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AXIOMS FOR THE THEORY OF RANDOM VARIABLE STRUCTURES: AN ELEMENTARY APPROACH

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Abstract. The theory of random variable structures was first studied by Ben Yaacov in [2]. Ben Yaacov's axiomatization of the theory of random variable structures used an early result on the completeness theorem for Łukasiewicz's $[0,1]$-valued propositional logic. In this paper, we give an elementary approach to axiomatizing the theory of random variable structures. Only well-known results from probability theory are required here.

1. Introduction

The study of the theory of random variable structures was initiated by Ben Yaacov in [2]. He proved that the class of random variable structures is elementary and gave axioms for the theory of random variable structures, but his axiomatization of the theory used an early result on the completeness theorem for Łukasiewicz's $[0,1]$-valued propositional logic. In this paper, we use only well-known results from probability theory to give an elementary approach to axiomatizing the theory of random variable structures. Our approach is built on the axiomatization of the theory of probability algebras (*e.g.*, see [5]).

In the rest of this section, we introduce the definitions and notations in this paper. In Section 2, we axiomatize the theory of random variable structures. Only basic measure theoretic probability theory is required. The main result is Theorem 2.10.

Definitions and notations

We follow the notations in [3, Chapter 16] (see [5] for more details). Let the signature L_{Pr} denote the set $\{\mathbf{0}, \mathbf{1}, \cdot^{\mathsf{C}}, \cap, \cup, \mu\}$, where $\mathbf{0}$ and $\mathbf{1}$ are constant symbols, $\cdot^{\mathsf{C}}$ is a unary function symbol, $\cap$ and $\cup$ are binary function symbols,

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and μ is a unary predicate symbol. Among those symbols, $\cdot^{\complement}$ and μ are 1-Lipschitz, and $\cap$ and $\cup$ are 2-Lipschitz. Let the theory of probability algebras Pr consist of the following axioms:

(i) boolean algebra axioms
(ii) measure axioms:

$$\mu(\mathbf{1}) = 1 \text{ and } \sup_x \sup_y \left|\frac{\mu(x)+\mu(y)}{2} - \frac{\mu(x\cup y)+\mu(x\cap y)}{2}\right| = 0$$

(iii) $\sup_x \sup_y \big|d(x,y) - \mu(x\triangle y)\big| = 0$, where $x\triangle y$ denotes the symmetric difference: $x\triangle y = (x\cap y^{\complement}) \cup (x^{\complement}\cap y)$.

The theory of atomless probability algebras APr consists of axioms in Pr and the following one:

(iv) $\sup_x \inf_y |\mu(x\cap y) - \frac{\mu(x)}{2}| = 0$.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. For $A_1, A_2 \in \mathcal{F}$, we write $A_1 \sim_\mu A_2$ if the symmetric difference $A_1\triangle A_2$ has measure zero. Clearly we see that $\sim_\mu$ is an equivalence relation. Let $\hat{\mathcal{F}}$ denote the collection of equivalence classes of $\mathcal{F}$ modulo $\sim_\mu$. We call elements in $\hat{\mathcal{F}}$ *events*. Naturally, $\hat{\mathcal{F}}$ is a σ-algebra and μ induces a well-defined countably additive probability measure on $\hat{\mathcal{F}}$. We call $\hat{\mathcal{F}}$ the *measure algebra associated to* $(\Omega, \mathcal{F}, \mu)$. The L_{Pr}-structure $\mathcal{M} = (\hat{\mathcal{F}}, \mathbf{0}, \mathbf{1}, \cdot^{\complement}, \cap, \cup, \mu)$ is called a *probability algebra*. It is called an *atomless probability algebra* if the probability space $(\Omega, \mathcal{F}, \mu)$ is atomless; that is, for every $F \in \mathcal{F}$ with $\mu(F) > 0$ there is $G \in \mathcal{F}$ with $G \subseteq F$ such that $0 < \mu(G) < \mu(F)$.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. Consider the set of all $\mathcal{F}$-measurable functions $f\colon \Omega \to [0,1]$. Define the L^1-*metric* $d_1(f,g) := \int_\Omega |f-g|d\mu$ for all $\mathcal{F}$-measurable $f, g\colon \Omega \to [0,1]$. The set of such functions together with d_1 forms a pseudometric space, which is denoted by $\mathcal{L}^1\big((\Omega, \mathcal{F}, \mu), [0,1]\big)$, or simply by $\mathcal{L}^1(\mu, [0,1])$. For all $f, g \in \mathcal{L}^1(\mu, [0,1])$, we say that f is equal to g *almost surely*, and write $f =_{a.s.} g$ (or $f = g \quad a.s.$), if f is equal to g up to a null set. We denote the equivalence class of f under $=_{a.s.}$ by $[f]_{a.s.}$. For each $F \in \mathcal{F}$, let χ_F denote the characteristic function of F, and let $\mathbf{1}_F$ denote $[\chi_F]_{a.s.}$. Let N be $\{f \mid \int_\Omega |f|d\mu = 0\} = \{f \mid f = 0 \quad a.s.\}$. Then the quotient space

