq --x \wedge -y$
$(G^{-}3(2)^{A})$ $--x \wedge -y \leq -(x \rightarrow y)$.

Definition 2.8 (Evaluation) Let $\mathcal{A}$ be an L_3-algebra. An

$\mathcal{A}$-*evaluation* is a function v : FOR → $\mathcal{A}$ satisfying: v(A → B) = v(A) → v(B), v(A ∧ B) = v(A) ∧ v(B), v(A ∨ B) = v(A) ∨ v(B), v(A & B) = v(A) * v(B), v(**F**) = ⊥, v(**f**) = f, v(~A) = ~v(A), v(-A) = -v(A) (and hence v(**t**) = t and v(**T**) = ⊤).

Definition 2.9 (Cintula (2006)) Let $\mathcal{A}$ be L_3-algebra, T a theory, A a formula, and K a class of L_3-algebras.

(i) (Tautology) A is a *t-tautology* in $\mathcal{A}$, briefly an $\mathcal{A}$-*tautology* (or $\mathcal{A}$-*valid*), if v(A) ≥ t for each $\mathcal{A}$-evaluation v.

(ii) (Model) An $\mathcal{A}$-evaluation v is an $\mathcal{A}$-*model* of T if v(A) ≥ t for each A ∈ T. By *Mod(T, $\mathcal{A}$)*, we denote the class of $\mathcal{A}$-models of T.

(iii) (Semantic consequence) A is a *semantic consequence* of T w.r.t. K, denoting by $T \vDash_K A$, if Mod(T, $\mathcal{A}$) = Mod(T ∪ {A}, $\mathcal{A}$) for each $\mathcal{A}$ ∈ K.

Definition 2.10 (**L_3**-algebra, Cintula (2006)) Let $\mathcal{A}$, T, and A be as in Definition 2.9. $\mathcal{A}$ is an ***L_3-algebra*** iff whenever A is L_3-provable in T (i.e. $T \vdash_{L_3} A$), it is a semantic consequence of T w.r.t. the set {$\mathcal{A}$} (i.e. $T \vDash_{\{A\}} A$), $\mathcal{A}$ a L_3-algebra. By $MOD^{(l)}(\mathbf{L_3})$, we denote the class of (linearly ordered) **L_3**-algebras. Finally, we write $T \vDash^{(l)}_{L_3} A$ in place of $T \vDash_{MOD^{(l)}(L_3)} A$.

Note that since each condition for an L_3-algebra has the form of an equation or can be defined in an equation, it can be ensured that the classes of all L_3-algebras are varieties.

We first show that classes of provably equivalent formulas form an L_3-algebra. Let T be a fixed theory over L_3. For each formula A, let $[A]_T$ be the set of all formulas ψ such that T $\vdash_{L_3}$ A ↔ B (formulas T-provably equivalent to A). A_T is the set of all the classes $[A]_T$. We define that $[A]_T \rightarrow [B]_T = [A \rightarrow B]_T$, $[A]_T * [B]_T = [A \& B]_T$, $[A]_T \wedge [B]_T = [A \wedge B]_T$, $[A]_T \vee [B]_T = [A \vee B]_T$, $\bot = [\mathbf{F}]_T$, $f = [\mathbf{f}]_T$, $\sim[A]_T = [\sim A]_T$, $-[A]_T = [-A]_T$ (and so $t = [\mathbf{t}]_T$ and $\top = [\mathbf{T}]_T$). By $\boldsymbol{A}_T$, we denote this algebra.

Proposition 2.11 For T a theory over L_3, $\mathbf{A}_T$ is a $\mathbf{L}_3$-algebra.

Proof: Note that SI, ∧-E, ∧-I, ∨-I, ∨-E, EF, and VE ensure that ∧ and ∨ satisfy (I) in Definition 2.6; that &-C, PP, and AS ensure that (II) holds; that RE and $PL_{\mathbf{t}}$ ensure that (III) and ($PL_{\mathbf{t}}^{A}$) hold; that the constant **f** ensures (IV) holds. The additional axioms for L_3 ensure that the corresponding algebraic conditions hold. It is obvious that $[A]_T \leq [B]_T$ iff T $\vdash_{L_3}$ A ↔ (A ∧ B) iff T $\vdash_{L_3}$ A → B. Finally recall that $\mathbf{A}_T$ is an $\mathbf{L}_3$-algebra iff T $\vdash_{L_3}$ B implies T $\vDash_{L_3}$ B, and observe that for A in T, since T $\vdash_{L_3}$ **t** → A, it follows that $[\mathbf{t}]_T \leq [A]_T$. Thus it is an $\mathbf{L}_3$-algebra. □

Proposition 2.12 (Cf. Tsinakis & Blount (2003)) Each L_3-algebra is a subdirect product of linearly ordered L_3-algebras.

