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TOTAL GRAPH OF A COMMUTATIVE SEMIRING WITH RESPECT TO IDENTITY-SUMMAND ELEMENTS

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Abstract. Let R be an I-semiring and $S(R)$ be the set of all identity-summand elements of R. In this paper we introduce the total graph of R with respect to identity-summand elements, denoted by $T(\Gamma(R))$, and investigate basic properties of $S(R)$ which help us to gain interesting results about $T(\Gamma(R))$ and its subgraphs.

1. Introduction

Associating a graph to an algebraic structure is a research subject and has attracted considerable attention. In fact, the research in this subject aims at exposing the relationship between algebra and graph theory and at advancing the application of one to the other.

In 1988, Beck [11] introduced the idea of a zero-divisor graph of a commutative ring R with identity. This notion was later redefined by Anderson and Livingston in [6]. Since then, there has been a lot of interest in this subject and various papers were published establishing different properties of these graphs as well as relations between graphs of various extensions. The total graph of a commutative ring was introduced by Anderson and Badawi in [3], as the graph with all elements of R as vertices, and two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in Z(R)$ where $Z(R)$ is the set of all zero divisor of R. In [4], Anderson and Badawi studied the subgraph $T_0(\Gamma(R))$ of $T(\Gamma(R))$ with vertices $R - \{0\}$. Recently, the study of graphs of rings are extended to include semirings as in [15, 16, 18, 19].

Semirings have proven to be useful in theoretical computer science, in particular for studying automata and formal languages, hence, ought to be in the literature [22, 24]. From now on let R be a commutative semiring with identity. In [18], the present authors introduced the identity-summand graph, denoted by $\Gamma(R)$, is the graph which vertices are all non-identity identity-summands of

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R and two distinct vertices joint by an edge when the sum of them is 1. We use the notation $S(R)$ to refer to the set of elements of R that are identity-summand (we use $S^*(R)$ to denote the set of non-identity identity-summands of R), we say that $r \in R$ is an identity-summand of R, if there exists $1 \neq a \in R$ such that $r + a = 1$.

Let R be an I-semiring (i.e., $1 + r = 1$ for each $r \in R$). Studying the $S(R)$ runs into the issue of a profound lack of algebraic structure, highlighted by a lack of closure under multiplication. This unfortunate lack of algebraic structure is the focus of our investigations in this paper. We define the total graph of a commutative semiring R with respect to identity-summand elements denoted by $T(\Gamma(R))$ and its subgraphs $S(\Gamma(R))$ and $S^*(\Gamma(R))$.

In Section 3, we show that $T(\Gamma(R))$ and $S^*(\Gamma(R))$ are not connected by giving an example but its subgraph $S(\Gamma(R))$ is always connected. Also by using I-semiring condition, it is proved that $\mathrm{diam}(S(\Gamma(R))) \in \{1, 2\}$ and $\mathrm{gr}(S(\Gamma(R))) \in \{3, \infty\}$. It is shown that $\mathrm{diam}(S(\Gamma(R))) = 1$ if and only if $S(R)$ is a co-ideal of R and $\mathrm{gr}(S(\Gamma(R))) = 3$ if and only if $|S(R)| \geq 4$. Moreover, we find chromatic number of $S(\Gamma(R))$.

In Section 4, $S^*(\Gamma(R))$ is investigated. At the first of this section, one of the important properties of $S(R)$ is introduced, which help us to gain interesting results about $S^*(\Gamma(R))$. It is shown that $S(R)$ is a union of all minimal prime co-ideals of R. It is proved that $S^*(\Gamma(R))$ is connected if and only if $|\mathrm{min(R)}| \neq 2$, $\mathrm{diam}(S^*(\Gamma(R))) \in \{1, 2\}$ and $\mathrm{gr}(S^*(\Gamma(R))) \in \{3, \infty\}$. Also it is investigated when $\mathrm{gr}(S^*(\Gamma(R))) = \mathrm{gr}(\Gamma(R))$ or $\mathrm{gr}(S^*(\Gamma(R))) = \mathrm{gr}(S(\Gamma(R)))$.

2. Preliminaries

In order to make this paper easier to follow, we recall various notions which will be used in the sequel. For a graph Γ by $E(\Gamma)$ and $V(\Gamma)$ we denote the set of all edges and vertices, respectively. A graph G is called connected if for any vertices x and y of G there is a path between x and y. Otherwise, G is called disconnected. The maximal connected subgraphs of G are its connected components. Here, maximal means that including any more vertices would yield a disconnected subgraph. Any graph is a union of its connected components. If the number of connected components of G is equal to one, then G is, of course, connected. The distance between two distinct vertices a and b, denoted by $\mathrm{d}(a, b)$, is the length of the shortest path connecting them (if such a path does not exist, then $d(a, b) = \infty$, also $d(a, a) = 0$). The diameter of graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is equal to $\sup\{d(a, b) : a, b \in V(\Gamma)\}$. A graph is complete if it is connected with diameter less than or equal to one. We denote the complete graph on n vertices by K_n. The girth of a graph Γ, denoted $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise; $\mathrm{gr}(\Gamma) = \infty$. For r a nonnegative integer, an r-partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite (i.e., 2-partite) with part sizes m and n is denoted by $K_{m,n}$. We will sometimes call $K_{1,n}$ a star graph. A clique of a graph is its maximal complete subgraph and the number of vertices in the largest clique of graph G, denoted by $w(G)$, is called the clique number of G.

A commutative semiring R is defined as an algebraic system $(R, +, .)$ such that $(R, +)$ and $(R, \cdot)$ are commutative semigroups, connected by $a(b + c) = ab + ac$ for all $a, b, c \in R$, and there exist $0, 1 \in R$ such that $r + 0 = r$ and $r0 = 0r = 0$ and $r1 = 1r = r$ for each $r \in R$. In this paper all semirings considered will be assumed to be commutative semirings with non-zero identity.

Definition 2.1. Let R be a semiring.

(1) A non-empty subset I of R is called *co-ideal*, if it is closed under multiplication and satisfies the condition $r + a \in I$ for all $a \in I$ and $r \in R$ (so $0 \in I$ if and only if $I = R$). A co-ideal I of R is called strong co-ideal provided that $1 \in I$ [17, 22].

(2) A co-ideal I of R is called *subtractive* if $x, xy \in I$, then $y \in I$ (so every subtractive co-ideal is a strong co-ideal) [17].

(3) A semiring R is called an I-semiring if $r + 1 = 1$ for all $r \in R$ [18].

(4) A proper co-ideal P of R is called *prime* if $x + y \in P$, then $x \in P$ or $y \in P$. The set of all prime (resp. minimal prime) co-ideals of R is denoted by co-Spec(R) (resp. $\min(R)$) [17].