$$L^1\big((\Omega, \mathcal{F}, \mu), [0,1]\big) = \mathcal{L}^1\big((\Omega, \mathcal{F}, \mu), [0,1]\big)/N$$

is a metric space with the L^1-metric d_1, called an L^1-*space*. It is well known that the space $L^1\big((\Omega, \mathcal{F}, \mu), [0,1]\big)$ with the L^1-metric d_1 is a complete metric space. When the underlying probability space is clear, $L^1\big((\Omega, \mathcal{F}, \mu), [0,1]\big)$ is often abbreviated as $L^1(\mu, [0,1])$, or just $L^1(\mathcal{F}, [0,1])$ when the underlying set Ω and the probability measure μ are clear. We write $L^1\big((\Omega, \mathcal{F}, \mu), \{0,1\}\big)$ for the set of equivalence classes of characteristic functions in the space $\mathcal{L}^1\big((\Omega, \mathcal{F}, \mu), [0,1]\big)$. Let $\mathbb{D}$ denote the dyadic numbers in $[0,1]$. We write $L^1\big((\Omega, \mathcal{F}, \mu), \mathbb{D}\big)$ for the set of equivalence classes of $\mathbb{D}$-valued simple functions in $\mathcal{L}^1\big((\Omega, \mathcal{F}, \mu), [0,1]\big)$.

Clearly,

$$L^1\big((\Omega,\mathcal{F},\mu),\{0,1\}\big)\subseteq L^1\big((\Omega,\mathcal{F},\mu),\mathbb{D}\big)\subseteq L^1\big((\Omega,\mathcal{F},\mu),[0,1]\big).$$

Moreover, we have that $L^1\big((\Omega,\mathcal{F},\mu),\{0,1\}\big)$ is closed in $L^1\big((\Omega,\mathcal{F},\mu),[0,1]\big)$, and $L^1\big((\Omega,\mathcal{F},\mu),\mathbb{D}\big)$ is dense in $L^1\big((\Omega,\mathcal{F},\mu),[0,1]\big)$. Let A be a subset of $L^1\big((\Omega,\mathcal{F},\mu),[0,1]\big)$. Let $\sigma(A)\subseteq\mathcal{F}$ denote the σ-subalgebra of $\mathcal{F}$-measurable sets generated by the random variables in the equivalence classes in A. We call *$\sigma(A)$ the σ-algebra generated by A*.

The elements in $L^1\big((\Omega,\mathcal{F},\mu),[0,1]\big)$ are not $\mathcal{F}$-measurable functions, but equivalence classes of them. In probability theory, most useful functions, relations, and maps (such as continuous functions, integrals, inequality relations, conditional expectations) on measurable functions are well-defined on the equivalence classes of those functions. Therefore, it causes no harm (and is more readable) to denote an equivalence class in $L^1\big((\Omega,\mathcal{F},\mu),[0,1]\big)$ by a member of the class.