Theorem 2.13 (Strong completeness) Let T be a theory over L_3, and A a formula. T $\vdash_{L_3}$ A iff T $\vDash_{L_3}$ A iff T $\vDash^{1}_{L_3}$ A.

Proof: (i) T $\vdash_{L_3}$ A iff T $\vDash_{L_3}$ A. The left-to-right direction follows from the Definition 2.8 and Proposition 2.11. The right-to-left direction is as follows: from Proposition 2.8, we obtain $\mathbf{A}_T \in$ MOD($\mathbf{L}_3$), and for $\mathbf{A}_T$-evaluation v defined as v(B) = $[B]_T$, it holds that v $\in$ Mod(T, $\mathbf{A}_T$). Thus, since from T $\vDash_{L_3}$ ϕ we obtain that $[A]_T$ = v(A) $\geq$ t, T $\vdash_{L_3}$ **t** $\rightarrow$ A. Then, since T $\vdash_{L_3}$ **t**, by (mp) T $\vdash_{L_3}$ A, as required.

(ii) T $\vDash_{L_3}$ A iff T $\vDash^1_{L_3}$ A. It follows from Proposition 2.12. □

Remark 2.14 Let Ls = {$\mathbf{IUML}^-$, $\mathbf{G}^{wB}$}. The system L ($\in$ Ls) is obtained from L_3 by eliminating the corresponding three-valued axiom scheme(s). Then, analogously, we can define L-algebras and then establish algebraic completeness for L.

3. Algebraic Kripke-style semantics

3.1. Semantics

Here, we consider a particular kind of binary relational Kripke-style semantics, which we shall call *algebraic* Kripke-style semantics, for L_3.

Definition 3.1 (i) (Kripke frame) A *Kripke frame* is a structure **X** = (X, $\leq$) such that (X, $\leq$) is a partially ordered set. The elements of **X** are called *nodes*.
(ii) (Algebraic Kripke frame) An *algebraic Kripke frame* is a structure **X** = (X, $\top$, $\bot$, t, f, $\leq$, $*$) such that (X, $\top$, $\bot$, $\leq$) is a bounded linearly ordered set with top and bottom elements $\top$, $\bot$, and (X, t, f, $\leq$, $*$) is a linearly ordered pointed

commutative monoid satisfying that for all x, y in X, the set {z: z * x $\leq$ y} has a supremum, denoted by x $\rightarrow$ y. This monoid is called *residuated*.

(iii) (UL frame) A *UL frame* is an algebraic Kripke frame satisfying (PL_t^A).

(iv) (MTL frame) An *MTL frame* is a UL frame satisfying (INT^A).

(v) (L frame) An *IUML⁻ frame* is a UL frame satisfying the additional algebraic conditions for an IUML⁻-algebra. A *G^{wB} frame* is an MTL frame satisfying the additional algebraic conditions for a G^{wB}-algebra. By an *L* frame, we ambiguously denote any of these frames.

(vi) (L_3 frame) An *L_3 frame* is an L frame where X consists of three elements, i.e., X = {$\top$, x, $\bot$}. By *X_3*, we denote such X.

Remark 3.2 We point out that Kripke's semantics for modal logics were not defined on ordered frames with further operators. In the case of the modal system **S4**, it is the order relation itself from which the modal operator is defined.

