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SOME RESULTS ON CONDITIONALLY UNIFORMLY STRONG MIXING SEQUENCES OF RANDOM VARIABLES

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Abstract. From the ordinary notion of uniformly strong mixing for a sequence of random variables, a new concept called conditionally uniformly strong mixing is proposed and the relation between uniformly strong mixing and conditionally uniformly strong mixing is answered by examples, that is, uniformly strong mixing neither implies nor is implied by conditionally uniformly strong mixing. A couple of equivalent definitions and some of basic properties of conditionally uniformly strong mixing random variables are derived, and several conditional covariance inequalities are obtained. By means of these properties and conditional covariance inequalities, a conditional central limit theorem stated in terms of conditional characteristic functions is established, which is a conditional version of the earlier result under the non-conditional case.

1. Introduction and definition

We will be working on a fixed probability space $(\Omega,\ \mathcal{A},\ P)$. Consider a sequence $\{X_n, n \geq 1\}$ of random variables and let

$$\mathcal{A}_1^k = \sigma\left(X_1, \ldots, X_k\right), \quad \mathcal{A}_{k+n}^\infty = \sigma\left(X_{k+n}, X_{k+n+1}, \ldots\right)$$

be the σ-algebras induced by the respective random variables, where k and n are arbitrary positive integers. Then $\{X_n,\ n \geq 1\}$ is said to be uniformly strong mixing or φ-mixing if there exists a nonnegative sequence $\varphi\left(n\right)$ converging to zero as $n \to \infty$ such that

$$\left|P\left(B\left|\mathcal{A}_1^k\right.\right) - P\left(B\right)\right| \leq \varphi\left(n\right) \tag{1.1}$$

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for all $B \in \mathcal{A}_{k+n}^{\infty}$ whenever $k \geq 1$, $n \geq 1$. This concept was proposed by Ibragimov [7] and condition (1.1) is essentially the extension to arbitrary processes of the ergodicity coefficient introduced by Dobrushin [3, 4] for Markov processes. An equivalent way of writing (1.1), due to Ibragimov [8], is that

$$|P(AB) - P(A)P(B)| \leq \varphi(n)P(A) \tag{1.2}$$

for every pair of $A \in \mathcal{A}_1^k$ and $B \in \mathcal{A}_{k+n}^{\infty}$ whenever $k \geq 1$, $n \geq 1$.

For uniformly strong mixing random variables, many sharp and elegant results are available in literature, including Chen and Wang [1] for complete moment convergence, Utev [19] for a central limit theorem, Peligrad [13] for a weak invariance principle, Sen [16] for weak convergence of empirical processes, Shao [17] for an almost sure invariance principle, Hu and Wang [6] for a large deviation principle, Kuczmaszewska [10] for a strong law of large numbers, Szewczak [18] for a Marcinkiewicz law, and the like.

Motivated by Prakasa Rao [14] extending the notion of strong mixing to that of conditionally strong mixing, further work related to which can be found in Yuan and Lei [21], we will now consider a new kind of mixing called conditionally uniformly strong mixing, which is an extension to the above non-conditional case.

Definition 1.1. Let $\mathcal{F}$ be a sub-σ-algebra of $\mathcal{A}$. A sequence $\{X_n,\ n \geq 1\}$ of random variables is called conditionally uniformly strong mixing or conditionally φ-mixing given $\mathcal{F}$ ($\mathcal{F}$-uniformly strong mixing, in short) if there exists a nonnegative $\mathcal{F}$-measurable sequence of random variables $\varphi_{\mathcal{F}}(n)$ converging to zero almost surely as $n \to \infty$ such that

$$\left|P\left(B \left| \mathcal{A}_1^k \vee \mathcal{F}\right.\right) - P(B|\mathcal{F})\right| \leq \varphi_{\mathcal{F}}(n) \text{ a.s.} \tag{1.3}$$

for every $B \in \mathcal{A}_{k+n}^{\infty}$ whenever $k \geq 1$, $n \geq 1$, where $\mathcal{A}_1^k \vee \mathcal{F}$ denotes the σ-algebra generated by $\mathcal{A}_1^k \cup \mathcal{F}$.

The essence behind this definition is that these random variables involved tend to be asymptotically $\mathcal{F}$-independent as they get further and further apart. This kind of process is certainly worth studying not only for its probabilistic interest, but also because of the potential importance in statistical applications. The sequence $\{X_n\}$ is automatically $\mathcal{F}$-uniformly strong mixing with $\varphi_{\mathcal{F}}(n) \equiv 0$, of course, provided it is $\mathcal{F}$-independent.

As with (1.1) being equivalent to (1.2), an equivalent way (see Proposition 3.2 below) to express (1.3) is

$$|P(A \cap B|\mathcal{F}) - P(A|\mathcal{F})P(B|\mathcal{F})| \leq \varphi_{\mathcal{F}}(n)P(A|\mathcal{F}) \text{ a.s.} \tag{1.4}$$

for each choice of events $A \in \mathcal{A}_1^k$ and $B \in \mathcal{A}_{k+n}^{\infty}$ whenever $k \geq 1$, $n \geq 1$, which, will be used in Section 2, is the key to many of our constructions of $\mathcal{F}$-uniformly strong mixing sequences.

For all $k \geq 1$ and $n \geq 1$, it follows that $\mathcal{A}_{k+n}^{\infty} \supset \mathcal{A}_{k+n+1}^{\infty}$. In order to avoid being distracted from the main path, we assume from now on, without further explicit mentioning, that the sequence $\{\varphi_{\mathcal{F}}(n)\}$ of mixing coefficients is almost surely monotonically non-increasing.

Conditionally uniformly strong mixing reduces to the ordinary (unconditionally) uniformly strong mixing if the conditional σ-algebra is taken as $\{\emptyset, \Omega\}$. Conditionally uniformly strong mixing may appear, at a first glance, to be synchronized with uniformly strong mixing. However, this is not the case because Examples 2.1 and 2.2 below show that uniformly strong mixing neither implies nor is implied by conditionally uniformly strong mixing. Hence a natural question is what results on uniformly strong mixing carry over to conditionally uniformly strong mixing? Prakasa Rao [14] pointed out that one does have to derive results under conditioning if there is a need even though the results and proofs of such results may be analogous to those under the non-conditioning setup. This motivates our original interest in investigating conditionally uniformly strong mixing sequences.

In the past few years, much effort has been dedicated to the extension of dependent variables to the conditional case, and a large number of elegant results are available. For example, Ordóñez Cabrera et al. [12] extended negative quadrant dependence to conditionally negative quadrant dependence, Yuan et al. [20] extended negative association to conditional negative association, Yuan and Xie [23] extended linearly negative quadrant dependence to conditionally linearly negative quadrant dependence, Prakasa Rao [14], Yuan and Yang [24] and Yuan and Zheng [25] extended association to conditional association, and these could provide important clues to various derivations for the subject of this paper.

Several examples indicating uniformly strong mixing neither implies nor is implied by conditionally uniformly strong mixing are constructed in Section 2, three equivalent definitions and some of basic properties are derived in Section 3 and several conditional covariance inequalities are established in Section 4. From these properties and conditional covariance inequalities, a conditional central limit theorem stated in terms of conditional characteristic functions is developed in Section 5.

Following Prakasa Rao [14], for the sake of convenience, we will use the notation $P^{\mathcal{F}}(A)$ to denote $P(A\,|\mathcal{F})$ and $E^{\mathcal{F}}X$ to denote $E(X\,|\mathcal{F})$. In addition, $Cov^{\mathcal{F}}(X,Y)$ stands for the conditional covariance of X and Y given $\mathcal{F}$, i.e.

$$Cov^{\mathcal{F}}(X,\ Y) = E^{\mathcal{F}}(XY) - E^{\mathcal{F}}X \cdot E^{\mathcal{F}}Y.$$

2. Several examples

As previously mentioned, uniformly strong mixing does not imply conditionally uniformly strong mixing, and vice versa. We start by an example of uniformly strong mixing random variables that are not conditionally uniformly strong mixing.

Example 2.1. Let $\Omega = (0,\ 1)$, $\mathcal{A}$ be the σ-algebra of Borel sets on Ω and P be the Lebesgue measure on $\mathcal{A}$. Setting, for every $n \geq 1$,

$$A_n = \bigcup_{i=0}^{2^n-1} \left(\frac{2i}{2^{n+1}},\ \frac{2i+1}{2^{n+1}} \right),$$

it can be readily proved that $\{A_n,\ n \geq 1\}$ forms a sequence of mutually independent events and that $P(A_n) = 1/2$ for any $n \geq 1$. If the random variables X_n are defined by $X_n = I_{A_n}$ where I_A denotes the indicator function, then $\{X_n,\ n \geq 1\}$ is a uniformly strong mixing sequence with mixing coefficients $\varphi(n) \equiv 0$.

Let $D = (0,\ 1/4)$ and let $\mathcal{F} = \{\emptyset,\ D,\ D^c,\ \Omega\}$ be the sub-σ-algebra generated by D. Some elementary calculations show that

$$\begin{aligned} P^{\mathcal{F}}(A_1) &= \begin{cases} P(A_1 | D),\ \omega \in D, \\ P(A_1 | D^c),\ \omega \in D^c \end{cases} \\ &= \begin{cases} 4P\left\{\left(0, \frac{1}{4}\right)\right\},\ \omega \in D, \\ \frac{4}{3}P\left\{\left(\frac{2}{4}, \frac{3}{4}\right)\right\},\ \omega \in D^c \end{cases} \\ &= \begin{cases} 1,\ \omega \in D, \\ \frac{1}{3},\ \omega \in D^c \end{cases} \text{a.s.}, \end{aligned}$$

$$\begin{aligned} P^{\mathcal{F}}(A_{k+n}) &= \begin{cases} P\left\{ \bigcup_{i=0}^{2^{k+n}-1} \left(\frac{2i}{2^{k+n+1}}, \frac{2i+1}{2^{k+n+1}}\right) | D \right\},\ \omega \in D, \\ P\left\{ \bigcup_{i=0}^{2^{k+n}-1} \left(\frac{2i}{2^{k+n+1}}, \frac{2i+1}{2^{k+n+1}}\right) | D^c \right\},\ \omega \in D^c \end{cases} \\ &= \begin{cases} 4P\left\{ \bigcup_{i=0}^{2^{k+n-2}-1} \left(\frac{2i}{2^{k+n+1}}, \frac{2i+1}{2^{k+n+1}}\right) \right\},\ \omega \in D, \\ \frac{4}{3}P\left\{ \bigcup_{i=2^{k+n-2}}^{3\cdot 2^{k+n-2}-1} \left(\frac{2i}{2^{k+n+1}}, \frac{2i+1}{2^{k+n+1}}\right) \right\},\ \omega \in D^c \end{cases} \\ &= \begin{cases} \frac{1}{2},\ \omega \in D, \\ \frac{1}{3},\ \omega \in D^c \end{cases} \text{a.s.} \end{aligned}$$