(5) If D is an arbitrary nonempty subset of R, then the set $F(D)$ consisting of all elements of R of the form $d_1d_2 \cdots d_n + r$ (with $d_i \in D$ for all $1 \leq i \leq n$ and $r \in R$) is a co-ideal of R containing D [17, 24].

(6) A semiring R is called *co-semidomain*, if $a+b = 1$ $(a, b \in R)$, then either $a = 1$ or $b = 1$ [17].

(7) We say that a subset $T \subseteq R$ is additively closed if $0 \in T$ and $a + b \in T$ for all $a, b \in T$.

(8) An ideal I of R is called *k-ideal* if $x, x+y \in I$, then $y \in I$, for all $x, y \in R$.

(9) An element $a \in R$ is called co-regular if a is not an identity-summand element and $Co - Reg(R) = R \setminus S(R)$.

The following theorem and proposition are used in the sequel and can be found in [18].

Proposition 2.2. *Let R be a commutative I-semiring. Then*

(1) ([18, Proposition 2.5]) *The following statements hold*:

(a) *If J is a co-ideal, then J is a strong co-ideal of R. Moreover, if $xy \in J$, then $x, y \in J$ for every $x, y \in R$. In particular, J is subtractive*;

(b) *The set* $(1 : x) = \{r \in R : r + x = 1\}$ *is a strong co-ideal of R for every $x \in S(R)$.*

(2) ([18, Theorem 2.8]) *If $\{P_\alpha\}_{\alpha\in\Lambda}$ is the set of all prime co-ideals of R, the following statements hold*:

(a) $\cap_{\alpha\in\Lambda} P_\alpha = \{1\}$;

(b) *If* Λ *is a finite set, then* $\cap_{i=1}^{n} P_i = \{1\}$ *and* $\{1\} \neq \cap_{1 \leq i \leq n, i \neq j} P_i$ *for each* $1 \leq j \leq n$.

Theorem 2.3. *Let* R *be an* I*-semiring. Then*

(1) ([18, Theorem 3.3]) *The following statements hold*:

(a) $\Gamma(R)$ *is a connected graph with* $\operatorname{diam}(\Gamma(R)) \leq 3$;

(b) *If* $|S^*(R)| \geq 3$, *then* $\Gamma(R)$ *is not a complete graph*;

(c) *If* $|S^*(R)| \geq 3$, *then* $\operatorname{diam}(\Gamma(R)) = 2$ *or* 3.

(2) ([18, Theorem 4.5]) $\Gamma(R)$ *is complete bipartite if and only if there exist two distinct prime co-ideals* P_1 *and* P_2 *of* R *such that* $P_1 \cap P_2 = \{1\}$.

(3) ([18, Theorem 5.4]) $w(\Gamma(R)) = |\min(R)|$.

(4) ([18, Theorem 3.5]) *If* $\Gamma(R)$ *contains a cycle, then* $\operatorname{gr}(\Gamma(R)) \leq 4$.

The proof of the following lemmas is well-known, but we give the details for convenience.

Lemma 2.4. *Let* R *be a semiring. If* $|S(R)| = 1$, *then* R *is a co-semidomain. Moreover, if* R *is an* I*-semiring which is not co-semidomain, then* $|S(R)| \neq 1, 2$.

Proof. Let $a + b = 1$ for some $a, b \in R$. Let $a \neq 1$. By definition of $S(R)$, $b \in S(R) = \{1\}$. Thus R is a co-semidomain. For the moreover statement, if $S(R) = \{1, a\}$, then we have $a + a = 1$. Hence $a = a.1 = a(1 + 1) = a + a = 1$, a contradiction. $\square$

Lemma 2.5. *Let* I *be a subtractive co-ideal of a semiring* R. *Then* $(I : a) = \{r \in R : r + a \in I\}$ *is a subtractive co-ideal of* R *for all* $a \in R$.

Proof. Clearly, $1 \in (I : a)$. If $x, y \in (I : a)$, then $x + a \in I$ and $y + a \in I$, implying $a^2 + ax + ay + xy \in I$. Since $(xy + a)(1 + a)(1 + y)(1 + x) \in I$, $xy + a \in I$ by (1). Thus $xy \in (I : a)$. As I is a co-ideal, $r + x + a \in I$ for each $r \in R$ and so $x + r \in (I : a)$ for each $r \in R$. This shows that $(I : a)$ is a co-ideal of R. Now let $xy, x \in (I : a)$. Then $xy + a + y + xa = (x + 1)(y + a) \in I$, which gives $y + a \in I$, and so $y \in (I : a)$, as desired. $\square$

3. Total graph of semirings

In this section, we introduce the total graph of a semiring R with respect to identity-summand elements.

Definition 3.1. Let R be an I-semiring. The total graph of R, denoted by $T(\Gamma(R))$, is the graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $xy \in S(R)$. $S(\Gamma(R))$ (resp. $Co - Reg(\Gamma(R))$, $S^*(\Gamma(R))$) denotes the subgraph of $T(\Gamma(R))$ with vertex set $S(R)$ (resp. $Co - Reg(R)$, $S^*(R)$).

Here we consider the following question: If R is a semiring, then do we have $T(\Gamma(R))$ is connected? Disconnectivity is a similarity between $\Gamma(R)$ and $T(\Gamma(R))$ for a commutative semiring R and also it is one important difference between them when R is an I-semiring.

In the following, we show that in general, for a semiring R, the total graph of identity-summand elements of R and it's subgraphs $S^*(\Gamma(R))$ and $Co-Reg(\Gamma(R))$ are not connected but the subgraph $S(\Gamma(R))$ of R is always connected. Theorem 3.7 shows that the condition "R is an I-semiring" is not enough to force $T(\Gamma(R))$ to be connected, but it is enough for $\Gamma(R)$ (Theorem 2.3).

Example 3.2. (1) Let $R = (\mathbb{Z}^+ \times \mathbb{Z}^+, +, \cdot)$. Then $S^*(R) = \{(1,0),(0,1)\}$. It can be easily seen that $T(\Gamma(R))$, $S^*(\Gamma(R))$ and $Co-Reg(\Gamma(R))$ are not connected and $S(\Gamma(R))$ is connected.

(2) Let $R = (\mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+, +, \cdot)$. Then $S^*(R) = \{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1)\}$. It can be easily seen that $S^*(\Gamma(R))$ is connected and R is not an I-semiring.

Among the subgraphs of $T(\Gamma(R))$, which are introduced in above, $S(\Gamma(R))$ is connected in general. So we characterize the diameter and girth of $S(\Gamma(R))$ in the following theorems. After that we use the "I-semiring" condition for R to prove some important properties of $S(R)$, which improve our result about the total graph of R in this paper.