A ($[0,1]$-valued) *random variable structure* is based on a set of the form $L^1\big((\Omega,\mathcal{F},\mu),[0,1]\big)$, where $(\Omega,\mathcal{F},\mu)$ is a probability space. It is called an *atomless random variable structure*, if its underlying probability space is atomless. We use the setting of continuous logic [3] ([4] is also a good reference) to discuss the model theory of random variable structures. Here we consider the signature $L_{\mathsf{RV}}=\{\mathbf{0},\neg,\dot-,\frac{1}{2},I\}$, where $\mathbf{0}$ is a constant symbol, $\dot-$ is a binary function symbol, $\neg$ and $\frac{1}{2}$ are unary function symbols, and I is a unary predicate symbol. Recall that on M^n, we take the maximum metric. Among those symbols, $\neg$ is 1-Lipschitz, $\frac{1}{2}$ is $\frac{1}{2}$-Lipschitz, $\dot-$ is 2-Lipschitz and I is 1-Lipschitz.

We interpret the symbols of L_{RV} in M as follows:

$\mathbf{0}^{\mathcal{M}}(\omega)=0$ for all $\omega\in\Omega$
$\neg^{\mathcal{M}}(f)=1-f$ for all $f\in M$
$(\dot-)^{\mathcal{M}}(f,g)=f\dot- g=\max(f-g,0)$ for all $f,g\in M$
$(\frac{1}{2})^{\mathcal{M}}f=f/2$ for all $f\in M$
$I^{\mathcal{M}}(f)=\int_\Omega f\,d\mu$ for all $f\in M$
$d^{\mathcal{M}}(f,g)=\int_\Omega |f-g|\,d\mu$ for all $f,g\in M$

Then $\mathcal{M}=\Big(L^1\big((\Omega,\mathcal{F},\mu),[0,1]\big),\mathbf{0},\neg,\dot-,\frac{1}{2},I,d\Big)$ is an L_{RV}-structure. Note that the L_{RV}-prestructure associated to $\mathcal{M}$ is $\Big(\mathcal{L}^1\big((\Omega,\mathcal{F},\mu),[0,1]\big),\mathbf{0},\neg,\dot-,\frac{1}{2},I,d\Big)$. Let $\mathcal{RV}$ denote the class of all random variable structures as L_{RV}-structures and let $\mathcal{ARV}$ denote the class of all atomless random variable structures as L_{RV}-structures. In Section 2, we show that the classes $\mathcal{RV}$ and $\mathcal{ARV}$ are elementary.

In the signature L_{RV}, we also use the following symbols as shorthand for expressions built from symbols in L_{RV}:

$$\mathbf{1}=\neg\mathbf{0}$$

$$x\dot+y=\neg(\neg x\dot- y)$$

$$x \wedge y = x \dot{-} (x \dot{-} y)$$
$$x \vee y = \neg(\neg x \wedge \neg y)$$
$$\frac{x}{2} = \frac{1}{2}x$$

By induction on n, we define

$$\frac{1}{2^n}x = \frac{x}{2^n} = \frac{1}{2}\frac{x}{2^{n-1}} \text{ for all } n \in \mathbb{N},$$
$$x_1 \dot{+} x_2 \dot{+} \cdots \dot{+} x_n = (x_1 \dot{+} x_2 \dot{+} \cdots \dot{+} x_{n-1}) \dot{+} x_n,$$
$$x_1 \wedge x_2 \wedge \cdots \wedge x_n = (x_1 \wedge x_2 \wedge \cdots \wedge x_{n-1}) \wedge x_n,$$

and

$$x_1 \vee x_2 \vee \cdots \vee x_n = (x_1 \vee x_2 \vee \cdots \vee x_{n-1}) \vee x_n.$$

Let $\mathbb{D}$ denote the set of dyadic numbers in [0,1]. Consider $r \in \mathbb{D}$. Suppose $r = \frac{m}{2^n}$, where $m, n \in \mathbb{N}$, $0 < m < 2^n$, and $2 \nmid m$. We define

$$rx = \underbrace{\frac{x}{2^n} \dot{+} \frac{x}{2^n} \dot{+} \cdots \dot{+} \frac{x}{2^n}}_{m \text{ times}}.$$

When $r = 0$ or 1, we write $0x$ for $\mathbf{0}$ and $1x$ for x.