and

$$\begin{aligned} & P^{\mathcal{F}}(A_1 \cap A_{k+n}) \\ &= \begin{cases} P\left\{ \left[\left(0, \frac{1}{4}\right) \cup \left(\frac{2}{4}, \frac{3}{4}\right)\right] \cap \left[\bigcup_{i=0}^{2^{k+n}-1} \left(\frac{2i}{2^{k+n+1}}, \frac{2i+1}{2^{k+n+1}}\right) \right] | D \right\},\ \omega \in D, \\ P\left\{ \left[\left(0, \frac{1}{4}\right) \cup \left(\frac{2}{4}, \frac{3}{4}\right)\right] \cap \left[\bigcup_{i=0}^{2^{k+n}-1} \left(\frac{2i}{2^{k+n+1}}, \frac{2i+1}{2^{k+n+1}}\right) \right] | D^c \right\},\ \omega \in D^c \end{cases} \\ &= \begin{cases} 4P\left\{ \left(0, \frac{1}{4}\right) \cap \left[\bigcup_{i=0}^{2^{k+n}-1} \left(\frac{2i}{2^{k+n+1}}, \frac{2i+1}{2^{k+n+1}}\right) \right] \right\},\ \omega \in D, \\ \frac{4}{3}P\left\{ \left(\frac{2}{4}, \frac{3}{4}\right) \cap \left[\bigcup_{i=0}^{2^{k+n}-1} \left(\frac{2i}{2^{k+n+1}}, \frac{2i+1}{2^{k+n+1}}\right) \right] \right\},\ \omega \in D^c \end{cases} \\ &= \begin{cases} 4P\left\{ \bigcup_{i=0}^{2^{k+n-2}-1} \left(\frac{2i}{2^{k+n+1}}, \frac{2i+1}{2^{k+n+1}}\right) \right\},\ \omega \in D, \\ \frac{4}{3}P\left\{ \bigcup_{i=2\cdot 2^{k+n-2}}^{3\cdot 2^{k+n-2}-1} \left(\frac{2i}{2^{k+n+1}}, \frac{2i+1}{2^{k+n+1}}\right) \right\},\ \omega \in D^c \end{cases} \end{aligned}$$

$$= \begin{cases} \frac{1}{2}, \ \omega \in D, \\ \frac{1}{6}, \ \omega \in D^c \end{cases} \text{ a.s.}$$

for any $n \geq 1$, so that $\left| P^{\mathcal{F}}(A_1 \cap A_{k+n}) - P^{\mathcal{F}}(A_1) P^{\mathcal{F}}(A_{k+n}) \right| = \frac{1}{6} P^{\mathcal{F}}(A_1)$ on D^c with $P(D^c) = \frac{3}{4}$. This indicates that $\{X_n, \ n \geq 1\}$ is not $\mathcal{F}$-uniformly strong mixing by means of (1.4).

Now we show an example of random variables that is conditionally uniformly strong mixing but not uniformly strong mixing.

Example 2.2. We continue to follow the example that appears in Yuan et al. [22]. Let $\{X_n, \ n \geq 1\}$ be the beta-Bernoulli process with parameters $a > 0$ and $b > 0$. In this case, $\{X_n\}$ is a sequence of conditionally independent indicator random variables given Θ with

$$P(X_n = 1 \,|\Theta = \theta) = \theta, \ 0 < \theta < 1$$

for each $n \geq 1$, where Θ is a beta random variable with left parameter a and right parameter b. Thus, Θ has probability density function $f_\Theta(\theta)$ given by

$$f_\Theta(\theta) = \frac{1}{B(a,b)} \theta^{a-1} (1-\theta)^{b-1}, \ 0 < \theta < 1.$$

Let $\mathcal{F} = \sigma(\Theta)$. Then the conditional independence of $\{X_n, \ n \geq 1\}$ with respect to $\mathcal{F}$ tells us that it is $\mathcal{F}$-uniformly strong mixing with mixing coefficients $\varphi_{\mathcal{F}}(n) \equiv 0$.

However, $\{X_n, \ n \geq 1\}$ does not possess the uniformly strong mixing property. This can be checked in the following manner. Let k be a positive integer satisfying $k \geq a + b$. For any $n \geq 1$, we find

$$\begin{aligned} P(X_{k+n} = 1) &= \int_0^1 P(X_{k+n} = 1 \,|\Theta = \theta) f_\Theta(\theta) d\theta \\ &= \frac{1}{B(a,b)} \int_0^1 \theta^a (1-\theta)^{b-1} d\theta \\ &= \frac{a}{a+b}, \\ P(X_1 = 1, X_2 = 1, \ldots, X_k = 1) &= \int_0^1 \prod_{i=1}^k P(X_i = 1 \,|\Theta = \theta) f_\Theta(\theta) d\theta \\ &= \frac{1}{B(a,b)} \int_0^1 \theta^{a+k-1} (1-\theta)^{b-1} d\theta \\ &= \frac{(a+k-1)(a+k-2)\cdots a}{(a+b+k-1)(a+b+k-2)\cdots(a+b)} \end{aligned}$$

and

$$\begin{aligned} &P(X_1 = 1, X_2 = 1, \ldots, X_k = 1, X_{k+n} = 1) \\ = &\frac{(a+k)(a+k-1)\cdots a}{(a+b+k)(a+b+k-1)\cdots(a+b)}. \end{aligned}$$

Consequently,

$$\begin{aligned} & P\left(X_{k+n}=1 \left| X_1=1, X_2=1, \ldots, X_k=1\right.\right) \\ = & \frac{P\left(X_1=1, X_2=1, \ldots, X_k=1, X_{k+n}=1\right)}{P\left(X_1=1, X_2=1, \ldots, X_k=1\right)} \\ = & \frac{a+k}{a+b+k}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} & \left|P\left(X_{k+n}=1 \left| X_1=1, X_2=1, \ldots, X_k=1\right.\right)-P\left(X_{k+n}=1\right)\right| \\ = & \frac{bk}{(a+b+k)(a+b)} \\ \geq & \frac{b}{2(a+b)}, \end{aligned}$$

so that

$$\sup_{k \geq 1} \sup_{B \in \mathcal{A}_{k+n}^{\infty}}\left|P^{\mathcal{A}_1^k}(B)-P(B)\right| \geq \frac{b}{2(a+b)} \nrightarrow \text{ as } n \rightarrow \infty$$

and the assertion is proved according to (1.1).

The preceding examples show that in general uniformly strong mixing neither implies nor is implied by conditionally uniformly strong mixing, while the forthcoming example indicates that both properties may hold.

Example 2.3. For every $1 \leq j < \infty$, let $\left\{Y_n^{(j)},\ n \geq 1\right\}$ be a stationary Markov chain satisfying condition (D_0) (see Doob [5], pages 221 and 192). Under (D_0), it is shown (see [5], page 217) that there exist $\gamma_j > 0$ and $0 < \rho_j < 1$ such that, for all $x \in \mathbb{R}$ and $B \in \mathcal{B}$ (the Borel σ-algebra in $\mathbb{R}$),

$$\left|P^{(n)}(x, B)-P(B)\right| \leq \gamma_j \rho_j^n \quad \text{for all sufficiently large } n,$$

where $P^{(n)}(x,\ B)$ is the n–step transition probability to B, given that the chain started at x and $P(\cdot)$ is the (unique stationary) initial distribution. As a result,

$$\left|P\left(B \left| Y_1^{(j)}, \ldots, Y_k^{(j)}\right.\right)-P(B)\right| \leq \gamma_j \rho_j^n \quad \text{for every } B \in \sigma\left(Y_{k+n}^{(j)}, Y_{k+n+1}^{(j)}, \ldots\right),$$

so that the chain is uniformly strong mixing with $\varphi_j(n) = \gamma_j \rho_j^n$ (for all sufficiently large n).

We assume that $\left\{X_n = Y_n^{(N)},\ n \geq 1\right\}$ is well-defined and N is a positive integer-valued random variable independent of $\left\{Y_n^{(j)}, n \geq 1,\ j \geq 1\right\}$. Taking $\mathcal{F} = \sigma(N)$, we have, for any $k \geq 1$ and $n \geq 1$,

$$\left[P\left(Y_{k+n}^{(N)} \leq y \left| \mathcal{A}_1^k \vee \mathcal{F}\right.\right)-P\left(Y_{k+n}^{(N)} \leq y \left| \mathcal{F}\right.\right)\right] I(N=j)$$

$$
\begin{aligned}
&= \left[P\left(Y_{k+n}^{(N)} \le y \,\middle|\, Y_1^{(N)},\ldots,Y_k^{(N)},N\right) - P\left(Y_{k+n}^{(N)} \le y \,|N\right)\right] I\left(N=j\right)\\
&= \left[P\left(Y_{k+n}^{(j)} \le y \,\middle|\, Y_1^{(j)},\ldots,Y_k^{(j)}\right) - P\left(Y_{k+n}^{(j)} \le y\right)\right] I\left(N=j\right)\\
&\quad \text{(by independence of } \left\{Y_n^{(j)}\right\} \textit{ and } N)\\
&= \left\{E\left[I\left(Y_{k+n}^{(j)} \le y\right) \,\middle|\, Y_k^{(j)}\right] - EI\left(Y_{k+n}^{(j)} \le y\right)\right\} I\left(N=j\right)\\
&\quad \text{(by the Markov property),}
\end{aligned}
$$

so that, on $(N=j)$,

$$
\begin{aligned}
&\left|P\left(Y_{k+n}^{(N)} \le y \,|\mathcal{A}_1^k \vee \mathcal{F}\right) - P\left(Y_{k+n}^{(N)} \le y \,|\mathcal{F}\right)\right|\\
&= \left|E\left[I\left(Y_{n+1}^{(j)} \le y\right) \,\middle|\, Y_1^{(j)}\right] - EI\left(Y_1^{(j)} \le y\right)\right|\\
&\quad \text{(by stationarity)}\\
&= \left|\int_R I_{(-\infty,y)}\left(z\right) P^{(n)}\left(Y_1^{(j)},dz\right) - \int_R I_{(-\infty,y)}\left(z\right) P\left(dz\right)\right|\\
&\le \int_R \left|P^{(n)}\left(Y_1^{(j)},dz\right) - P\left(dz\right)\right|\\
&\le \left\|P^{(n)}\left(Y_1^{(j)},\cdot\right) - P\left(\cdot\right)\right\|\\
&\le r_j \rho_j^n .
\end{aligned}
$$

It can now be shown that, for any $B \in \mathcal{A}_{k+n}^{\infty}$,

$$
\left|P\left(B\,|\mathcal{A}_1^k \vee \mathcal{F}\right) - P\left(B\,|\mathcal{F}\right)\right| \le \sum_{j=1}^{\infty} r_j \rho_j^n I\left(N=j\right),
$$

and therefore $\left\{X_n = Y_n^{(N)},\ n \ge 1\right\}$ is $\mathcal{F}$-uniformly strong mixing according to (1.3).

3. Equivalent definitions and basic properties

To facilitate immediate use of conditionally uniformly strong mixing, we first present its three equivalent definitions. These results are useful in various derivations and are interesting in their right, so we distinguish them as propositions.

Proposition 3.1. *A sequence $\{X_n,\ n \ge 1\}$ is $\mathcal{F}$-uniformly strong mixing with coefficient $\{\varphi_{\mathcal{F}}(n)\}$ if and only if*

$$
\left|P^{\mathcal{F}}\left(\tilde{A} \cap B\right) - P^{\mathcal{F}}\left(\tilde{A}\right) P^{\mathcal{F}}\left(B\right)\right| \le \varphi_{\mathcal{F}}\left(n\right) P^{\mathcal{F}}\left(\tilde{A}\right) \quad a.s. \tag{3.1}
$$

for each choice of events $\tilde{A} \in \mathcal{A}_1^k \vee \mathcal{F}$ and $B \in \mathcal{A}_{k+n}^{\infty}$ whenever $k \ge 1$, $n \ge 1$.