Theorem 3.3. *Let R be a semiring. Then $S(\Gamma(R))$ is connected and*

$$\mathrm{diam}(S(\Gamma(R))) \in \{1,2\}.$$

Proof. Since $1 \in S(R)$, $x-1-y$ is a path in $S(\Gamma(R))$ for each $x, y \in S(R)$. So $S(\Gamma(R))$ is connected and $\mathrm{diam}(S(\Gamma(R))) \in \{1,2\}$. □

Theorem 3.4. *Let R be a semiring. Then* $\mathrm{gr}(S(\Gamma(R))) \in \{3,\infty\}$.

Proof. If $xy \in S(R)$ for some $x, y \in S^*(R)$, then $1-x-y-1$ is a cycle in $S(\Gamma(R))$, hence $\mathrm{gr}(S(\Gamma(R))) = 3$. Let $xy \notin S(R)$ for each $x, y \in S^*(R)$. So $S(\Gamma(R))$ does not contain any cycle. Hence $\mathrm{gr}(S(\Gamma(R))) = \infty$. □

Remark 3.5. Let R be an I-semiring, then $S(R)$ is closed under additive operation.

Lemma 3.6. *Let R be an I-semiring. The following statements hold*:

(i) *For each $x, y \in R$, if $xy = 1$, then $x = 1$ and $y = 1$;*

(ii) *If $\{1\} \neq S(R)$ is finite, then $S(R)$ is not a co-ideal of R.*

(iii) *$S(R)$ is a union of prime co-ideals of R.*

(iv) *$Co-Reg(R)$ is an ideal of R.*

(v) *$S(R)$ is a co-ideal of R if and only if $Co-Reg(R)$ is a prime ideal of R.*

Proof. (i) Let $xy = 1$ for some $x, y \in R$. Since R is an I-semiring, $x = x + xy = x + 1 = 1$ and by the similar way $y = 1$.

(ii) Let $S(R) = \{a_1, a_2, \ldots, a_n\}$. Suppose, on the contrary, $S(R)$ is a co-ideal of R. So $a_1 a_2 \cdots a_n \in S(R)$, hence $a_1 a_2 \cdots a_n = a_i$ for some $1 \leq i \leq n$. By (i), $a_i \neq 1$. Since $a_i \in S(R)$, there exists $1 \neq a_j \in S(R)$ such that $a_i + a_j = 1$.

So $a_j = a_1a_2 \cdots a_n + a_j = a_i + a_j = 1$, a contradiction. Thus $S(R)$ is not a co-ideal of R.

(iii) Let $\sum$ be the set of all co-ideals in which every element is an identity summand. If $S(R) = \{1\}$, then R is a co-semidomain and there is nothing to prove. So assume that $S(R) \neq \{1\}$. Since for each $x \in S^*(R)$, $F(\{x\}) \in \sum$, $\sum \neq \emptyset$. It is clear that $\sum$ has a maximal element by Zorn's Lemma. Let P be a maximal element of $\sum$. We show that P is prime in R. Let $x + y \in P$ and $x, y \notin P$. Since $P \subset (P : x)$ and P is maximal in $\sum$, $(P : x) \notin \sum$. So there exists $z \in (P : x)$, which is not identity-summand. We claim that $(P : z) \in \sum$. Let $1 \neq w \in (P : z)$, so $w+z \in P$. Since z is not identity-summand , $w+z \neq 1$. Thus $w + z + u = 1$ for some $1 \neq u \in R$. Since z is not identity-summand, $w + u = 1$. Thus w is an identity-summand, and $(P : z) \in \sum$. This is a contradiction with maximality of P, because $P \subset (P : z)$, so P is a prime co-ideal of R. It is clear that $S(R)$ is a union of all maximal elements of $\sum$. So $S(R)$ is a union of prime co-ideals of R.

(iv) By (iii), $S(R) = \cup_{i\in\Lambda} P_i$, where $P_i \in co - Spec(R)$ for each $i \in \Lambda$. Let $x, y \in R \setminus S(R) = Co-Reg(R)$. We show that $x+y \in R \setminus S(R)$. If $x+y \in S(R)$, then $x + y \in P_i$ for some $i \in \Lambda$. Since P_i is a prime co-ideal of R, $x \in P_i$ or $y \in P_i$, a contradiction. Hence $x + y \in R \setminus S(R)$. Now, let $x \in R \setminus S(R), r \in R$. By (i), $xr \neq 1$. If $xr \in S(R)$, then there exists $1 \neq y \in S(R)$ such that $xr \in (1 : y)$. Since $(1 : y)$ is a co-ideal of R, $x \in (1 : y) \subseteq S(R)$ by Proposition 2.2(1), a contradiction. Thus $xr \in R \setminus S(R)$ and $R \setminus S(R)$ is an ideal of R.

(v) Let $S(R)$ be a co-ideal of R and $ab \in Co-Reg(R)$. If $a, b \notin Co-Reg(R)$, then $a, b \in S(R)$ gives $ab \in S(R)$, a contradiction. Conversely, let $Co-Reg(R)$ be a prime ideal of R. We show $S(R)$ is a co-ideal. It suffices to show that $S(R)$ is closed under multiplication. Let $a, b \in S(R)$. If $ab \notin S(R)$, then $ab \in Co - Reg(R)$, which implies $a \in Co - Reg(R)$ or $b \in Co - Reg(R)$, since $Co - Reg(R)$ is a prime ideal of R, a contradiction. Hence $ab \in S(R)$ for each $a, b \in S(R)$ (Since R is an I-semiring $a + r \in S(R)$ for each $r \in R$ and $a \in S(R)$). $\square$

Now, in the following theorem, we investigate total graph of a commutative semiring based on identity-summand elements by the result which we gain from Lemma 3.6 . It is clear that $V(T(\Gamma(R))) = Co - Reg(R) \cup S(R)$.

Theorem 3.7. *Let R be an I-semiring. Then*

(i) *No element of $Co - Reg(R)$ is adjacent to any element of $S(R)$*;

(ii) *$Co - Reg(\Gamma(R))$ is totally disconnected*;

(iii) *$T(\Gamma(R))$ is always disconnected*;

(iv) *$T(\Gamma(R))$ is an empty graph if and only if R is a co-semidomain. Moreover, for each I-semiring R, $Co - Reg(\Gamma(R))$ is an empty graph.*

Proof. (i) Let $x \in Co - Reg(R)$. If x is adjacent to y for some $y \in S(R)$, then $xy \in S(R)$. By Lemma 3.6(i), $xy \neq 1$, so $xy \in (1 : r)$ for some $1 \neq r \in S(R)$. By Proposition 2.2(1), $x \in (1 : r) \subseteq S(R)$, a contradiction.

(ii) Let $x, y \in Co - Reg(R)$ are adjacent in $T(\Gamma(R))$. Hence $xy \in S(R)$. By the similar argument in (i), $x, y \in S(R)$, a contradiction. So for each $x, y \in Co - Reg(R)$, x, y are not adjacent.

(iii) By (ii), $Co - Reg(\Gamma(R))$ is totally disconnected, so $T(\Gamma(R))$ is always disconnected.