2. Axioms for RV

In this section, we give axioms for the theory of (atomless) random variable structures. Only basic measure theoretic probability theory is assumed. The main result is Theorem 2.10.

The theory RV consists of the following axioms:

(E1): $\sup_x \inf_y \max\Big(I(y \wedge \neg y), \big|I(x \wedge \neg x) - d(x,y)\big|\Big) = 0$

(E2): $\sup_x \Big|I(x \wedge \neg x) - \inf_y \big(I(y \wedge \neg y) \dot{+} d(x,y)\big)\Big| = 0$

(APPR): for all $n \in \mathbb{N}$,

$$\sup_x \inf_{y_1,\ldots,y_{2^n}} \big(d(x, \frac{1}{2^n}y_1 \dot{+} \frac{1}{2^n}y_2 \dot{+} \cdots \dot{+} \frac{1}{2^n}y_{2^n}) \dot{+} \max_{1 \le i \le 2^n} I(y_i \wedge \neg y_i)\big) \dot{-} \frac{1}{2^n} = 0$$

(ADD): $\sup_x \sup_y \frac{1}{2}\big|I(x) - \big(I(x \dot{-} y) + I(y \wedge x)\big)\big| = 0$

(C): $\sup_x I(\mathbf{0} \dot{-} x) = 0$; $\sup_x d(x \dot{-} \mathbf{0}, x) = 0$; $|I(\mathbf{1}) - 1| = 0$

(H1): $\sup_x \sup_y d\big(\frac{x \dot{-} y}{2}, \frac{x}{2} \dot{-} \frac{y}{2}\big) = 0$

(H2): $\sup_x \sup_y d((\frac{x}{2} \dot{+} \frac{y}{2}) \dot{-} \frac{x}{2}, \frac{y}{2}) = 0$; $\sup_x d(\frac{x}{2} \dot{+} \frac{x}{2}, x) = 0$

(H3): $\sup_x \sup_y d\big(\frac{x}{2} \vee \frac{y}{2}, \frac{1}{2}(x \vee y)\big) = 0$

(H4): $\sup_x \sup_y I\big((\frac{1}{2}x \wedge y) \dot{-} (x \wedge y)\big) = 0$

(H5): $\sup_x \sup_y \frac{1}{2}(\frac{x}{2} \dot{+} \frac{y}{2}) = \frac{x}{4} \dot{+} \frac{y}{4}$

(MET): $\sup_x \sup_y \frac{1}{2}\big|d(x,y) - \big(I(x \dot{-} y) + I(y \dot{-} x)\big)\big| = 0$

(N): $d(\neg\mathbf{1}, \mathbf{0}) = 0$; $\sup_x \sup_y d(x \dot{-} y, \neg y \dot{-} \neg x) = 0$

(P1): $\sup_{x_1} \sup_{x_2} \sup_{y_1} \sup_{y_2} d(x_1 \dot{+} y_1, x_2 \dot{+} y_2) \dot{-} \big(d(x_1,x_2) \dot{+} d(y_1,y_2)\big) = 0$

(P2): $\sup_x \sup_y \Big(d(x \dot{+} y, x \vee y) \dot{-} \max\big(I(x \wedge \neg x), I(y \wedge \neg y)\big)\Big) = 0$