Proof. Assuming that (1.3) holds, so that

$$-\varphi_{\mathcal{F}}(n) \leq P^{\mathcal{A}_1^k \vee \mathcal{F}}(B) - P^{\mathcal{F}}(B) \leq \varphi_{\mathcal{F}}(n) \text{ a.s.}$$

Multiplying each side of this inequality by $I_{\tilde{A}}$ and then taking conditional expectation given $\mathcal{F}$, we get

$$-\varphi_{\mathcal{F}}(n) P^{\mathcal{F}}\left(\tilde{A}\right) \leq P^{\mathcal{F}}\left(\tilde{A} \cap B\right) - P^{\mathcal{F}}\left(\tilde{A}\right) P^{\mathcal{F}}(B) \leq \varphi_{\mathcal{F}}(n) P^{\mathcal{F}}\left(\tilde{A}\right) \text{ a.s.,}$$

which is just (3.1).

Conversely, suppose now that (3.1) holds and that (1.3) is violated. Then there must exist a set $B \in \mathcal{A}_{k+n}^{\infty}$ such that

$$\left|P^{\mathcal{A}_1^k \vee \mathcal{F}}(B) - P^{\mathcal{F}}(B)\right| > \varphi_{\mathcal{F}}(n)$$

on a set $\tilde{A}$ with $\tilde{A} \in \mathcal{A}_1^k \vee \mathcal{F}$ and $P\left(\tilde{A}\right) > 0$. Set

$$\tilde{A}^{+} = \tilde{A} \cap \left\{P^{\mathcal{A}_1^k \vee \mathcal{F}}(B) - P^{\mathcal{F}}(B) > \varphi_{\mathcal{F}}(n)\right\}$$

and

$$\tilde{A}^{-} = \tilde{A} \cap \left\{P^{\mathcal{A}_1^k \vee \mathcal{F}}(B) - P^{\mathcal{F}}(B) < -\varphi_{\mathcal{F}}(n)\right\}.$$

Then at least one of $P\left(\tilde{A}^{+}\right)$ and $P\left(\tilde{A}^{-}\right)$ must be positive. Let $P\left(\tilde{A}^{+}\right) > 0$, then

$$E\left\{\left[P^{\mathcal{A}_1^k \vee \mathcal{F}}(B) - P^{\mathcal{F}}(B)\right] I_{\tilde{A}^{+}}\right\} > E\left\{\varphi_{\mathcal{F}}(n) I_{\tilde{A}^{+}}\right\},$$

which implies

$$E^{\mathcal{F}}\left\{\left[P^{\mathcal{A}_1^k \vee \mathcal{F}}(B) - P^{\mathcal{F}}(B)\right] I_{\tilde{A}^{+}}\right\} > E^{\mathcal{F}}\left\{\varphi_{\mathcal{F}}(n) P^{\mathcal{F}}(B) I_{\tilde{A}^{+}}\right\} \text{ a.s.}$$

on some $A \in \mathcal{A}$ with $P(A) > 0$, or equivalently,

$$P^{\mathcal{F}}\left(\tilde{A}^{+} \cap B\right) - P^{\mathcal{F}}\left(\tilde{A}^{+}\right) P^{\mathcal{F}}(B) > \varphi_{\mathcal{F}}(n) P^{\mathcal{F}}\left(\tilde{A}^{+}\right) P^{\mathcal{F}}(B) \text{ a.s.}$$

on such a set A, so that (3.1) is violated for the set $\tilde{A}^{+}$ and that is a contradiction. Finally, (1.3) is verified in the same way for $P\left(\tilde{A}^{-}\right) > 0$. □

With the help of the previous proposition, we next fulfill our promise made in Section 1 and prove the equivalence between (1.3) and (1.4), which has been used in Example 2.1.

Proposition 3.2. *A sequence* $\{X_n,\ n \geq 1\}$ *is* $\mathcal{F}$*-uniformly strong mixing if and only if* (1.4) *holds.*

Proof. We only need to show that (1.4) implies (3.1). For fixed $k \geq 1$ and $n \geq 1$, let

$$\mathcal{E} = \left\{\bigcup_{i=1}^{m}\left(A_i^{(1)} \cap A_i^{(2)}\right) :\ A_i^{(1)} \in \mathcal{A}_1^k,\ A_i^{(2)} \in \mathcal{F},\ i = 1, \ldots, m,\ m \geq 1\right\},$$

then $\mathcal{E}$ is an algebra. Let $B \in \mathcal{A}_{k+n}^{\infty}$ and let

$$\mathcal{A}' = \left\{ \tilde{A} \in \mathcal{A} : \left| P^{\mathcal{F}}\left(\tilde{A} \cap B\right) - P^{\mathcal{F}}\left(\tilde{A}\right) P^{\mathcal{F}}(B) \right| \le \varphi_{\mathcal{F}}(n) P^{\mathcal{F}}\left(\tilde{A}\right) \text{ a.s.} \right\},$$

then $\mathcal{A}'$ is a monotone class. If we can prove that $\mathcal{A}' \supset \mathcal{E}$, then from Theorem 1.3.1 in [2] it follows that

$$\mathcal{A}' \supset m(\mathcal{E}) = \sigma(\mathcal{E}) = \mathcal{A}_1^k \vee \mathcal{F},$$

where $m(\mathcal{E})$ denotes the monotone class generated by $\mathcal{E}$, and therefore (3.1) holds.

We now focus our attention on the proof of $\mathcal{A}' \supset \mathcal{E}$. For this purpose we use the conditional version of the inclusion-exclusion formula (c.f. Lin and Bai [11], Chapter 1.1) to conclude that

$$\begin{aligned}
& P^{\mathcal{F}}\left\{\left[\bigcup_{i=1}^{m}\left(A_i^{(1)} \cap A_i^{(2)}\right)\right] \cap B\right\} - P^{\mathcal{F}}\left\{\bigcup_{i=1}^{m}\left(A_i^{(1)} \cap A_i^{(2)}\right)\right\} P^{\mathcal{F}}(B) \\
= {} & P^{\mathcal{F}}\left\{\bigcup_{i=1}^{m}\left(A_i^{(1)} \cap A_i^{(2)} \cap B\right)\right\} - P^{\mathcal{F}}\left\{\bigcup_{i=1}^{m}\left(A_i^{(1)} \cap A_i^{(2)}\right)\right\} P^{\mathcal{F}}(B) \\
= {} & \sum_{i=1}^{m}\left[P^{\mathcal{F}}\left(A_i^{(1)} \cap B\right) - P^{\mathcal{F}}\left(A_i^{(1)}\right) P^{\mathcal{F}}(B)\right] I_{A_i^{(2)}} \\
& - \sum_{1 \le i < j \le m}\left[P^{\mathcal{F}}\left(A_i^{(1)} \cap A_j^{(1)} \cap B\right) - P^{\mathcal{F}}\left(A_i^{(1)} \cap A_j^{(1)}\right) P^{\mathcal{F}}(B)\right] I_{A_i^{(2)} \cap A_j^{(2)}} \\
& + \cdots \\
& + (-1)^{m-1}\left[P^{\mathcal{F}}\left(A_1^{(1)} \cap \cdots \cap A_m^{(1)} \cap B\right) - P^{\mathcal{F}}\left(A_1^{(1)} \cap \cdots \cap A_m^{(1)}\right) P^{\mathcal{F}}(B)\right] \\
& \times I_{A_1^{(2)} \cap \cdots \cap A_m^{(2)}} \text{ a.s.}
\end{aligned}$$

For any $\omega \in \cup_{i=1}^{m} A_i^{(2)}$, we may (by relabeling indices if necessary) assume $\omega \in \cap_{i=1}^{r} A_i^{(2)}$ but $\omega \notin \cup_{i=r+1}^{m} A_i^{(2)}$, $1 \le r \le m$, and then obtain from the above inequality and (1.4),

$$\begin{aligned}
& P^{\mathcal{F}}\left\{\left[\bigcup_{i=1}^{m}\left(A_i^{(1)} \cap A_i^{(2)}\right)\right] \cap B\right\}(\omega) - P^{\mathcal{F}}\left\{\bigcup_{i=1}^{m}\left(A_i^{(1)} \cap A_i^{(2)}\right)\right\} P^{\mathcal{F}}(B)(\omega) \\
= {} & \sum_{i=1}^{r}\left[P^{\mathcal{F}}\left(A_i^{(1)} \cap B\right) - P^{\mathcal{F}}\left(A_i^{(1)}\right) P^{\mathcal{F}}(B)\right](\omega) \\
& - \sum_{1 \le i < j \le r}\left[P^{\mathcal{F}}\left(A_i^{(1)} \cap A_j^{(1)} \cap B\right) - P^{\mathcal{F}}\left(A_i^{(1)} \cap A_j^{(1)}\right) P^{\mathcal{F}}(B)\right](\omega) \\
& + \cdots \\
& + (-1)^{r-1}\left[P^{\mathcal{F}}\left(A_1^{(1)} \cap A_2^{(1)} \cap \cdots \cap A_r^{(1)} \cap B\right)\right. \\
& \left. - P^{\mathcal{F}}\left(A_1^{(1)} \cap A_2^{(1)} \cap \cdots \cap A_r^{(1)}\right) P^{\mathcal{F}}(B)\right](\omega)
\end{aligned}$$

$$= \left\{P^{\mathcal{F}}\left[\left(\bigcup_{i=1}^{r} A_i^{(1)}\right)\cap B\right] - P^{\mathcal{F}}\left(\bigcup_{i=1}^{r} A_i^{(1)}\right) P^{\mathcal{F}}(B)\right\}(\omega)$$
$$\leq \left[\varphi_{\mathcal{F}}(n)\, P^{\mathcal{F}}\left(\bigcup_{i=1}^{r} A_i^{(1)}\right)\right](\omega)$$
$$= \left[\varphi_{\mathcal{F}}(n)\, P^{\mathcal{F}}\left\{\bigcup_{i=1}^{r}\left(A_i^{(1)}\cap A_i^{(2)}\right)\right\}\right](\omega)$$
$$\leq \left[\varphi_{\mathcal{F}}(n)\, P^{\mathcal{F}}\left\{\bigcup_{i=1}^{m}\left(A_i^{(1)}\cap A_i^{(2)}\right)\right\}\right](\omega),$$

that is,

$$P^{\mathcal{F}}\left\{\left[\bigcup_{i=1}^{m}\left(A_i^{(1)}\cap A_i^{(2)}\right)\right]\cap B\right\} - P^{\mathcal{F}}\left\{\bigcup_{i=1}^{m}\left(A_i^{(1)}\cap A_i^{(2)}\right)\right\} P^{\mathcal{F}}(B)$$
$$\leq \varphi_{\mathcal{F}}(n)\, P^{\mathcal{F}}\left\{\bigcup_{i=1}^{m}\left(A_i^{(1)}\cap A_i^{(2)}\right)\right\}$$

on $\cup_{i=1}^{m} A_i^{(2)}$. But the left-hand side of the inequality above equals to 0 outside $\cup_{i=1}^{m} A_i^{(2)}$, so we have

$$P^{\mathcal{F}}\left\{\left[\bigcup_{i=1}^{m}\left(A_i^{(1)}\cap A_i^{(2)}\right)\right]\cap B\right\} - P^{\mathcal{F}}\left\{\bigcup_{i=1}^{m}\left(A_i^{(1)}\cap A_i^{(2)}\right)\right\} P^{\mathcal{F}}(B)$$
$$\leq \varphi_{\mathcal{F}}(n)\, P^{\mathcal{F}}\left\{\bigcup_{i=1}^{m}\left(A_i^{(1)}\cap A_i^{(2)}\right)\right\} \text{ a.s.,}$$

and similarly

$$P^{\mathcal{F}}\left\{\left[\bigcup_{i=1}^{m}\left(A_i^{(1)}\cap A_i^{(2)}\right)\right]\cap B\right\} - P^{\mathcal{F}}\left\{\bigcup_{i=1}^{m}\left(A_i^{(1)}\cap A_i^{(2)}\right)\right\} P^{\mathcal{F}}(B)$$
$$\geq -\varphi_{\mathcal{F}}(n)\, P^{\mathcal{F}}\left\{\bigcup_{i=1}^{m}\left(A_i^{(1)}\cap A_i^{(2)}\right)\right\} \text{ a.s.}$$