(v) Is clear by (ii).

The moreover statement is clear by (ii). □

The following result is an immediate consequence of Theorem 3.7

Corollary 3.8. *Let R be an I-semiring. Then $T(\Gamma(R))$ contains $|Co-Reg(R)|$ $+1$ connected component.*

By Theorem 3.7, even if R is an I-semiring, $T(\Gamma(R))$ is not connected. Hence we investigate it's connected subgraph $S(\Gamma(R))$.

Theorem 3.9. *Let R be an I-semiring which is not a co-semidomain. The following statements hold*:

(i) $S(\Gamma(R))$ *is complete if and only if* $S(R)$ *is a co-ideal of* R;

(ii) $\operatorname{diam}(S(\Gamma(R))) = 1$ *if and only if* $S(R)$ *is a co-ideal of* R.

(iii) $\operatorname{diam}(S(\Gamma(R))) = 2$ *if and only if* $S(R)$ *is not a co-ideal of* R.

(iv) $|S(R)| = 3$ *if and only if* $\operatorname{gr}(S(\Gamma(R))) = \infty$.

(v) $|S(R)| \geq 4$ *if and only if* $\operatorname{gr}(S(\Gamma(R))) = 3$.

Proof. (i) $S(\Gamma(R))$ is complete if and only if $xy \in S(R)$ for every $x, y \in S(R)$, if and only if $S(R)$ is a co-ideal of R.

(ii) Is clear from (i).

(iii) For any $x, y \in S(R)$, there is the path $x - 1 - y$ in $S(\Gamma(R))$, hence $\operatorname{diam}(S(\Gamma(R))) = 2$ if and only if $xy \notin S(R)$ for some $x, y \in S(R)$ if and only if $S(R)$ is not a co-ideal of R.

(iv) Let $S(R) = \{1, a_1, a_2\}$. By Lemma 3.6(ii), $S(R)$ is not a co-ideal of R, hence a_1, a_2 are not adjacent, so we have the path $a_1 - 1 - a_2$, which gives $\operatorname{gr}(S(\Gamma(R))) = \infty$.

Conversely, let $\operatorname{gr}(S(\Gamma(R))) = \infty$. Since R is not a co-semidomain, $|S(R)| \neq 1, 2$ by Lemma 2.4. We show that $|S(R)| = 3$. Suppose, on the contrary, $|S(R)| \geq 4$. Since $\operatorname{diam}(\Gamma(R)) = 2$ or 3 by Theorem 2.3(1), there exist $a_i, a_j \in S^*(R)$ such that $d(a_i, a_j) = 2$. Thus there is $a_k \in S^*(R)$ such that $a_i - a_k - a_j$ is a path in $\Gamma(R)$. Hence $a_i, a_j \in (1 : a_k)$, which gives $a_ia_j \in (1 : a_k) \subseteq S(R)$, because $(1 : a_k)$ is a co-ideal of R by Proposition 2.2(1). So $1 - a_i - a_j - 1$ is a cycle in $S(\Gamma(R))$ and $\operatorname{gr}(S(\Gamma(R))) = 3$, a contradiction.

(v) Let $|S(R)| \geq 4$. We show that there exist at least two elements $a_i, a_j \in S^*(R)$ such that $a_ia_j \in S^*(R)$. Since $\Gamma(R)$ is not complete and $\operatorname{diam}(\Gamma(R)) = 2$ or 3 by Theorem 2.3(1), there exist $a_i, a_j \in S^*(R)$ such that $d(a_i, a_j) = 2$ in $\Gamma(R)$. So there exists $a_k \in S^*(R)$ such that $a_i - a_k - a_j$ is a path in $\Gamma(R)$. Thus $a_i, a_j \in (1 : a_k)$, which gives $a_ia_j \in (1 : a_k) \subseteq S(R)$, because $(1 : a_k)$ is a co-ideal of R by Proposition 2.2(1). This implies $1 - a_i - a_j - 1$ is a cycle in $S(\Gamma(R))$ and $\operatorname{gr}(S(\Gamma(R))) = 3$.

Conversely, let $\mathrm{gr}(S(\Gamma(R))) = 3$, we show that $|S(R)| \geq 4$. By Lemma 2.4, $|S(R)| \neq 1, 2$. Suppose, on the contrary, $S(R) = \{1, a_1, a_2\}$. Since $\mathrm{gr}(S(\Gamma(R))) = 3$, a_1 and a_2 are adjacent in $S(\Gamma(R))$ which gives $a_1 a_2 \in S(R)$. Hence $S(R)$ is a co-ideal of R, which is a contradiction by Lemma 3.6(ii). Therefore $|S(R)| \geq 4$. □

Example 3.10. Let $R = (\mathbb{Z}^+, \mathrm{gcd}, \mathrm{lcm})$. It is clear that $S(R) = R \setminus \{0\}$ is a co-ideal of R and $S(\Gamma(R))$ is a complete graph.

Theorem 3.11. *Let R be an I-semiring. Then*

(i) $\omega(S(\Gamma(R))) = \max\{|P_\alpha| : P_\alpha$*'s are maximal elements of* $\sum$ *in Lemma 3.6(iii)*$\}$.

(ii) $\omega(\Gamma(R)) \leq \omega(S(\Gamma(R)))$.

Proof. (i) Let T be a clique of $S(\Gamma(R))$, then $F(T)$ is a co-ideal of R which $F(T) \subseteq S(R)$. Hence there exists a co-ideal P which is maximal in $\sum$ ($\sum$ is defined in Lemma 3.6(iii)) such that $T \subseteq F(T) \subseteq P$. Since each P in $\sum$ is a complete subgraph of $S(\Gamma(R))$ and T is a maximal complete subgraph of $S(\Gamma(R))$, $T = F(T) = P$. Thus the clique of $S(\Gamma(R))$ is one of the maximal co-ideals in $\sum$ which has maximal number of elements.

(ii) If $|\min(R)| = 1$, then there is nothing to prove. If $|\min(R)| = 2$, then $\omega(\Gamma(R)) = 2$ by Theorem 2.3(3). It is clear that $\omega(S(\Gamma(R))) \geq 2$, because $S(\Gamma(R))$ is connected by Theorem 3.3. Let $T \subseteq S(R)$ be a clique in $\Gamma(R)$. We show that T is a clique in $S(\Gamma(R))$. Let $x, y \in T$ and $|\min(R)| \geq 3$. So $\omega(\Gamma(R)) \geq 3$ by Theorem 2.3(3). Hence there exists $z \in T$ such that $x, y \neq z$. Since T is a clique, $y, x \in (1 : z)$ and so $xy \in (1 : z) \subseteq S(R)$. Hence $xy \in S(R)$, so T is a clique in $S(\Gamma(R))$. □

In general the equality of Theorem 3.11 is not true, as the following example shows.