(P3): $\sup_x \sup_y \sup_z d\big((x \dotplus y) \dotplus z, x \dotplus (y \dotplus z)\big) = 0$
(S1): $\sup_x \sup_y \sup_z d\big((z \vee y) \dot{-} x, (z \dot{-} x) \vee y\big) \dot{-} I(x \wedge y) = 0$
(S2): $\sup_x \sup_y \sup_z I\Big(\big((x \dotplus y) \wedge z\big) \dot{-} \big((x \wedge z) \dotplus (y \wedge z)\big)\Big) = 0$
(L1): $\sup_x \sup_y d(x \vee y, y \vee x) = 0$; $\quad \sup_x \sup_y d(x \wedge y, y \wedge x) = 0$
(L2): $\sup_x \sup_y \sup_z d\big(x \vee (y \vee z), (x \vee y) \vee z\big) = 0$
(L3): $\sup_x \sup_y \sup_z d\big(x \wedge (y \wedge z), (x \wedge y) \wedge z\big) = 0$
(L4): $\sup_x \sup_y d\big(x \vee (x \wedge y), x\big) = 0$; $\quad \sup_x \sup_y d\big(x \wedge (x \vee y), x\big) = 0$
(L5): $\sup_x \sup_y \sup_z d\big(x \vee (y \wedge z), (x \vee y) \wedge (x \vee z)\big) = 0$
(L6): $\sup_x \sup_y \sup_z d\big(x \wedge (y \vee z), (x \wedge y) \vee (x \wedge z)\big) = 0$

Axioms (L1) to (L6) are the axioms for distributive lattices.

Let ARV be RV together with the following axiom:

(NA): $\sup_x \inf_y \big(\max(I(y \wedge \neg y), \big|I(y \wedge x) - \frac{I(x)}{2}\big|)\big) = 0$

Proposition 2.1. *Every random variable structure $\mathcal{M} = (M, \mathbf{0}, \neg, \dot{-}, \frac{1}{2}, I, d)$ is a model of* RV*. Further, if $\mathcal{M}$ is an atomless random variable structure, then it is a model of* ARV*.*

Proof. Assume $M = L^1\big((\Omega, \mathcal{F}, \mu), [0,1]\big)$ for some probability space $(\Omega, \mathcal{F}, \mu)$. By [3, Theorem 3.7], it suffices to consider axioms in the L_{RV}-prestructure $M_0 = \mathcal{L}^1\big((\Omega, \mathcal{F}, \mu), [0,1]\big)$. Note that $f \wedge g = \min(f, g)$ and $f \vee g = \max(f, g)$ for all $f, g \in M_0$. Most axioms are easy to verify and some of them are just arithmetic. We will check Axioms (E1), (E2), (APPR), and leave the rest to the readers.

(E1) and (E2): We consider

$$\mathcal{X} = \{f \in M_0 \mid f \text{ is a characteristic function}\}.$$

For all $f \in M_0$, we have

$$|f(\omega) - \chi_{\{f \geq \frac{1}{2}\}}(\omega)| \leq |f(\omega) - \chi_A(\omega)| \text{ for all } A \in \mathcal{F} \text{ and all } \omega \in \Omega,$$

whereby $\mathrm{dist}(f, \mathcal{X}) = d(f, \chi_{\{f \geq \frac{1}{2}\}})$. Also we note that

$$d(f, \chi_{\{f \geq \frac{1}{2}\}}) = \int_\Omega |f - \chi_{\{f \geq \frac{1}{2}\}}| d\mu = \int_\Omega |f \wedge (1 - f)| d\mu = I^{\mathcal{M}_0}(f \wedge \neg f),$$

whereby $\mathrm{dist}(f, \mathcal{X}) = I^{\mathcal{M}}(f \wedge \neg f)$. Then to verify Axioms (E1) and (E2), we need only check that

$$\sup_x \inf_y \max\big(\mathrm{dist}(y, \mathcal{X}), \big|\mathrm{dist}(x, \mathcal{X}) - d(x, y)\big|\big) = 0$$

and

$$\sup_x \big|\mathrm{dist}(x, \mathcal{X}) - \inf_y \big(\mathrm{dist}(y, \mathcal{X}) \dotplus d(x, y)\big)\big| = 0.$$

Both are clear here.