The last two inequalities yield

$$\left|P^{\mathcal{F}}\left\{\left[\bigcup_{i=1}^{m}\left(A_i^{(1)}\cap A_i^{(2)}\right)\right]\cap B\right\} - P^{\mathcal{F}}\left\{\bigcup_{i=1}^{m}\left(A_i^{(1)}\cap A_i^{(2)}\right)\right\} P^{\mathcal{F}}(B)\right|$$
$$\leq \varphi_{\mathcal{F}}(n)\, P^{\mathcal{F}}\left\{\bigcup_{i=1}^{m}\left(A_i^{(1)}\cap A_i^{(2)}\right)\right\} \text{ a.s.,}$$

this concludes the proof of the fact that $\mathcal{A}' \supset \mathcal{E}$. □

Following this direction we can develop the third characterization of a conditionally uniformly strong mixing sequence, which will be used in Section 4.

Proposition 3.3. *A sequence $\{X_n, n \geq 1\}$ is $\mathcal{F}$-uniformly strong mixing with coefficient $\{\varphi_{\mathcal{F}}(n)\}$ if and only if*

$$\left|P^{\mathcal{F}}\left(\tilde{A}\cap\tilde{B}\right) - P^{\mathcal{F}}\left(\tilde{A}\right)P^{\mathcal{F}}\left(\tilde{B}\right)\right| \leq \varphi_{\mathcal{F}}(n)\, P^{\mathcal{F}}\left(\tilde{A}\right) \quad \text{a.s.} \tag{3.2}$$

for each choice of events $\tilde{A} \in \mathcal{A}_1^k \vee \mathcal{F}$ *and* $\tilde{B} \in \mathcal{A}_{k+n}^\infty \vee \mathcal{F}$ *whenever* $k \geq 1$, $n \geq 1$.

Proof. We only need to show that (3.1) implies (3.2). For fixed $k \geq 1$ and $n \geq 1$, let

$$\mathcal{D} = \left\{ \bigcup_{i=1}^{m} \left(B_i^{(1)} \cap B_i^{(2)} \right) : \; B_i^{(1)} \in \mathcal{A}_{k+n}^\infty, \; B_i^{(2)} \in \mathcal{F}, \; i = 1, \ldots, m, \; m \geq 1 \right\},$$

then $\mathcal{D}$ is an algebra. Let $\tilde{A} \in \mathcal{A}_1^k \vee \mathcal{F}$ and let

$$\mathcal{A}'' = \left\{ \tilde{B} \in \mathcal{A} : \left| P^{\mathcal{F}}\left(\tilde{A} \cap \tilde{B}\right) - P^{\mathcal{F}}\left(\tilde{A}\right) P^{\mathcal{F}}\left(\tilde{B}\right) \right| \leq \varphi_{\mathcal{F}}(n) P^{\mathcal{F}}\left(\tilde{A}\right) \; a.s. \right\},$$

then $\mathcal{A}''$ is a monotone class. Analogously to the proof of Proposition 3.2, one can show that $\mathcal{A}'' \supset \mathcal{D}$, implying that

$$\mathcal{A}'' \supset m(\mathcal{D}) = \sigma(\mathcal{D}) = \mathcal{A}_{k+n}^\infty \vee \mathcal{F},$$

which yields (3.2). □

We now develop several basic properties of $\mathcal{F}$-uniformly strong mixing random variables. The following property is a direct consequence of Proposition 3.2.

Proposition 3.4. *If* $\{X_n, n \geq 1\}$ *is* $\mathcal{F}$*-uniformly strong mixing and* $f(\cdot)$ *is an arbitrary real function, then* $\{f(X_n), \; n \geq 1\}$ *is also* $\mathcal{F}$*-uniformly strong mixing with the same coefficient* $\{\varphi_{\mathcal{F}}(n)\}$.

The next property is immediate from the definition of conditionally uniformly strong mixing by noting that the mixing coefficients are assumed to be non-increasing.

Proposition 3.5. *If* $\{X_n, \; n \geq 1\}$ *is an* $\mathcal{F}$*-uniformly strong mixing sequence with coefficient* $\{\varphi_{\mathcal{F}}(n)\}$, *then so is* $\left\{\hat{X}_n = \sum_{k=i_{n-1}+1}^{i_n} X_k, \; n \geq 1\right\}$, *where* $0 = i_0 < i_1 < i_2 < \cdots$.

The following property will be used frequently in subsequent sections.

Proposition 3.6. *If* $\{X_n, \; n \geq 1\}$ *is an* $\mathcal{F}$*-uniformly strong mixing sequence with coefficient* $\{\varphi_{\mathcal{F}}(n)\}$, *and* $\{Y_n, \; n \geq 1\}$ *is an* $\mathcal{F}$*-measurable sequence, then*

(i) *Each of the relations* (1.3) *and* (1.4) *holds for all* $A \in \mathcal{A}_1^k \vee \mathcal{G}_1^k$, $B \in \mathcal{A}_{n+k}^\infty \vee \mathcal{G}_{n+k}^\infty$ *and* $k \geq 1$, $n \geq 1$, *where* $\mathcal{G}_j^l = \sigma(Y_i, \; j \leq i \leq l)$, $1 \leq j \leq l \leq \infty$;

(ii) *Each of* $\{X_n Y_n, \; n \geq 1\}$ *and* $\{X_n \pm Y_n, \; n \geq 1\}$ *is* $\mathcal{F}$*-uniformly strong mixing*;

(iii) *The sequence* $\left\{X_n - E^{\mathcal{F}} X_n, \; n \geq 1\right\}$ *is* $\mathcal{F}$*-centered* (*i.e. conditional expectation with respect to* $\mathcal{F}$ *equals zero*) *and* $\mathcal{F}$*-uniformly strong mixing.*

Proof. Part (iii) follows immediately from part (ii). Part (ii) is a direct consequence of (i) from the observations that

$$\sigma(X_i Y_i, \; j \leq i \leq l) \subset \mathcal{A}_j^l \vee \mathcal{G}_j^l, \; 1 \leq j \leq l \leq \infty$$

and

$$\sigma\left(X_i \pm Y_i,\ j \le i \le l\right) \subset \mathcal{A}_j^l \vee \mathcal{G}_j^l,\ 1 \le j \le l \le \infty,$$

whereas part (i) follows from Proposition 3.2 since $\mathcal{G}_1^k \subset \mathcal{F}$ and $\mathcal{G}_{n+k}^{\infty} \subset \mathcal{F}$ for any $k \ge 1$ and $n \ge 1$. □

4. Conditional covariance inequalities

As regards to covariance inequalities for uniformly strong mixing random variables, there are plenty of results in literature such as Lin and Bai [11], Roussas and [15], etc. Inequalities of this kind are potentially useful for Ioannides obtaining limit theorems, especially strong laws of large numbers and central limit theorems.

In this section, we derive some covariance inequalities for conditionally uniformly strong mixing random variables, which are conditional versions of the corresponding ones in the non-conditional case.

Theorem 4.1. *Assume that $\{X_n,\ n \ge 1\}$ is an $\mathcal{F}$-uniformly strong mixing sequence with coefficient $\{\varphi_{\mathcal{F}}(n)\}$ and Y and Z are $\mathcal{A}_1^k$ and $\mathcal{A}_{k+n}^{\infty}$-measurable random variables, respectively. If*

$$E^{\mathcal{F}}|Y|^p < \infty \text{ a.s. and } E^{\mathcal{F}}|Z|^q < \infty \text{ a.s. for } p,q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1,$$

then

$$\left|E^{\mathcal{F}}YZ - E^{\mathcal{F}}Y \cdot E^{\mathcal{F}}Z\right| \le 2\varphi_{\mathcal{F}}^{1/p}(n)\left(E^{\mathcal{F}}|Y|^p\right)^{1/p}\left(E^{\mathcal{F}}|Z|^q\right)^{1/q} \text{ a.s.} \tag{4.1}$$

Remark 4.2. If Y and Z are complex-valued random variables, then inequality (4.1) becomes

$$\left|E^{\mathcal{F}}YZ - E^{\mathcal{F}}Y \cdot E^{\mathcal{F}}Z\right| \le 8\varphi_{\mathcal{F}}^{1/p}(n)\left(E^{\mathcal{F}}|Y|^p\right)^{1/p}\left(E^{\mathcal{F}}|Z|^q\right)^{1/q} \text{ a.s.}$$

Proof of Theorem 4.1. The basic approach of this proof is based on 10.1.d in [11] and Theorem 5.1 in [15] but the details are quite different. Suppose first that Y and Z are simple random variables. Specifically, let

$$Y = \sum_i a_i I_{A_i}, \quad Z = \sum_j b_j I_{B_j},$$

where both $\sum_i$ and $\sum_j$ are finite sums and $A_i \cap A_r = \emptyset\ (i \ne r)$, $B_j \cap B_l = \emptyset\ (j \ne l)$, $A_i, A_r \in \mathcal{A}_1^k$, $B_j, B_l \in \mathcal{A}_{k+n}^{\infty}$. Then, by the telescopic property of conditional expectation,

$$\begin{aligned} E^{\mathcal{F}}YZ - E^{\mathcal{F}}Y \cdot E^{\mathcal{F}}Z &= \sum_{i,j} a_i b_j \left[E^{\mathcal{F}} I_{A_i} I_{B_j} - E^{\mathcal{F}} I_{A_i} E^{\mathcal{F}} I_{B_j}\right] \\ &= \sum_{i,j} a_i b_j E^{\mathcal{F}}\left[I_{A_i}\left(E^{A_i \vee \mathcal{F}} I_{B_j} - E^{\mathcal{F}} I_{B_j}\right)\right] \\ &= E^{\mathcal{F}}\left[\sum_i a_i I_{A_i} \sum_j b_j \left(E^{A_i \vee \mathcal{F}} I_{B_j} - E^{\mathcal{F}} I_{B_j}\right)\right] \end{aligned}$$

$$= E^{\mathcal{F}}\left(\sum_i a_i \xi_i I_{A_i}\right),$$

where

$$\xi_i = \sum_j b_j \left(E^{A_i \vee \mathcal{F}} I_{B_j} - E^{\mathcal{F}} I_{B_j}\right). \tag{4.2}$$