Example 3.12. Let $R = (\{0, 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}, \mathrm{gcd}, \mathrm{lcm})$. We can easily see that $\omega(\Gamma(R)) = 3$ and $\omega(S(\Gamma(R))) = 5$.

Theorem 3.13. *Let R be an I-semiring. Then $S(\Gamma(R))$ is not a cycle graph.*

Proof. By the proof of Theorem 3.4, if $S(\Gamma(R))$ contains a cycle, then

$$\mathrm{gr}(\mathrm{S}(\Gamma(\mathrm{R}))) = 3.$$

Hence if $S(\Gamma(R))$ is a cycle graph, then $\mathrm{gr}(\mathrm{S}(\Gamma(\mathrm{R}))) = 3$. So $|S(R)| = 3$, a contradiction by Theorem 3.9. □

4. Total graph of semirings without identity element

In this section, we refine our results on $\mathrm{diam}(\mathrm{S}^*(\Gamma(\mathrm{R})))$, $\mathrm{gr}(\mathrm{S}^*(\Gamma(\mathrm{R})))$ and the relation between $S^*(\Gamma(R))$ and $S(\Gamma(R))$. At first we prove one of the most important properties of $S(R)$ which will be used in the sequel. We prove this property of $S(R)$ for I-semirings and we show that it maybe not true, when R is not an I-semiring.

Let T be a subset of R. We call , the set of elements of $R \setminus T$ by T^c.

Theorem 4.1. *Let R be an I-semiring, then $S(R) = \cup P_\alpha$, where P_α's are all minimal prime co-ideals of R.*

Proof. Let $P_\alpha \in \min(R)$ and $x \in P_\alpha$. Set $T = \{y + ix : y \in P^c, i \in \mathbb{N} \cup \{0\}\}$(Note that $0x = 0$). Then T is an additively closed subset of R which properly contains P^c. We show P^c is maximal with respect to property not containing 1. By Zorn's lemma, there exists maximal additively closed subset M of R with respect to the property of not containing 1 such that $P^c \subseteq M$. By Zorn's lemma there exists a strong co-ideal Q, which is maximal with respect to the property of not meeting M. We claim Q is prime. Let $a + b \in Q$ and $a, b \notin Q$. Therefore $Q \subset F(Q \cup \{a\})$ and $Q \subset F(Q \cup \{b\})$. Thus $F(Q \cup \{a\}) \cap T \neq \emptyset$ and $F(Q \cup \{b\}) \cap T \neq \emptyset$. Let $x \in F(Q \cup \{a\}) \cap T$ and $y \in F(Q \cup \{b\}) \cap T$. Then $x + y \in F(Q \cup \{a\}) \cap F(Q \cup \{b\}) \cap T$, because $F(Q \cup \{a\})$ and $F(Q \cup \{b\})$ are co-ideals of R and T is an additively closed subset of R . We claim that $F(Q \cup \{a\}) \cap F(Q \cup \{b\}) = Q$. Clearly $Q \subseteq F(Q \cup \{a\}) \cap F(Q \cup \{b\})$. For the reverse inclusion, let $z \in F(Q \cup \{a\}) \cap F(Q \cup \{b\})$. Then $z = r_1 + c_1 a^n = r_2 + c_2 b^m$ for some $r_1, r_2 \in R$, $c_1, c_2 \in Q$ and $n, m \in \mathbb{N}$. Since $c_1(a + b)^n = c_1 a^n + bt \in Q$ for some $t \in R$, $z + bt = r_1 + c_1 a^n + bt \in Q$. Hence $bt \in (Q : z)$. By Proposition 2.3, $b \in (Q : z)$. Therefore $c_2 b^m \in (Q : z)$. As $(Q : z)$ is a co-ideal, $z = r_2 + c_2 b^m \in (Q : z)$. Thus $z + z \in Q$, which gives $z \in Q$, since Q is subtractive. Therefore $F(Q \cup \{a\}) \cap F(Q \cup \{b\}) = Q$. Hence $x + y \in Q \cap T$ gives a contradiction. Thus $a \in Q$ or $b \in Q$, as needed. Since $Q \cap M = \emptyset$, $Q \subseteq M^c \subseteq P$. Since Q is prime and $P \in \min(R)$, $Q = M^c = P$. Hence $P^c = M$ is maximal with respect to the property of not containing 1. Thus $1 \in T$. Hence there exists a positive integer i and $y \notin P$ such that $y + ix = 1$. So $x \in S(R)$(note that $ix = x$ when $i \neq 0$ because $ix = x(1 + \cdots + 1) = x$).

Conversely, let $x \in S(R)$, so there exists $1 \neq y \in S(R)$ such that $x + y = 1$. Since $y \neq 1$, there exists $P_\alpha \in \min(R)$ such that $y \notin P_\alpha$, because $\cap_{P_\alpha \in \min(R)} P_\alpha = \{1\}$, by Proposition 2.2(2). Since $x + y = 1 \in P_\alpha$ and $y \notin P_\alpha$, $x \in P_\alpha$. Thus $S(R) = \cup_{P_\alpha \in \min(R)} P_\alpha$. □

In the following example, it is shown that the condition "R is an I-semiring" in Theorem 4.1 cannot be omitted.

Example 4.2. Let $R = (\mathbb{Z}^+ \times \mathbb{Z}^+, +, \cdot)$. Let I be a co-ideal of R and $(a, b) \in I$. Then $(a + r, b + s) \in I$ for each $(r, s) \in R$. So each co-ideal of R is infinite. It can be easily seen that $S(R) = \{(1, 1), (1, 0), (0, 1)\}$. So $S(R) \neq \cup P_\alpha$, where P_α's are all minimal prime co-ideals of R.

Remark 4.3. We can prove easily by using mathematical induction on n: Let $P_1, P_2, \ldots, P_n$ be subtractive prime co-ideals of a semiring R. If I is a strong co-ideal of R such that $I \subseteq \cup_{i=1}^n P_i$, then $I \subseteq P_r$ for some $1 \leq r \leq n$. This is useful in the proof of next propositions.

Theorem 4.4. *Let R be an I-semiring which is not co-semidomain. Then $S^*(\Gamma(R))$ is connected if and only if* $|\min(R)| \neq 2$. *Moreover if $S^*(\Gamma(R))$ is connected, then* $\mathrm{diam}(S^*(\Gamma(R))) \in \{1, 2\}$.

Proof. Let $S^*(\Gamma(R))$ be connected. Suppose, on the contrary, $\min(R) = \{P_1, P_2\}$, then $S(R) = P_1 \cup P_2$ by Theorem 4.1. By Proposition 2.2(2), $P_1 \cap P_2 = \{1\}$. If $1 \neq x \in P_1$ and $1 \neq y \in P_2$, then $xy \notin S(R)$, because if $xy \in S(R)$, then $xy \in P_1$ or $xy \in P_2$ which implies $x \in P_2 \cap P_1 = \{1\}$ or $y \in P_1 \cap P_2 = \{1\}$, by Proposition 2.2(1), a contradiction. So $xy \notin S(R)$, which implies non of elements of P_1 and P_2 are adjacent in $S(\Gamma(R))$. So $S^*(\Gamma(R))$ is not connected, a contradiction.