An *evaluation* or *forcing* on an algebraic Kripke frame is a relation $\Vdash$ between nodes and propositional variables, and arbitrary formulas subject to the conditions below: for every propositional variable p,

(AHC) if x $\Vdash$ p and y $\leq$ x, then y $\Vdash$ p;

(min) $\bot$ $\Vdash$ p; and

for arbitrary formulas,

(t) x $\Vdash$ **t** iff x $\leq$ t;
(f) x $\Vdash$ **f** iff x $\leq$ f;
($\bot$) x $\Vdash$ **F** iff x = $\bot$;
($\wedge$) x $\Vdash$ A $\wedge$ B iff x $\Vdash$ A and x $\Vdash$ B;
($\vee$) x $\Vdash$ A $\vee$ B iff x $\Vdash$ A or x $\Vdash$ B;
(&) x $\Vdash$ A & B iff there are y, z in X such that y $\Vdash$ A, z $\Vdash$ B, and x $\leq$ y * z;
($\rightarrow$) x $\Vdash$ A $\rightarrow$ B iff for all y in X, if y $\Vdash$ A, then x * y $\Vdash$ B.

Definition 3.3 (i) (Algebraic Kripke model) An *algebraic Kripke model* is a pair (**X**, $\Vdash$), where **X** is an algebraic Kripke frame and $\Vdash$ is a forcing on **X**.
(ii) (L model) An *L model* is a pair (**X**, $\Vdash$), where **X** is an L frame and $\Vdash$ is a forcing on **X**.
(iii) (L_3 model) An L_3 model is a pair (**X**, $\Vdash$), where **X** is an L_3 frame and $\Vdash$ is an evaluation on **X**.

Definition 3.4 (Cf. Montagna & Sacchetti (2004)) Given an L_3 model (**X**, $\Vdash$), a node x of **X** and a formula A, we say that *x forces A* to express x $\Vdash$ A. We say that A is *true* in (**X**, $\Vdash$) if t $\Vdash$ A, and that ϕ is *valid* in the frame **X** (expressed by **X** models A) if A is true in (**X**, $\Vdash$) for every evaluation $\Vdash$ on **X**.

Definition 3.5 An L_3 frame **X** is an $\mathbf{L_3}$ frame iff all axioms of L_3 are valid in **X**. We say that an L_3 model (**X**, $\Vdash$) is an $\boldsymbol{L_3}$

model if **X** is an $\mathbf{L}_3$ frame.

3.2. Soundness and completeness for L_3

First, we introduce the following lemma.

Lemma 3.6 (i) (Hereditary Lemma, HL) Let **X** be an L_3 frame. For any sentence A and for all nodes x, y in **X**, if x $\Vdash$ A and y $\leq$ x, then y $\Vdash$ A.
(ii) Let $\Vdash$ be an evaluation on an L_3 frame and A a sentence. Then the set {x in X : x $\Vdash$ A} has a maximum.
(iii) $\top \Vdash A \rightarrow B$ iff for all x in **X**, if x $\Vdash$ A, then x $\Vdash$ B.

Proof: Easy. □

Proposition 3.7 (Soundness) If $\vdash_{L_3}$ A, then A is valid in every L_3 frame.

Proof: Since X_3 in **X** is {1, ½, 0} (up to isomorphism), We henceforth regard X_3 as the set {1, ½, 0}. We prove (RM3(2)) and ($G^{-}3(1)$) as examples:

(RM3) In checking for (t =) ½ $\Vdash A \vee (A \rightarrow B)$, it suffices to show that ½ $\Vdash$ A or ½ $\Vdash A \rightarrow B$. We instead assume ½ $\nVdash$ A and show ½ $\Vdash A \rightarrow B$. Let ½ $\nVdash$ A. Then 0 $\Vdash$ A and so 0 $\Vdash$ B. Hence, since ½ * 0 = 0, by ($\rightarrow$), we obtain that ½ $\Vdash A \rightarrow B$.