Set

$$\xi = \sum_i \xi_i I_{A_i}, \tag{4.3}$$

then

$$\left|E^{\mathcal{F}}\left(\sum_i a_i \xi_i I_{A_i}\right)\right| = \left|E^{\mathcal{F}}(Y\xi)\right| \le E^{\mathcal{F}}|Y\xi| \le \left(E^{\mathcal{F}}|Y|^p\right)^{1/p}\left(E^{\mathcal{F}}|\xi|^q\right)^{1/q},$$

so that

$$\left|E^{\mathcal{F}} YZ - E^{\mathcal{F}} Y \cdot E^{\mathcal{F}} Z\right| \le \left(E^{\mathcal{F}}|Y|^p\right)^{1/p}\left(E^{\mathcal{F}}|\xi|^q\right)^{1/q}. \tag{4.4}$$

By (4.2) and (4.3),

$$\begin{aligned}
E^{\mathcal{F}}|\xi|^q &= E^{\mathcal{F}}\left(\sum_i I_{A_i}|\xi_i|^q\right) \\
&\le E^{\mathcal{F}}\left[\sum_i I_{A_i}\left(\sum_j |b_j|\left|E^{A_i\vee\mathcal{F}} I_{B_j} - E^{\mathcal{F}} I_{B_j}\right|\right)^q\right] \\
&\le E^{\mathcal{F}}\left\{\sum_i I_{A_i}\left[\sum_j |b_j|\left(E^{A_i\vee\mathcal{F}} I_{B_j} + E^{\mathcal{F}} I_{B_j}\right)^{1/q}\right.\right. \\
&\qquad \left.\left. \cdot\left|E^{A_i\vee\mathcal{F}} I_{B_j} - E^{\mathcal{F}} I_{B_j}\right|^{1/p}\right]^q\right\} \\
&\le E^{\mathcal{F}}\left\{\sum_i I_{A_i}\left[\sum_j |b_j|^q\left(E^{A_i\vee\mathcal{F}} I_{B_j} + E^{\mathcal{F}} I_{B_j}\right)\right]\right. \\
&\qquad \left. \cdot\left[\sum_j \left|E^{A_i\vee\mathcal{F}} I_{B_j} - E^{\mathcal{F}} I_{B_j}\right|\right]^{q/p}\right\}.
\end{aligned} \tag{4.5}$$

Here, we have used an elementary inequality

$$\sum_j |\alpha_j\beta_j| \le \left(\sum_j |\alpha_j|^p\right)^{1/p}\left(\sum_j |\beta_j|^q\right)^{1/q}$$

for nonnegative numbers p, q with $1/p+1/q=1$. Next, define the sets J^+ and J^- by

$$J^+ = \left\{j : E^{A_i\vee\mathcal{F}} I_{B_j} - E^{\mathcal{F}} I_{B_j} > 0 \text{ a.s.}\right\},$$

$$J^{-}=\left\{j : E^{A_i\vee\mathcal{F}}I_{B_j}-E^{\mathcal{F}}I_{B_j}<0\text{ a.s.}\right\},$$

then

$$\begin{aligned}(4.6)\qquad &\sum_j\left|E^{A_i\vee\mathcal{F}}I_{B_j}-E^{\mathcal{F}}I_{B_j}\right|\\ &=\sum_{j\in J^+}\left(E^{A_i\vee\mathcal{F}}I_{B_j}-E^{\mathcal{F}}I_{B_j}\right)+\sum_{j\in J^-}\left(E^{\mathcal{F}}I_{B_j}-E^{A_i\vee\mathcal{F}}I_{B_j}\right)\\ &=\left|E^{A_i\vee\mathcal{F}}\left(\sum_{j\in J^+}I_{B_j}\right)-E^{\mathcal{F}}\left(\sum_{j\in J^+}I_{B_j}\right)\right|\\ &\quad+\left|E^{\mathcal{F}_i}\left(\sum_{j\in J^-}I_{B_j}\right)-E^{A_i\vee\mathcal{F}}\left(\sum_{j\in J^-}I_{B_j}\right)\right|.\end{aligned}$$

But by the conditionally uniformly strong mixing property,

$$\begin{aligned}(4.7)\qquad &\left|E^{A_i\vee\mathcal{F}}\left(\sum_{j\in J^+}I_{B_j}\right)-E^{\mathcal{F}}\left(\sum_{j\in J^+}I_{B_j}\right)\right|\\ &=\left|E^{A_i\vee\mathcal{F}}\left[E^{\mathcal{A}_1^k\vee\mathcal{F}}\left(\sum_{j\in J^+}I_{B_j}\right)\right]-E^{A_i\vee\mathcal{F}}\left[E^{\mathcal{F}}\left(\sum_{j\in J^+}I_{B_j}\right)\right]\right|\\ &\le E^{A_i\vee\mathcal{F}}\left|E^{\mathcal{A}_1^k\vee\mathcal{F}}\left(\sum_{j\in J^+}I_{B_j}\right)-E^{\mathcal{F}}\left(\sum_{j\in J^+}I_{B_j}\right)\right|\\ &\le\varphi_{\mathcal{F}}(n),\end{aligned}$$

and similarly,

$$(4.8)\qquad \left|E^{\mathcal{F}}\left(\sum_{j\in J^-}I_{B_j}\right)-E^{A_i\vee\mathcal{F}}\left(\sum_{j\in J^-}I_{B_j}\right)\right|\le\varphi_{\mathcal{F}}(n).$$

Putting (4.7) and (4.8) into (4.6), and then putting (4.6) into (4.5), we have

$$\begin{aligned}E^{\mathcal{F}}|\xi|^q&\le 2^{q/p}\varphi_{\mathcal{F}}^{q/p}(n)\,E^{\mathcal{F}}\left\{\sum_i I_{A_i}\left[\sum_j|b_j|^q\left(E^{A_i\vee\mathcal{F}}I_{B_j}+E^{\mathcal{F}}I_{B_j}\right)\right]\right\}\\ &=2^{q/p}\varphi_{\mathcal{F}}^{q/p}(n)\sum_i\sum_j|b_j|^q\left(E^{\mathcal{F}}I_{A_i\cap B_j}+E^{\mathcal{F}}I_{A_i}\cdot E^{\mathcal{F}}I_{B_j}\right)\\ &=2^{1+q/p}\varphi_{\mathcal{F}}^{q/p}(n)\sum_j|b_j|^q\,E^{\mathcal{F}}I_{B_j}\\ &=2^q\varphi_{\mathcal{F}}^{q/p}(n)\,E^{\mathcal{F}}|Z|^q\ \text{ a.s.},\end{aligned}$$

which coupled with (4.4) leads to (4.1) in the case where Y and Z are simple random variables.

For the general case, similarly to the proofs of Lemmas 4.1 and 4.2 in Roussas and Ioannides [15], we can conclude that there exist simple random variables Y_n and Z_n such that, as $n \to \infty$,

$$E^{\mathcal{F}} Y_n \to E^{\mathcal{F}} Y,\ E^{\mathcal{F}} Z_n \to E^{\mathcal{F}} Z,$$
$$E^{\mathcal{F}} |Y_n|^p \to E^{\mathcal{F}} |Y|^p,\ E^{\mathcal{F}} |Z_n|^q \to E^{\mathcal{F}} |Z|^q,$$

and

$$E^{\mathcal{F}} Y_n Z_n \to E^{\mathcal{F}} YZ.$$

The desired inequality follows from the above relations and the result proved for Y_n and Z_n. □

The right-hand side in the result of Theorem 4.1 contains the factor $\varphi_{\mathcal{F}}^{1/p}(n)$ with $p > 1$. In certain situations, a closer bound can be replaced by eliminating the exponent $1/p$. More precisely, one has:

Theorem 4.3. *Assume that $\{X_n, n \geq 1\}$ is an $\mathcal{F}$-uniformly strong mixing sequence with coefficient $\{\varphi_{\mathcal{F}}(n)\}$, Y and Z are $\mathcal{A}_1^k$ and $\mathcal{A}_{k+n}^{\infty}$-measurable random variables, respectively. If $|Y| \leq Y_{\mathcal{F}}$ a.s. and $|Z| \leq Z_{\mathcal{F}}$ a.s., where $Y_{\mathcal{F}}$ and $Z_{\mathcal{F}}$ are $\mathcal{F}$-measurable random variables, then*

$$\left|E^{\mathcal{F}} YZ - E^{\mathcal{F}} Y \cdot E^{\mathcal{F}} Z\right| \leq 2\varphi_{\mathcal{F}}(n)\, Y_{\mathcal{F}} Z_{\mathcal{F}} \ \text{a.s.} \tag{4.9}$$

Remark 4.4. If Y and Z are complex-valued random variables, then inequality (4.9) becomes

$$\left|E^{\mathcal{F}} YZ - E^{\mathcal{F}} Y \cdot E^{\mathcal{F}} Z\right| \leq 8\varphi_{\mathcal{F}}(n)\, Y_{\mathcal{F}} Z_{\mathcal{F}} \text{ a.s.}$$

Proof of Theorem 4.3. By the telescopic property of conditional expectation,

$$\begin{aligned} &\left|E^{\mathcal{F}} YZ - E^{\mathcal{F}} Y \cdot E^{\mathcal{F}} Z\right| \qquad (4.10)\\ &= \left|E^{\mathcal{F}}\left(E^{\mathcal{A}_1^k \vee \mathcal{F}} YZ\right) - E^{\mathcal{F}} Y \cdot E^{\mathcal{F}} Z\right| \\ &= \left|E^{\mathcal{F}}\left[Y\left(E^{\mathcal{A}_1^k \vee \mathcal{F}} Z\right)\right] - E^{\mathcal{F}}\left(Y E^{\mathcal{F}} Z\right)\right| \\ &= \left|E^{\mathcal{F}}\left[Y\left(E^{\mathcal{A}_1^k \vee \mathcal{F}} Z - E^{\mathcal{F}} Z\right)\right]\right| \\ &\leq Y_{\mathcal{F}} E^{\mathcal{F}}\left|E^{\mathcal{A}_1^k \vee \mathcal{F}} Z - E^{\mathcal{F}} Z\right| \\ &= Y_{\mathcal{F}} E^{\mathcal{F}}\left[\left(E^{\mathcal{A}_1^k \vee \mathcal{F}} Z - E^{\mathcal{F}} Z\right) I_{A^+}\right] - Y_{\mathcal{F}} E^{\mathcal{F}}\left[\left(E^{\mathcal{A}_1^k \vee \mathcal{F}} Z - E^{\mathcal{F}} Z\right) I_{A^-}\right], \end{aligned}$$

where

$$A^+ = \left\{E^{\mathcal{A}_1^k \vee \mathcal{F}} Z - E^{\mathcal{F}} Z \geq 0\right\}, \quad A^- = \left\{E^{\mathcal{A}_1^k \vee \mathcal{F}} Z - E^{\mathcal{F}} Z < 0\right\},$$