Conversely, suppose that $|\min(R)| \neq 2$. By Lemma 2.4, $|\min(R)| \neq 1$. Therefore $|\min(R)| \geq 3$. We show that $P_i \cap P_j \neq \{1\}$ for each $P_i, P_j \in \min(R)$. Let $P_i \cap P_j = \{1\}$ for some $P_i, P_j \in \min(R)$. We show $S(R) = P_i \cup P_j$. Let $x \in S^*(R) \setminus P_i \cup P_j$, so there exists $y \in S^*(R)$ such that $x + y = 1 \in P_i \cap P_j$. Since $x \notin P_i, P_j$, we have $y \in P_i \cap P_j = \{1\}$, a contradiction. Thus $S(R) \subseteq P_i \cup P_j$, hence $S(R) = P_i \cup P_j$ which implies $\min(R) = \{P_i, P_j\}$, because if there exists $P_k \in \min(R) \setminus \{P_i, P_j\}$, then $P_k \subseteq S(R) = P_i \cup P_j$ by Theorem 4.1, which implies $P_k \subseteq P_i$ or $P_k \subseteq P_j$ by Remark 4.3, a contradiction. Thus $P_i \cap P_j \neq \{1\}$ for each minimal prime co-ideals P_i, P_j of R. Now, let $x, y \in S^*(R)$. If $xy \in S^*(R)$, then $d(x, y) = 1$. Let $xy \notin S^*(R)$, so $x \in P_i$ and $y \in P_j$ where P_i, P_j are distinct minimal prime co-ideals of R. Choose $1 \neq z \in P_i \cap P_j$, then $x - z - y$ is a path in $S^*(\Gamma(R))$ and $d(x, y) = 2$. $\square$

We now state Theorem 4.5 which shows the relationship between the prime co-ideals which are contained in $S(R)$ and prime ideals which contain $Co-Reg(R)$. This theorem help us to characterize the relationship between $\mathrm{diam}(\Gamma(R))$, $|\min(R)|$ and the connectivity of $S^*(\Gamma(R))$ in Proposition 4.6 and Theorem 4.8.

Theorem 4.5. *Let R be an I-semiring and $S(R) = \cup_{i \in \Lambda} P_i$. Then*

(i) *$Q_i = R \setminus P_i$ is a prime k-ideal of R for each $i \in \Lambda$.*

(ii) *$Co - Reg(R) = \cap_{i \in \Lambda} Q_i$.*

Proof. (i) We show that $R \setminus P_i = Q_i$ is a prime ideal of R for each $i \in \Lambda$. Let $x, y \in Q_i$, if $x + y \notin Q_i$, then $x + y \in P_i$. Since P_i is a prime co-ideal of R, $x \in P_i$ or $y \in P_i$, a contradiction, so $x + y \in Q_i$ for each $x, y \in Q_i$. Let $r \in R$ and $x \in Q_i$, we show that $rx \in Q_i$. Let $rx \notin Q_i$, hence $rx \in P_i$ gives $x \in P_i$ by Proposition 2.2(1), a contradiction. So Q_i is an ideal of R. Now, let $xy \in Q_i$ and $x, y \notin Q_i$. So $x, y \in P_i$, which gives $xy \in P_i$, because P_i is a co-ideal of R, a contradiction. So Q_i is a prime ideal of R. Now, we show that Q_i is a k-ideal. Let $x, x + y \in Q_i$, we show that $y \in Q_i$. If $y \notin Q_i$, then $y \in P_i$, gives $x + y \in P_i$, because P_i is a co-ideal of R, a contradiction. So $y \in Q_i$ and Q_i is a k-ideal of R.

(ii) Is clear. $\square$

Proposition 4.6. *Let R be an I-semiring with* $\mathrm{diam}(\Gamma(R)) = 2$. *Then the following statements hold*:

(i) *If there exists an element in $S(R)$ that is contained in the unique minimal prime co-ideal of R, then* $|\min(R)| = 2$.

(ii) *If* $\min(R)$ *is a finite set, then* $|\min(R)| = 2$.

Proof. Note that since $\mathrm{diam}(\Gamma(R)) = 2$, $|S(R)| \geq 3$, which implies $|\min(R)| \neq 1$ by Lemma 2.4.

(i) Suppose $x \in P_1$ such that P_1 is the unique minimal prime co-ideal of R which contains x. Suppose, on the contrary, there are at least two other minimal prime co-ideals P_2, P_3. We claim that $P_2 \setminus P_1 \cup P_3 \neq \emptyset$. If not, then $P_2 \subseteq P_1 \cup P_3$. Hence by Remark 4.3, $P_2 \subseteq P_1$ or $P_2 \subseteq P_3$, a contradiction. Let $y \in P_2 \setminus P_1 \cup P_3$. We show $xy \in Co - Reg(R)$. Since $x \notin \cup_{i \neq 1, P_i \in \min(R)} P_i$, $x \in \cap_{i \neq 1} Q_i$ ($Q_i = R \setminus P_i$ is an ideal for each i), which gives $xy \in \cap_{i \neq 1} Q_i$. Since $y \notin P_1$, $y \in Q_1$, which gives $xy \in Q_1$. Thus $xy \in \cap_{\Lambda} Q_i = Co - Reg(R)$. By assumption $\mathrm{diam}(\Gamma(R)) = 2$. If $d(x, y) = 1$ in $\Gamma(R)$, then $x + y = 1 \in P_3$ which gives $x \in P_3$ or $y \in P_3$, because P_3 is a prime co-ideal of R, a contradiction. If $d(x, y) = 2$ in $\Gamma(R)$, then $x, y \in (1 : r)$ for some $r \in S^*(R)$, which gives $xy \in (1 : r) \subseteq S(R)$ by Proposition 2.2(1), a contradiction with $xy \in Co - Reg(R)$. Hence $|\min(R)| = 2$.

(ii) Let $\min(R) = \{P_1, P_2, \ldots, P_n\}$. By Remark 4.3, $P_1 \not\subseteq P_2 \cup \cdots \cup P_n$. Hence there exists an element in $S(R)$ that is contained in a unique minimal prime co-ideal P_1 of R. Thus $|\min(R)| = 2$ by (i). □

The following example shows that in general Proposition 4.6 is not true in the case $\mathrm{diam}(\Gamma(R)) = 3$.