(G^{-}3(1)) It suffices to show that for all x in X_3 such that $x \Vdash (A \to B) \wedge -(A \to B)$, $x \Vdash --A \wedge -B$. If x = 1, it does not hold that $x \Vdash (A \to B) \wedge -(A \to B)$, and if x = 0, it holds that $x \Vdash --A \wedge -B$. Thus we consider the case that x = ½. We assume that $½ \Vdash (A \to B) \wedge -(A \to B)$, and show that $½ \Vdash --A \wedge -B$. By the supposition and ($\wedge$), $½ \Vdash A \to B$ and $½ \Vdash -(A \to B)$. Then $1 \nVdash --(A \to B)$ and so $1 \Vdash -(A \to B)$. Hence, $1 \nVdash A \to B$. Then there is y such that $y \Vdash A$ and $y \nVdash B$. Let y be 1. Then $1 \Vdash A$ and $1 \nVdash B$, and so $1 \Vdash --A$ and $1 \Vdash -B$. Hence $1 \Vdash --A \wedge -B$, and so $½ \Vdash --A \wedge -B$. y = ½ is not the case. (Otherwise, $½ \Vdash A$ and $½ \nVdash B$. But then $max\{x: x \Vdash B\} = 0$ and so $½ \nVdash A \to B$, a contradiction.) Obviously, it is not the case that y = 0. Therefore, $½ \Vdash --A \wedge -B$, as wished.

The proof for the other cases is left to the interested reader. □

This proposition ensures that L_3 frames are $\mathbf{L_3}$ frames. Moreover, the next proposition connects L_3 semantics and algebraic semantics (cf. see Montagna & Sacchetti (2004)).

Proposition 3.8 (i) The $\{\top, \bot, t, f, \leq, *, \to\}$ reduct of a linearly ordered L_3-algebra $\mathbf{A}$ is an L_3 frame.
(ii) Let $\mathbf{X} = (X, \top, \bot, t, f, \leq, *, \to)$ be an L_3 frame. Then the structure $\mathbf{A} = (X, \top, \bot, t, f, max, min, *, \to)$ is an L_3-algebra (where *max* and *min* are meant w.r.t. $\leq$).
(iii) Let $\mathbf{X}$ be the $\{\top, \bot, t, f, \leq, *, \to\}$ reduct of a linearly

ordered L_3-algebra $\mathbf{A}$, and let v be an evaluation in $\mathbf{A}$. Let for every atomic formula p and for every $x \in \mathbf{A}$, $x \Vdash p$ iff $x \leq v(p)$. Then $(\mathbf{X}, \Vdash)$ is an L_3 model, and for every formula A and for every $x \in \mathbf{A}$, we obtain that: $x \Vdash A$ iff $x \leq v(A)$.

(iv) Let $(\mathbf{X}, \Vdash)$ be an L_3 model, and let $\mathbf{A}$ be the L_3-algebra defined as in (ii). Define, for every atomic formula p, $v(p) = \max\{x \text{ in } X : x \Vdash p\}$. Then, for every formula A, $v(A) = \max\{x \text{ in } X : x \Vdash A\}$.

Proof: The proof for (i) and (ii) is easy. Since (iv) follows almost directly from (iii) and Lemma 3.6 (iii), we prove (iii). As regards to claim (iii), we consider the induction steps corresponding to the cases where $A = B \,\&\, C$ and $A = B \rightarrow C$. (The proof for the other cases are trivial.)

Suppose $A = B \,\&\, C$. By the condition (&), $x \Vdash B \,\&\, C$ iff there are $y, z \in X$ such that $y \Vdash B$, $z \Vdash C$, and $x \leq y * z$, hence by the induction hypothesis, $y \Vdash B$ and $z \Vdash C$ iff $y \leq v(B)$ and $z \leq v(C)$. Then, it holds true that $x \leq y * z \leq v(B) * v(C) = v(B \,\&\, C)$. Conversely, if $x \leq v(B) * v(C) = v(B \,\&\, C)$, then take $y = v(B)$ and $z = v(C)$. Then we have $x \leq y * z$, $y \Vdash B$, and $z \Vdash C$, therefore $x \Vdash B \,\&\, C$.

Suppose $A = B \rightarrow C$. By the condition ($\rightarrow$), $x \Vdash B \rightarrow C$ iff for all $y \in X$, if $y \Vdash B$, then $x * y \Vdash C$, hence by the induction hypothesis, $y \Vdash B$ only if $x * y \Vdash C$ iff $y \leq v(B)$ only if $x * y \leq v(C)$, therefore iff $x * v(B) \leq v(C)$, therefore by residuation, iff $x \leq v(B) \rightarrow v(C) = v(B \rightarrow C)$, as desired. □

Theorem 3.9 (Strong completeness) L_3 is strongly complete w.r.t. the class of all L_3 frames

Proof: It follows from Proposition 3.8 and Theorem 2.13. □

4. Concluding remark

As is known, Kripke-style semantics for many-valued predicate logics (as well as propositional logics) have been introduced (see Montagna & Ono (2002), Montagna & Sacchetti (2003; 2004)). A trivial generalization of Kripke-style semantics for such predicate logics in Montagna & Ono (2002), Montagna & Sacchetti (2003; 2004) gives us similar Kripke-style semantics for the first-order extensions of L_3. We leave this generalization to the interested reader.