so that A^+, $A^- \in \mathcal{A}_1^k \vee \mathcal{F}$. But

$$E^{\mathcal{F}}\left[\left(E^{\mathcal{A}_1^k \vee \mathcal{F}} Z - E^{\mathcal{F}} Z\right) I_{A^+}\right] \tag{4.11}$$

$$
\begin{aligned}
&= \left| E^{\mathcal{F}} \left[\left(E^{\mathcal{A}_1^k \vee \mathcal{F}} Z - E^{\mathcal{F}} Z \right) I_{A^+} \right] \right| \\
&= \left| E^{\mathcal{F}} \left[E^{\mathcal{A}_1^k \vee \mathcal{F}} \left(Z I_{A^+} \right) \right] - E^{\mathcal{F}} Z \cdot E^{\mathcal{F}} I_{A^+} \right| \\
&= \left| E^{\mathcal{F}} \left(Z I_{A^+} \right) - E^{\mathcal{F}} Z \cdot E^{\mathcal{F}} I_{A^+} \right| \\
&= \left| E^{\mathcal{A}_{k+n}^\infty \vee \mathcal{F}} \left[E^{\mathcal{F}} \left(Z I_{A^+} \right) \right] - E^{\mathcal{F}} \left(Z E^{\mathcal{F}} I_{A^+} \right) \right| \\
&= \left| E^{\mathcal{F}} \left[E^{\mathcal{A}_{k+n}^\infty \vee \mathcal{F}} \left(Z I_{A^+} \right) - Z E^{\mathcal{F}} I_{A^+} \right] \right| \\
&= \left| E^{\mathcal{F}} \left[Z \left(E^{\mathcal{A}_{k+n}^\infty \vee \mathcal{F}} I_{A^+} - E^{\mathcal{F}} I_{A^+} \right) \right] \right| \\
&\le Z_{\mathcal{F}} E^{\mathcal{F}} \left| E^{\mathcal{A}_{k+n}^\infty \vee \mathcal{F}} I_{A^+} - E^{\mathcal{F}} I_{A^+} \right| \\
&= Z_{\mathcal{F}} E^{\mathcal{F}} \left[\left(E^{\mathcal{A}_{k+n}^\infty \vee \mathcal{F}} I_{A^+} - E^{\mathcal{F}} I_{A^+} \right) I_{B^+} \right] \\
&\quad - Z_{\mathcal{F}} E^{\mathcal{F}} \left[\left(E^{\mathcal{A}_{k+n}^\infty \vee \mathcal{F}} I_{A^+} - E^{\mathcal{F}} I_{A^+} \right) I_{B^-} \right],
\end{aligned}
$$

where

$$B^+ = \left\{ E^{\mathcal{A}_{k+n}^\infty \vee \mathcal{F}} I_{A^+} - E^{\mathcal{F}} I_{A^+} \ge 0 \right\}, \ B^- = \left\{ E^{\mathcal{A}_{k+n}^\infty \vee \mathcal{F}} I_{A^+} - E^{\mathcal{F}} I_{A^+} < 0 \right\},$$

so that B^+, $B^- \in \mathcal{A}_{k+n}^\infty \vee \mathcal{F}$. Furthermore,

$$E^{\mathcal{F}} \left[\left(E^{\mathcal{A}_{k+n}^\infty \vee \mathcal{F}} I_{A^+} - E^{\mathcal{F}} I_{A^+} \right) I_{B^+} \right] = P^{\mathcal{F}} \left(A^+ \cap B^+ \right) - P^{\mathcal{F}} \left(A^+ \right) P^{\mathcal{F}} \left(B^+ \right),$$

and similarly

$$E^{\mathcal{F}} \left[\left(E^{\mathcal{A}_{k+n}^\infty \vee \mathcal{F}} I_{A^+} - E^{\mathcal{F}} I_{A^+} \right) I_{B^-} \right] = P^{\mathcal{F}} \left(A^+ \cap B^- \right) - P^{\mathcal{F}} \left(A^+ \right) P^{\mathcal{F}} \left(B^- \right).$$

By taking advantage of Proposition 3.3 and applying (4.11), we conclude that

$$
\begin{aligned}
(4.12) \qquad & E^{\mathcal{F}} \left[\left(E^{\mathcal{A}_1^k \vee \mathcal{F}} Z - E^{\mathcal{F}} Z \right) I_{A^+} \right] \\
&\le Z_{\mathcal{F}} \left| P^{\mathcal{F}} \left(A^+ \cap B^+ \right) - P^{\mathcal{F}} \left(A^+ \right) P^{\mathcal{F}} \left(B^+ \right) \right| \\
&\quad + Z_{\mathcal{F}} \left| P^{\mathcal{F}} \left(A^+ \cap B^- \right) - P^{\mathcal{F}} \left(A^+ \right) P^{\mathcal{F}} \left(B^- \right) \right| \\
&\le 2 Z_{\mathcal{F}} \varphi_{\mathcal{F}} (n) P^{\mathcal{F}} \left(A^+ \right),
\end{aligned}
$$

and symmetrically

$$(4.13) \qquad -E^{\mathcal{F}} \left[\left(E^{\mathcal{A}_1^k \vee \mathcal{F}} Z - E^{\mathcal{F}} Z \right) I_{A^-} \right] \le 2 Z_{\mathcal{F}} \varphi_{\mathcal{F}} (n) P^{\mathcal{F}} \left(A^- \right).$$

Inequality (4.9) is proved by means of (4.10), (4.12) and (4.13). □

Our third conditional covariance inequality is a natural multivariate extension of Theorem 4.1.

Theorem 4.5. *Assume that* $\{X_n, n \ge 1\}$*is an* $\mathcal{F}$*-strong mixing sequence with coefficient* $\{\varphi_{\mathcal{F}}(n)\}$ *and assume that integers* s_i, t_i, $i = 1, 2, \ldots, n$ *satisfy*

$$1 = s_1 < t_1 < s_2 < t_2 < \cdots < s_n < t_n \text{ with } s_{i+1} - t_i \ge \tau, \ i = 1, \ldots, n-1.$$

If Y_i are $\mathcal{A}_{s_i}^{t_i}$-measurable random variables such that

$$E^{\mathcal{F}}\left|Y_i\right|^{p_i} < \infty \text{ a. s. for } p_i > 1,\ i = 1, \ldots, n \text{ with } \frac{1}{p_1} + \cdots + \frac{1}{p_n} = 1,$$

then

$$\left|E^{\mathcal{F}}\prod_{i=1}^{n} Y_i - \prod_{i=1}^{n} E^{\mathcal{F}} Y_i\right| \le 2\,(n-1)\,\varphi_{\mathcal{F}}^{1/r_n}(\tau)\prod_{i=1}^{n}\left(E^{\mathcal{F}}\left|Y_i\right|^{p_i}\right)^{1/p_i} \text{ a.s.,} \tag{4.14}$$

where $r_n = \max\{p_1, \ldots, p_n\}$.

Proof. It is routine to verify that

$$\left|E^{\mathcal{F}}\prod_{i=1}^{n} Y_i - \prod_{i=1}^{n} E^{\mathcal{F}} Y_i\right| \le \left|E^{\mathcal{F}}\left[Y_1\left(\prod_{i=2}^{n} Y_i\right)\right] - E^{\mathcal{F}} Y_1 \cdot E^{\mathcal{F}}\prod_{i=2}^{n} Y_i\right| + E^{\mathcal{F}}\left|Y_1\right|\left|E^{\mathcal{F}}\prod_{i=2}^{n} Y_i - \prod_{i=2}^{n} E^{\mathcal{F}} Y_i\right|. \tag{4.15}$$

Inequality (4.14) holds for $n = 2$, by means of Theorem 4.1. Assuming it to be true for $n - 1$, we have

$$\left|E^{\mathcal{F}}\left[Y_1\left(\prod_{i=2}^{n} Y_i\right)\right] - E^{\mathcal{F}} Y_1 \cdot E^{\mathcal{F}}\prod_{i=2}^{n} Y_i\right| \le 2\varphi_{\mathcal{F}}^{1/p_1}(\tau)\left(E^{\mathcal{F}}\left|Y_1\right|^{p_1}\right)^{1/p_1}\left(E^{\mathcal{F}}\left|\prod_{i=2}^{n} Y_i\right|^{p}\right)^{1/p},$$

where $1/p = 1/p_2 + \cdots + 1/p_n < 1$. Applying the conditional Hölder inequality with $q_i = p_i/p$, $i = 1, 2, \ldots, n$ (so that $1/q_2 + \cdots + 1/q_n = 1$), we have

$$E^{\mathcal{F}}\left|\prod_{i=2}^{n} Y_i\right|^{p} \le \prod_{i=2}^{n}\left(E^{\mathcal{F}}\left|Y_i\right|^{pq_i}\right)^{1/q_i} = \prod_{i=2}^{n}\left(E^{\mathcal{F}}\left|Y_i\right|^{p_i}\right)^{1/q_i},$$

and therefore

$$\left(E^{\mathcal{F}}\left|\prod_{i=2}^{n} Y_i\right|^{p}\right)^{1/p} \le \prod_{i=2}^{n}\left(E^{\mathcal{F}}\left|Y_i\right|^{p_i}\right)^{1/p_i}.$$

Thus, inserting this in the previous majorization, it follows

$$\left|E^{\mathcal{F}}\left[Y_1\left(\prod_{i=2}^{n} Y_i\right)\right] - E^{\mathcal{F}} Y_1 \cdot E^{\mathcal{F}}\prod_{i=2}^{n} Y_i\right| \le 2\varphi_{\mathcal{F}}^{1/r_n}(\tau)\prod_{i=1}^{n}\left(E^{\mathcal{F}}\left|\prod_{i=1}^{n} Y_i\right|^{p_i}\right)^{1/p_i}. \tag{4.16}$$

Next, apply the induction hypothesis with the same q_i, $i = 1, 2, \ldots, n$ as defined above to obtain

$$\left| E^{\mathcal{F}} \prod_{i=2}^{n} Y_i - \prod_{i=2}^{n} E^{\mathcal{F}} Y_i \right| \leq 2\,(n-2)\,\varphi_{\mathcal{F}}^{1/r_{n-1}^*}(\tau) \prod_{i=2}^{n} \left(E^{\mathcal{F}} |Y_i|^{q_i} \right)^{1/q_i}, \tag{4.17}$$

where $r_{n-1}^* = \max\{q_2, \ldots, q_n\}$. Since, $q_i < p_i$, $i = 2, \ldots, n$, $r_{n-1}^* < r_n$ and $\left(E^{\mathcal{F}} |Y_i|^{q_i}\right)^{1/q_i} \leq \left(E^{\mathcal{F}} |Y_i|^{q_i}\right)^{1/p_i}$, $i = 2, \ldots, n$. Thus (4.17) yields

$$\left| E^{\mathcal{F}} \prod_{i=2}^{n} Y_i - \prod_{i=2}^{n} E^{\mathcal{F}} Y_i \right| \leq 2\,(n-2)\,\varphi_{\mathcal{F}}^{1/r_n}(\tau) \prod_{i=2}^{n} \left(E^{\mathcal{F}} |Y_i|^{p_i} \right)^{1/p_i},$$

and therefore the second member on the right-hand side of (4.15) becomes

$$\begin{aligned} E^{\mathcal{F}} |Y_1| & \left| E^{\mathcal{F}} \prod_{i=2}^{n} Y_i - \prod_{i=2}^{n} E^{\mathcal{F}} Y_i \right| \\ & \leq 2\,(n-2)\,\varphi_{\mathcal{F}}^{1/r_n}(\tau) \prod_{i=1}^{n} \left(E^{\mathcal{F}} |Y_i|^{p_i} \right)^{1/p_i}. \end{aligned} \tag{4.18}$$

The desired result (4.14) are provided by (4.16) and (4.18) by way of (4.15). □

Our fourth conditional covariance inequality is a natural multivariate extension of Theorem 4.3.