Example 4.7. Let $R = (\{0, 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}, \mathrm{gcd}, \mathrm{lcm})$ (take $\mathrm{gcd}(0, 0) = 0$ and $\mathrm{lcm}(0, 0) = 0$). Then $S(R) = P_1 \cup P_2 \cup P_3$, where $P_1 = \{1, 2, 4, 5, 10, 20\}$, $P_2 = \{1, 2, 3, 4, 6, 12\}$ and $P_3 = \{1, 3, 5, 15\}$ are minimal prime co-ideals of R. As we see $\mathrm{diam}(\Gamma(R)) = 3$ and P_3 is the unique minimal prime co-ideal of R which contains 15.

Theorem 4.8. *Let R be an I-semiring which is not co-semidomain. If R has only finitely many minimal prime co-ideals, then*

(i) $\mathrm{diam}(\Gamma(R)) = 2$ *if and only if* $S^*(\Gamma(R))$ *is not connected and* $|S^*(R)| \geq 3$.

(ii) $\mathrm{diam}(\Gamma(R)) = 3$ *if and only if* $S^*(\Gamma(R))$ *is connected.*

Proof. (i) Let $\mathrm{diam}(\Gamma(R)) = 2$. So $|\min(R)| = 2$ by Proposition 4.6. Hence $S^*(\Gamma(R))$ is not connected by Theorem 4.4. It is clear that if $\mathrm{diam}(\Gamma(R)) = 2$ then $|S^*(R)| \geq 3$. Conversely, assume that $S^*(\Gamma(R))$ is not connected and $|S^*(R)| \geq 3$, so $|\min(R)| = 2$ by Theorem 4.4. Hence $\Gamma(R)$ is a complete bipartite graph by Theorem 2.3(2), which at least one of the parts has more than one vertex. Thus $\mathrm{diam}(\Gamma(R)) = 2$.

(ii) Let $\mathrm{diam}(\Gamma(R)) = 3$. So $\Gamma(R)$ is not complete bipartite. Hence $|\min(R)| \neq 2$ by Theorem 2.3(2) and Proposition 2.2(2). So $S^*(\Gamma(R))$ is connected.

Conversely, assume that $\mathrm{diam}(\Gamma(R)) \neq 3$. By (i), $\mathrm{diam}(\Gamma(R)) \neq 2$. Hence $\mathrm{diam}(\Gamma(R)) = 1$ and $|S^*(R)| = 2$ by Theorem 2.3(1). Since $S^*(\Gamma(R))$ is connected, $S(R)$ is a co-ideal of R, a contradiction by Lemma 3.6(ii). □

Theorem 4.9. *Let R be an I-semiring. Then* $\mathrm{gr}(S^*(\Gamma(R))) \in \{3, \infty\}$.

Proof. By Theorem 4.1, $S(R) = \cup P_i$, where P_i's are minimal prime co-ideals of R. If $|\min(R)| = 1$, then there is nothing to prove. So we consider two cases:

Case 1: $|\min(R)| = 2$, then $S(R) = P_1 \cup P_2$. If $|P_1| \geq 4$ or $|P_2| \geq 4$, then $\mathrm{gr}(S^*(\Gamma(R))) = 3$, because $P_i \setminus \{1\}$ is a complete subgraph of $S^*(\Gamma(R))$ for each $P_i \in min(R)$. If $|P_1|, |P_2| \leq 3$, then there is no cycle in P_1 and P_2. Also there is no cycle between the elements of P_1 and P_2 (because non of elements of P_1 and P_2 are adjacent by the proof of Theorem 4.4). Hence there is no cycle in $S^*(\Gamma(R))$, so $\mathrm{gr}(S^*(\Gamma(R))) = \infty$.

Case 2: $|\min(R)| \geq 3$. At first we show that for each $P_i \in \min(R)$, $|P_i| \geq 3$. Suppose,on the contrary, there exists $P_i \in \min(R)$ such that $P_i = \{1, a\}$. By the proof of Theorem 4.4, $P_i \cap P_j \neq \{1\}$ for each $P_j \in \min(R)$, hence $P_i \cap P_j = \{1, a\}$, which implies $P_i \subseteq P_j$, a contradiction. So $|P_i| \geq 3$ for any minimal prime co-ideal P_i of R . If $|P_i| \geq 4$ for some $P_i \in \min(R)$, then $\mathrm{gr}(S^*(\Gamma(R))) = 3$, because $P_i \setminus \{1\}$ is a complete subgraph of $S^*(\Gamma(R))$. Let $|P_i| = 3$ for each $P_i \in \min(R)$. Let $P_i = \{1, x_1, x_2\} \in \min(R)$. Since $x_1 \neq 1$, there exists $P_j \in \min(R)$ such that $x_1 \notin P_j$ (by Proposition 2.2(2), $\cap_{P\in\min(R)} P = \{1\}$), hence $P_i \cap P_j = \{1, x_2\}$, because $P_i \cap P_j \neq \{1\}$. Since $x_2 \neq 1$, there exists $P_j \neq P_k \in \min(R)$, such that $x_2 \notin P_k$, hence $P_i \cap P_k = \{1, x_1\}$. On the other hand, $P_k \cap P_j \neq \{1\}$, $x_2 \in P_j \setminus P_k$ and $x_1 \in P_k \setminus P_j$, so there exists $1 \neq a \in S(R)$ such that $a \in P_j \cap P_k$. Thus $P_j = \{1, x_1, a\}$ and $P_k = \{1, x_2, a\}$. So $x_1 - a - x_2 - x_1$ is a cycle in $S^*(\Gamma(R))$ and $\mathrm{gr}(S^*(\Gamma(R))) = 3$. □

Theorem 4.10. *Let R be an I-semiring which is not a co-semidomain and $S^*(\Gamma(R))$ a complete graph, then* $\min(R)$ *is infinite.*

Proof. Since $S^*(\Gamma(R))$ is a complete graph, $S(R)$ is a co-ideal of R, hence $S(R)$ is infinite by Lemma 3.6(ii). We show that $\min(R)$ is infinite. Suppose $\min(R)$ is finite, so $S(R) = \cup_{i=1}^{n} P_i$, where $P_i^{;}$ s are minimal prime co-ideals of R by Theorem 4.1. Since $S(R)$ is a co-ideal of R, $S(R) = P_i$ for some $1 \leq i \leq n$ by Remark 4.3. So P_i is the only minimal prime co-ideal of R by Theorem 4.1 . Thus $S(R) = P_i = \{1\}$ by Proposition 2.2(2), which gives R is a co-semidomain, a contradiction. □

Theorem 4.11. *Let R be an I-semiring which is not co-semidomain. The following statements hold*:

(i) *If* $|\min(R)| \neq 2$, *then* $\mathrm{diam}(S^*(\Gamma(R))) = \mathrm{diam}(S(\Gamma(R)))$;

(ii) *If* $|S(R)| \neq 4, 5$, *then* $\mathrm{gr}(S(\Gamma(R))) = \mathrm{gr}(S^*(\Gamma(R)))$.