We investigated algebraic Kripke-style semantics for three-valued paraconsistent systems. We proved soundness and completeness theorems. But we did not provide non-algebraic Kripke-style semantics for them. We will investigate it in a subsequent paper.

References

Cintula, P. (2006), "Weakly Implicative (Fuzzy) Logics I: Basic properties", *Archive for Mathematical Logic*, pp. 673-704.

Diaconescu, D., and Georgescu, G. (2007), "On the forcing semantics for monoidal t-norm based logic", *Journal of Universal Computer Science* 13, pp. 1550-1572.

Dunn, J. M.(1986), "Relevance logic and entailment", in *Handbook of Philosophical Logic*, vol III, D. Gabbay and F. Guenthner (eds.), Dordrecht, D. Reidel Publ. Co., pp. 117-224.

Dunn, J. M. (2000), "Partiality and its Dual", *Studia Logica* 66, pp. 5-40.

Galatos, N., Jipsen, P., Kowalski, T., and Ono, H. (2007), *Residuated lattices: an algebraic glimpse at substructural logic*s, Amsterdam, Elsevier.

Metcalfe, G., and Montagna, F. (2007), "Substructural Fuzzy Logics", *Journal of Symbolic Logic* 72, pp. 834-864.

Montagna, F. and Ono, H. (2002), "Kripke semantics, undecidability and standard completeness for Esteva and Godo's Logic MTL∀", *Studia Logica* 71, pp. 227-245.

Montagna, F. and Sacchetti, L. (2003), "Kripke-style semantics for many-valued logics", *Mathematical Logic Quaterly* 49, pp. 629-641.

Montagna, F. and Sacchetti, L. (2004), "Corrigendum to "Kripke-style semantics for many-valued logics", *Mathematical Logic Quaterly* 50, pp. 104-107.

Tsinakis, C., and Blount, K. (2003), "The structure of residuated lattices", *International Journal of Algebra and Computation*

13, pp. 437-461.

Yang, E. (2012a), "(Star-based) three-valued Kripke-style semantics for pseudo- and weak-Boolean logics", *Logic Journal of the IGPL* 20, pp. 187-206.

Yang, E. (2012b), "Kripke-style semantics for UL", *Korean Journal of Logic* 15(1), pp. 1-15.

Yang, E. (2012c), "**R**, fuzzy **R**, and algebraic Kripke-style semantics", *Korean Journal of Logic* 15(2), pp. 207-221.

Yang, E. (2013), "**R** and Relevance principle revisited", *Journal of Philosophical Logic* 42, pp. 767-782.

Yang, E. (2014), "Algebraic Kripke-style semantics for weakening-free fuzzy logics", *Korean Journal of Logic* 17(1), pp. 181-195.

Yang, E. (201+), "Two kinds of (binary) Kripke-style semantics for three-valued logic", *Logique et Analyse*, To appear.

전북대학교 철학과, 비판적사고와논술연구소

Department of Philosophy & Institute of Critical Thinking and Writing, Chonbuk National University

eunsyang@jbnu.ac.kr

3치 초일관 논리를 위한 대수적 크립키형 의미론

양 은 석

이 글에서 우리는 3치 초일관 논리를 위한 한 종류의 크립키형 의미론 즉 대수적 크립키형 의미론을 다룬다. 이를 위하여 먼저 두 3치 체계를 소개하고 그에 상응하는 대수를 정의한 후 이 두 체계가 대수적으로 완전하다는 것을 보인다. 다음으로 이 체계들을 위한 대수적 크립키형 의미론을 소개하고 이를 대수적 의미론과 연관짓는다.

주요어: (대수적) 크립키형 의미론, 대수적 의미론, 3치 논리, 초일관 논리