Theorem 4.6. *Assume that $\{X_n, n \geq 1\}$ is an $\mathcal{F}$-uniformly strong mixing sequence with coefficient $\{\varphi_{\mathcal{F}}(n)\}$ and assume that integers s_i, t_i, $i = 1, 2, \ldots, n$ satisfy*

$$1 = s_1 < t_1 < s_2 < t_2 < \cdots < s_n < t_n \text{ with } s_{i+1} - t_i \geq \tau,\ i = 1, 2, \ldots, n-1.$$

If $|Y_i| \leq Y_{i,\mathcal{F}}$ a.s.,$i = 1, 2, \ldots, n$, where $Y_{i,\mathcal{F}}$ is $\mathcal{F}$-measurable random variables, then

$$\left| E^{\mathcal{F}} \prod_{i=1}^{n} Y_i - \prod_{i=1}^{n} E^{\mathcal{F}} Y_i \right| \leq 2\,(n-1)\,\varphi_{\mathcal{F}}(\tau) \prod_{i=1}^{n} Y_{i,\mathcal{F}} \ \text{a.s.}$$

Proof. The result holds for $n = 2$, by means of Theorem 4.3. Assuming it to be true for $n-1$, we have

$$\begin{aligned} & \left| E^{\mathcal{F}} \prod_{i=1}^{n} Y_i - \prod_{i=1}^{n} E^{\mathcal{F}} Y_i \right| \\ \leq & \left| E^{\mathcal{F}} \left[\left(\prod_{i=1}^{n-1} Y_i \right) Y_n \right] - E^{\mathcal{F}} \prod_{i=1}^{n-1} Y_i \cdot E^{\mathcal{F}} Y_n \right| \\ & + E^{\mathcal{F}} |Y_n| \left| E^{\mathcal{F}} \prod_{i=1}^{n-1} Y_i - \prod_{i=1}^{n-1} E^{\mathcal{F}} Y_i \right| \end{aligned}$$

$$\leq 2\varphi_{\mathcal{F}}(\tau)\left(\prod_{i=1}^{n-1} Y_{i,\mathcal{F}}\right) Y_{n,\mathcal{F}} + Y_{n,\mathcal{F}} \cdot 2(n-2)\varphi_{\mathcal{F}}(\tau)\prod_{i=1}^{n-1} Y_{i,\mathcal{F}}$$

$$= 2(n-1)\prod_{i=1}^{n} Y_{i,\mathcal{F}}.$$

□

The following Theorems 4.5′ and 4.6′ are essentially extensions of Theorems 4.5 and 4.6 to the case of complex-valued random variables, respectively. They can be proved by induction, replacing appeals to Theorems 4.1 and 4.3 with appeals to Remarks 4.2 and 4.4, respectively.

Theorem 4.5′. *Assume that* $\{X_n, n \geq 1\}$ *is an* $\mathcal{F}$*-strong mixing sequence of complex-valued random variables with coefficient* $\{\varphi_{\mathcal{F}}(n)\}$ *and assume that integers* s_i, t_i, $i = 1, 2, \ldots, n$ *satisfy*

$$1 = s_1 < t_1 < s_2 < t_2 < \cdots < s_n < t_n \text{ with } s_{i+1} - t_i \geq \tau,\ i = 1, 2, \ldots, n-1.$$

If Y_i *are* $\mathcal{A}_{s_i}^{t_i}$*-measurable random variables such that*

$$E^{\mathcal{F}} |Y_i|^{p_i} < \infty \text{ a.s. for } p_i > 1,\ i = 1, \ldots, n \text{ with } \frac{1}{p_1} + \cdots + \frac{1}{p_n} = 1,$$

then

$$\left| E^{\mathcal{F}} \prod_{i=1}^{n} Y_i - \prod_{i=1}^{n} E^{\mathcal{F}} Y_i \right| \leq 8(n-1)\varphi_{\mathcal{F}}^{1/r_n}(\tau)\prod_{i=1}^{n}\left(E^{\mathcal{F}}|Y_i|^{p_i}\right)^{1/p_i} \text{ a.s.}$$

Theorem 4.6′. *Assume that* $\{X_n, n \geq 1\}$ *is an* $\mathcal{F}$*- uniformly strong mixing sequence of complex-valued random variables with coefficient* $\{\varphi_{\mathcal{F}}(n)\}$ *and assume that integers* s_i, t_i, $i = 1, 2, \ldots, n$ *satisfy*

$$1 = s_1 < t_1 < s_2 < t_2 < \cdots < s_n < t_n \quad \text{with} \quad s_{i+1} - t_i \geq \tau,\ i = 1, 2, \ldots, n-1.$$

If $|Y_i| \leq Y_{i,\mathcal{F}}$ *a.s.,* $i = 1, 2, \ldots, n$*, where* $Y_{i,\mathcal{F}}$ *are* $\mathcal{F}$*-measurable random variables, then*

$$\left| E^{\mathcal{F}} \prod_{i=1}^{n} Y_i - \prod_{i=1}^{n} E^{\mathcal{F}} Y_i \right| \leq 8(n-1)\varphi_{\mathcal{F}}(\tau)\prod_{i=1}^{n} Y_{i,\mathcal{F}} \text{ a.s.}$$

5. Conditional central limit theorem

A sequence $\{X_n,\ n \geq 1\}$ is called $\mathcal{F}$-stationary if for all $1 \leq t_1 < \cdots < t_k < \infty$ and $r \geq 1$, the joint distribution of $(X_{t_1}, \ldots, X_{t_k})$ conditioned on $\mathcal{F}$ is the same as the joint distribution of $(X_{t_1+r}, \ldots, X_{t_k+r})$ conditioned on $\mathcal{F}$ a.s.

We establish two lemmas prior to our conditional central limit theorem. The first one is an analogue of Lemma 4.1 in Yuan and Lei [21], which follows immediately from the result on page 124 of Knopp [9], we here give another proof for the sake of completeness.

Lemma 5.1. $\sum_{n=1}^{\infty} \varphi_{\mathcal{F}}^{p}(n) < \infty$ *a.s. for some* $0 < p < 1$ *implies* $n\varphi_{\mathcal{F}}(n) \to 0$ *a.s. as* $n \to \infty$.

Proof. The definition of definite integral ensures

$$\frac{1}{n}\sum_{j=1}^{n}\left(\frac{j}{n}\right)^{p} \geq \int_{0}^{1} x^{p}dx = \frac{1}{1+p} \geq \frac{1}{n^{1-p}} \ for \ n \geq (1+p)^{1/(1-p)},$$

which gives

$$\sum_{j=1}^{n} j^{p} \geq n^{2p} \text{ for } \ n \geq (1+p)^{1/(1-p)}. \tag{5.1}$$

On the other hand, applying the Kronecker lemma with $\sum_{n=1}^{\infty}\varphi_{\mathcal{F}}^{p}(n) < \infty$ a.s., we get

$$n^{-p}\sum_{j=1}^{n} j^{p}\varphi_{\mathcal{F}}^{p}(j) \to 0 \text{ as } n \to \infty,$$

which together with (5.1) yields

$$\varphi_{\mathcal{F}}^{p}(n)\, n^{p} = \varphi_{\mathcal{F}}^{p}(n)\, n^{-p}\cdot n^{2p} \leq \varphi_{\mathcal{F}}^{p}(n)\, n^{-p}\sum_{j=1}^{n} j^{p} \leq n^{-p}\sum_{j=1}^{n} j^{p}\varphi_{\mathcal{F}}^{p}(j) \to 0 \text{ a.s.},$$

which is tantamount to the conclusion. □

Lemma 5.2. *Let $\{X_n,\ n \geq 1\}$ be a sequence of $\mathcal{F}$-uniformly strong mixing random variables satisfying $\sum_{n=1}^{\infty}\varphi_{\mathcal{F}}^{1/2}(n) < \infty$ a.s. If $E^{\mathcal{F}}X_1 = 0$ a.s. and $E^{\mathcal{F}}X_1^2 < \infty$ a.s., then there exists a nonnegative $\mathcal{F}$-measurable random variable $\xi_{\mathcal{F}}$ such that for all $n \geq 1$,*

$$E^{\mathcal{F}}\left(\sum_{i=1}^{n} X_i\right)^{2} \leq \xi_{\mathcal{F}}\sum_{i=1}^{n} E^{\mathcal{F}}X_i^{2}.$$

Proof. By Theorem 4.1 for $p = q = 2$, we conclude that

$$\begin{aligned} E^{\mathcal{F}}\left(\sum_{i=1}^{n} X_i\right)^{2} &= \sum_{i=1}^{n} E^{\mathcal{F}}X_i^{2} + 2\sum_{1\leq i<j\leq n} E^{\mathcal{F}}X_iX_j \\ &\leq \sum_{i=1}^{n} E^{\mathcal{F}}X_i^{2} + 4\sum_{1\leq i<j\leq n}\varphi_{\mathcal{F}}^{1/2}(j-i)\left(E^{\mathcal{F}}X_i^{2}\right)^{1/2}\left(E^{\mathcal{F}}X_j^{2}\right)^{1/2} \\ &\leq \sum_{i=1}^{n} E^{\mathcal{F}}X_i^{2} + 2\sum_{i=1}^{n-1}\sum_{k=1}^{n-i}\varphi_{\mathcal{F}}^{1/2}(k)\left(E^{\mathcal{F}}X_i^{2} + E^{\mathcal{F}}X_{k+i}^{2}\right) \\ &\leq \left(1 + 4\sum_{k=1}^{\infty}\varphi_{\mathcal{F}}^{1/2}(k)\right)\sum_{i=1}^{n} E^{\mathcal{F}}X_i^{2} \text{ a.s.,} \end{aligned}$$

verifying the result by taking $\xi_{\mathcal{F}} = 1 + 4\sum_{k=1}^{\infty}\varphi_{\mathcal{F}}^{1/2}(k)$. □

Our conditional central limit theorem stated in terms of conditional characteristic functions reads as follows, whose non-conditional version is Theorem 1.5 in Ibragimov [8].