Proof. (i) If $\mathrm{diam}(S(\Gamma(R))) = 1$, then $S(R)$ is a co-ideal of R. So $a_i a_j \in S^*(R)$ for each $a_i, a_j \in S^*(R)$ by Lemma 3.6(i), which implies $\mathrm{diam}(S^*(\Gamma(R))) = 1$. Hence $\mathrm{diam}(S(\Gamma(R))) = \mathrm{diam}(S^*(\Gamma(R)))$.

If $\mathrm{diam}(S(\Gamma(R))) = 2$, then there exist $a_i, a_j \in S^*(R)$ such that $a_i a_j \notin S^*(R)$. By Theorem 4.4, $\mathrm{diam}(S^*(\Gamma(R))) \in \{1, 2\}$. So there exists $a_k \in S^*(R)$ such that $a_i - a_k - a_j$ is a path in $S^*(\Gamma(R))$. Hence $\mathrm{diam}(S^*(\Gamma(R))) = 2$, so $\mathrm{diam}(S(\Gamma(R))) = \mathrm{diam}(S^*(\Gamma(R)))$.

(ii) By Lemma 2.4, $|S(R)| \neq 1, 2$. If $|S(R)| = 3$, then $\mathrm{gr}(S(\Gamma(R))) = \infty$ by Theorem 3.9. Since $|S^*(R)| = 2$, $S^*(\Gamma(R))$ contains no cycle, hence $gr(S^*(\Gamma(R))) = \infty$. So $\mathrm{gr}(S(\Gamma(R))) = \mathrm{gr}(S^*(\Gamma(R)))$. If $|S(R)| \geq 6$, then $\mathrm{gr}(S(\Gamma(R))) = 3$ by Theorem 3.9. We show that $\mathrm{gr}(S^*(\Gamma(R))) = 3$. If $|\min(R)| \geq 3$, then $\mathrm{gr}(S^*(\Gamma(R))) = 3$ by the proof of Theorem 4.9. If $|\min(R)| = 2$ and P_1, P_2 are two minimal prime co-ideals of R, then at least one of the P_i' s has more than 3 vertex, thus $\mathrm{gr}(S^*(\Gamma(R))) = 3$, because each $P_i \setminus \{1\}$ is a complete subgraph of $S^*(\Gamma(R))$. Hence $\mathrm{gr}(S(\Gamma(R))) = \mathrm{gr}(S^*(\Gamma(R)))$. □

The following example shows that if $|S(R)| = 4$ or 5, maybe $\mathrm{gr}(S(\Gamma(R))) \neq \mathrm{gr}(S^*(\Gamma(R)))$.

Example 4.12. (i) Let $R = (\{0, 1, 2, 4, 5, 10, 20\}, \mathrm{gcd}, \mathrm{lcm})$(take $\mathrm{gcd}(0, 0) = 0$ and $\mathrm{lcm}(0, 0) = 0$), then $S(R) = \{1, 2, 4, 5\}$. We can easily see that $\mathrm{gr}(S(\Gamma(R))) = 3$ and $\mathrm{gr}(S^*(\Gamma(R))) = \infty$.

(ii) Let $R = (\{0, 1, 2, 3, 4, 6, 9, 12, 18, 36\}, \mathrm{gcd}, \mathrm{lcm})$ (take $\mathrm{gcd}(0, 0) = 0$ and $\mathrm{lcm}(0, 0) = 0$), then $S(R) = \{1, 2, 3, 4, 9\}$. As we see $\mathrm{gr}(S(\Gamma(R))) = 3$ and $\mathrm{gr}(S^*(\Gamma(R))) = \infty$.

The following theorem shows the relationship between the girth of $\Gamma(R)$ and $S^*(\Gamma(R))$.

Theorem 4.13. *Let R be an I-semiring which is not a co-semidomain. Then* $\mathrm{gr}(S^*(\Gamma(R))) = \mathrm{gr}(\Gamma(R))$ *if and only if one of the following conditions hold*:

(i) $|\min(R)| \neq 2$.

(ii) $\min(R) = \{P_1, P_2\}$ *with* $|P_1| + |P_2| \leq 5$.

Proof. Assume that (i) holds. Since R is not a co-semidomain, $|\min(R)| \neq 1$. So $|\min(R)| \geq 3$. Thus $\omega(\Gamma(R)) \geq 3$ by Theorem 2.3(3), which implies $\mathrm{gr}(\Gamma(R)) = 3$. Also $\mathrm{gr}(S^*(\Gamma(R))) = 3$ by the proof of Theorem 4.9, so $\mathrm{gr}(S^*(\Gamma(R))) = \mathrm{gr}(\Gamma(R))$.

Now, assume that (ii) holds. Hence $\Gamma(R)$ is a complete bipartite graph with two parts $P_1 \setminus \{1\}$ and $P_2 \setminus \{1\}$ by Theorems 2.3(2). By assumption $|P_1| = |P_2| = 2$ or $|P_1| = 2$ and $|P_2| = 3$. So $\mathrm{gr}(\Gamma(R)) = \infty$. Also by the proof of Theorem 4.9, $\mathrm{gr}(S^*(\Gamma(R))) = \infty$. Thus $\mathrm{gr}(S^*(\Gamma(R))) = \mathrm{gr}(\Gamma(R)) = \infty$.

Conversely, let $\mathrm{gr}(\Gamma(R)) = \mathrm{gr}(S^*(\Gamma(R)))$. We consider two cases:

Case 1: $\mathrm{gr}(\Gamma(R)) = \mathrm{gr}(S^*(\Gamma(R))) = 3$. First, we show that $|\min(R)| \neq 2$. If $|\min(R)| = 2$, then $\Gamma(R)$ is a complete bipartite graph by Theorem 2.3(2), hence $\mathrm{gr}(\Gamma(R)) = 4$ or ∞, a contradiction with $\mathrm{gr}(\Gamma(R)) = \mathrm{gr}(S(\Gamma(R))) = 3$. So $|\min(R)| \neq 2$. Hence $|\min(R)| \geq 3$, because R is not co-semidomain.

Case 2: If $\mathrm{gr}(\Gamma(R)) = \mathrm{gr}(S^*(\Gamma(R))) = \infty$, then $\min(R) = \{P_1, P_2\}$ with $|P_1|, |P_2| \leq 3$ by the proof of Theorem 4.9. Hence $\Gamma(R)$ is a complete bipartite graph with two parts $P_1 \setminus \{1\}$ and $P_2 \setminus \{1\}$ by Theorems 2.3(2). We show that at least one of the P_i's has less than 3 element. If both of P_1 and P_2 have more than two elements then $\mathrm{gr}(\Gamma(R)) = 4$ (because $\Gamma(R)$ is a complete bipartite graph), which is a contradiction. Thus $|\min(R)| = 2$ and $|P_1| + |P_2| \leq 5$. $\square$

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