Theorem 5.3. *Assume that* $\{X_n,\ n \geq 1\}$ *is a sequence of* $\mathcal{F}$*-uniformly strong mixing and* $\mathcal{F}$*-stationary random variables with*

$$\sum_{n=1}^{\infty} \varphi_{\mathcal{F}}^{1/2}(n) < \infty \ a.s., \tag{5.2}$$

and if $E^{\mathcal{F}} |X_1|^2 < \infty$ *a.s., then*

$$\sigma_{\mathcal{F}}^2 := E^{\mathcal{F}} \left(X_1 - E^{\mathcal{F}} X_1\right)^2 + 2 \sum_{i=2}^{\infty} Cov^{\mathcal{F}}(X_1, X_i) < \infty \ a.s., \tag{5.3}$$

$$n^{-1} E^{\mathcal{F}} \left(S_n - E^{\mathcal{F}} S_n\right)^2 \to \sigma_{\mathcal{F}}^2 \ a.s. \tag{5.4}$$

and the series converges absolutely almost surely. Furthermore, if $\sigma_{\mathcal{F}} > 0$ *almost surely, then*

$$E^{\mathcal{F}} \exp\left[\frac{it\left(S_n - E^{\mathcal{F}} S_n\right)}{\sqrt{n}\sigma_{\mathcal{F}}}\right] \to \exp\left(-\frac{t^2}{2}\right) \ a.s. \ as\ n \to \infty \tag{5.5}$$

In particular,

$$\frac{S_n - E^{\mathcal{F}} S_n}{\sqrt{n}\sigma_{\mathcal{F}}} \to N(0,1) \ in\ distribution. \tag{5.6}$$

Proof. Relation (5.6) follows from (5.5) by using the dominated convergence theorem and the continuity theorem for characteristic functions.

Noticing that $\left\{X_n - E^{\mathcal{F}} X_n,\ n \geq 1\right\}$ is $\mathcal{F}$-centered and $\mathcal{F}$-uniformly strong mixing with the same coefficient $\{\varphi_{\mathcal{F}}(n)\}$ in view of Proposition 3.6 (iii), without loss of generality we may assume that $E^{\mathcal{F}} X_n = 0$ for all $n \geq 1$. Taking into account the conditional stationarity, we need only to assume that $E^{\mathcal{F}} X_1 = 0$. By Theorem 4.1,

$$\sum_{j=2}^{\infty} \left|Cov^{\mathcal{F}}(X_1, X_j)\right| \leq 2E^{\mathcal{F}} X_1^2 \sum_{j=1}^{\infty} \varphi_{\mathcal{F}}^{1/2}(j),$$

and so the series in (5.3) converges absolutely almost surely according to (5.2). From the conditional stationarity of $\{X_n\}$,

$$E^{\mathcal{F}} S_n^2 = nE^{\mathcal{F}} X_1^2 + 2\sum_{j=2}^{n} (n - j + 1)\, Cov^{\mathcal{F}}(X_1, X_j).$$

Using the convergence of $\sum_{j=1}^{\infty} Cov^{\mathcal{F}}(X_1, X_j)$ together with the Kronecker lemma,

$$n^{-1} \sum_{j=1}^{n} jCov^{\mathcal{F}}(X_1, X_j) \to 0 \ a.s.,$$

and we have

$$\frac{1}{n}E^{\mathcal{F}}S_n^2-\sigma_{\mathcal{F}}^2=2\sum_{j=n+1}^{\infty}Cov^{\mathcal{F}}\left(X_1,X_j\right)-\frac{2}{n}\sum_{j=2}^{n}\left(j-1\right)Cov^{\mathcal{F}}\left(X_1,X_j\right)\to 0\text{ a.s.}$$

as $n\to\infty$, which is tantamount to (5.4).

Next we prove (5.5). Let $b_n=\left[n^{3/4}\right]$ and $l_n=\left[n^{1/4}\right]$. If r_n is the largest integer j such that $(j-1)\left(b_n+l_n\right)+b_n<n$, then

$$b_n\sim n^{3/4},\ l_n\sim n^{1/4},\ r_n\sim n^{1/4}. \tag{5.7}$$

Let

$$U_{nj}=X_{(j-1)(b_n+l_n)+1}+\cdots+X_{(j-1)(b_n+l_n)+b_n},\ 1\le j\le r_n$$

and

$$V_{nj}=\begin{cases}X_{(j-1)(b_n+l_n)+b_n+1}+\cdots+X_{j(b_n+l_n)},\ 1\le j<r_n,\\ X_{(j-1)(b_n+l_n)+b_n+1}+\cdots+X_n,\ j=r_n,\end{cases}$$

then $S_n=\sum_{j=1}^{r_n}U_{nj}+\sum_{j=1}^{r_n}V_{nj}$, that is, the sum S_n is split into alternate blocks of length b_n(the big blocks) and l_n(the little blocks). We will prove that $\sum_{j=1}^{r_n}V_{nj}$ is small in comparison with $\sum_{j=1}^{r_n}U_{nj}$ but large enough that the U_{nj} are nearly $\mathcal{F}$-independent. Note that the family $\{V_{nj},1\le j\le r_n\}$ is $\mathcal{F}$-uniformly strong mixing by Proposition 3.5. It follows from the conditional stationarity and Lemma 5.2 that

$$\begin{aligned}&\left|E^{\mathcal{F}}\exp\left(it\frac{S_n}{\sqrt{n}\sigma_{\mathcal{F}}}\right)-E^{\mathcal{F}}\exp\left(it\frac{\sum_{j=1}^{r_n}U_{nj}}{\sqrt{n}\sigma_{\mathcal{F}}}\right)\right|\\ \le&\frac{|t|}{\sqrt{n}\sigma_{\mathcal{F}}}E^{\mathcal{F}}\left|S_n-\sum_{j=1}^{r_n}U_{nj}\right|\\ =&\frac{|t|}{\sqrt{n}\sigma_{\mathcal{F}}}E^{\mathcal{F}}\left|\sum_{j=1}^{r_n}V_{nj}\right|\\ \le&\frac{\sqrt{2}\,|t|}{\sqrt{n}\sigma_{\mathcal{F}}}\left[E^{\mathcal{F}}\left(\sum_{j=1}^{r_n-1}V_{nj}\right)^2+E^{\mathcal{F}}V_{nr_n}^2\right]^{1/2}\\ \le&\frac{\sqrt{2}\,|t|}{\sqrt{n}\sigma_{\mathcal{F}}}\left[\left(r_n-1\right)\xi_{\mathcal{F}}E^{\mathcal{F}}V_{n1}^2+E^{\mathcal{F}}V_{nr_n}^2\right]^{1/2}.\end{aligned}$$

Now, by Lemma 5.2 again and (5.7),

$$\frac{1}{n}\left(r_n-1\right)\xi_{\mathcal{F}}E^{\mathcal{F}}V_{n1}^2\le\frac{1}{n}r_nl_n\xi_{\mathcal{F}}^2E^{\mathcal{F}}X_1^2\to 0\text{ a.s.},$$

$$\frac{1}{n}E^{\mathcal{F}}V_{nr_n}^2\le\frac{1}{n}\left[n-\left(r_n-1\right)\left(b_n+l_n\right)-b_n\right]\xi_{\mathcal{F}}E^{\mathcal{F}}X_1^2$$

$$\leq \frac{1}{n}\left(b_n + l_n\right)\xi_{\mathcal{F}} E^{\mathcal{F}} X_1^2 \to 0 \text{ a.s.}$$

Putting these relations derived above together, we conclude that

$$E^{\mathcal{F}} \exp\left(it\frac{S_n}{\sqrt{n}\sigma_{\mathcal{F}}}\right) - E^{\mathcal{F}} \exp\left(it\frac{\sum_{j=1}^{r_n} U_{nj}}{\sqrt{n}\sigma_{\mathcal{F}}}\right) \to 0 \text{ a.s.}$$

To verify (5.5), it suffices to show that

$$E^{\mathcal{F}} \exp\left(it\frac{\sum_{j=1}^{r_n} U_{nj}}{\sqrt{n}\sigma_{\mathcal{F}}}\right) \to \exp\left(-\frac{t^2}{2}\right) \text{ a.s. as } n \to \infty. \tag{5.8}$$

For this purpose, let $U'_{nj}, 1 \leq j \leq r_n$, be $\mathcal{F}$-independent random variables such that U'_{nj} and U_{nj} have the same distribution with respect to $\mathcal{F}$. Noting that the family $\{U_{nj}/(\sqrt{n}\sigma_{\mathcal{F}}), 1 \leq j \leq r_n\}$ is $\mathcal{F}$-uniformly strong mixing by Propositions 3.4, 3.5 and 3.6(ii), we have

$$\begin{aligned}
&\left|E^{\mathcal{F}} \exp\left(it\frac{\sum_{j=1}^{r_n} U_{nj}}{\sqrt{n}\sigma_{\mathcal{F}}}\right) - E^{\mathcal{F}} \exp\left(it\frac{\sum_{j=1}^{r_n} U'_{nj}}{\sqrt{n}\sigma_{\mathcal{F}}}\right)\right| \\
&= \left|E^{\mathcal{F}} \exp\left(it\frac{\sum_{j=1}^{r_n} U_{nj}}{\sqrt{n}\sigma_{\mathcal{F}}}\right) - \prod_{j=1}^{r_n} E^{\mathcal{F}} \exp\left(it\frac{U_{nj}}{\sqrt{n}\sigma_{\mathcal{F}}}\right)\right| \\
&\leq 8\left(r_n - 1\right)\varphi_{\mathcal{F}}\left(l_n + 1\right) \\
&\leq 8 r_n \varphi_{\mathcal{F}}\left(l_n\right) \\
&\to 0 \text{ a.s.}
\end{aligned}$$

by using Theorem 4.6′, Lemma 5.1 and (5.7). Now

$$\begin{aligned}
&\left|E^{\mathcal{F}} \exp\left(it\frac{\sum_{j=1}^{r_n} U'_{nj}}{\sqrt{n}\sigma_{\mathcal{F}}}\right) - E^{\mathcal{F}} \exp\left(it\frac{\sum_{j=1}^{r_n} U'_{nj}}{\sqrt{r_n E^{\mathcal{F}} U'^2_{n1}}}\right)\right| \\
&\leq |t| \sqrt{E^{\mathcal{F}}\left(\sum_{j=1}^{r_n} U'_{nj}\right)^2} \left|\frac{1}{\sqrt{n}\sigma_{\mathcal{F}}} - \frac{1}{\sqrt{r_n E^{\mathcal{F}} U'^2_{n1}}}\right| \\
&\leq |t| \left|\frac{\sqrt{r_n E^{\mathcal{F}} U'^2_{n1}}}{\sqrt{n}\sigma_{\mathcal{F}}} - 1\right| \\
&= |t| \left|\frac{\sqrt{r_n E^{\mathcal{F}} U^2_{n1}}}{\sqrt{n}\sigma_{\mathcal{F}}} - 1\right|,
\end{aligned}$$

the last term tends to 0 since $\frac{\sqrt{r_n E^{\mathcal{F}} U^2_{n1}}}{\sqrt{n}\sigma_{\mathcal{F}}} \sim \frac{\sqrt{r_n b_n \sigma^2_{\mathcal{F}}}}{\sqrt{n}\sigma_{\mathcal{F}}} \to 1$ by (5.4) and (5.7), and therefore (5.8) is an immediate consequence of

$$E^{\mathcal{F}} \exp\left(it\frac{\sum_{j=1}^{r_n} U'_{nj}}{\sqrt{r_n E^{\mathcal{F}} U'^2_{n1}}}\right) \to \exp\left(-\frac{t^2}{2}\right) \text{ a.s.,}$$

which follows by Theorem 8 in Prakasa Rao [14]. □

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