(APPR): This axiom is an approximation result from real analysis. For all $n \in \mathbb{N}$ and $f \in M_0$, let $g_i = \chi_{\{f \geq \frac{i-1}{2^n}\}}$, for every $1 \leq i \leq 2^n$. Then

$$\begin{aligned}\frac{1}{2^n} g_1 \dotplus \cdots \dotplus \frac{1}{2^n} g_{2^n} &= \sum_{i=1}^{2^n} \frac{1}{2^n} g_i \\ &= \frac{1}{2^n}\chi_{\{0 \leq f < \frac{1}{2^n}\}} + \frac{2}{2^n}\chi_{\{\frac{1}{2^n} \leq f < \frac{2}{2^n}\}} + \cdots + \frac{2^n}{2^n}\chi_{\{\frac{2^n-1}{2^n} \leq f\}}.\end{aligned}$$

Thus

$$\begin{aligned}d\big(f, \frac{1}{2^n} g_1 \dotplus \cdots \dotplus \frac{1}{2^n} g_{2^n}\big) &= d\big(f, \sum_{i=1}^{2^n} \frac{i}{2^n}\chi_{\{\frac{i-1}{2^n} \leq f < \frac{i}{2^n}\}}\big) \\ &= \int_\Omega \big| f - \sum_{i=1}^{2^n} \frac{i}{2^n}\chi_{\{\frac{i-1}{2^n} \leq f < \frac{i}{2^n}\}}\big| d\mu \\ &\leq \int_\Omega \sum_{i=1}^{2^n} \frac{1}{2^n}\chi_{\{\frac{i-1}{2^n} < f \leq \frac{i}{2^n}\}} d\mu = \frac{1}{2^n}\mu(\Omega) = \frac{1}{2^n}.\end{aligned}$$

Also note that $I(g_i \wedge \neg g_i) = 0$ for every $1 \leq i \leq 2^n$. Consequently, (APPR) is true in $\mathcal{M}$.

When $(\Omega, \mathcal{F}, \mu)$ is atomless, clearly $\Big(L^1\big((\Omega, \mathcal{F}, \mu), [0,1]\big), \mathbf{0}, \neg, \dot{-}, \frac{1}{2}, I, d\Big)$ satisfies (NA). Hence it is a model of ARV. □

Indeed, RV also axiomatizes the class $\mathcal{RV}$ (see Theorem 2.10) and then ARV axiomatizes the class $\mathcal{ARV}$ (see Corollary 2.11), which are the main results from this section. Toward the proof of Theorem 2.10, we prove the following results about models of RV. In the following arguments, we interpret symbols of L_{RV} in a given model $\mathcal{M}$ of RV without putting $\mathcal{M}$ explicitly into the notations, for easier readability.

Fact 2.2. *Let $\mathcal{M}$ be a model of* RV*. For all $x, y \in M$, we have the following properties*:

(i) $I(x) = 0$ *if and only if* $x = \mathbf{0}$.
(ii) $\mathbf{0} \dot{-} x = \mathbf{0}$ *and* $x \dot{-} \mathbf{1} = \mathbf{0}$.
(iii) $\neg x = \mathbf{1} \dot{-} x$ *and* $\neg\neg x = x$.
(iv) $x \wedge \mathbf{0} = \mathbf{0}$ *and* $x \wedge \mathbf{1} = x$.
(v) $x \vee \mathbf{0} = x$ *and* $x \vee \mathbf{1} = \mathbf{1}$.
(vi) $x \dot{-} x = \mathbf{0}$.
(vii) $I(\neg x) = 1 - I(x)$.
(viii) $\frac{I(x)}{2} = I(\frac{x}{2})$.
(ix) $x \dotplus y = y \dotplus x$.
(x) $d(\frac{x}{2}, \frac{y}{2}) = \frac{1}{2} d(x, y)$.
(xi) $\mathbf{1} \dotplus x = x \dotplus \mathbf{1} = \mathbf{1}$ *and* $\mathbf{0} \dotplus x = x \dotplus \mathbf{0} = x$.
(xii) *If* $\frac{x}{2} \dotplus \frac{y}{2} = \frac{\mathbf{1}}{2}$, *then* $x = \neg y$.

Proof. (i) By (C), we have $\mathbf{0} \dot{-} \mathbf{0} = \mathbf{0}$ and $0 = I(\mathbf{0} \dot{-} \mathbf{0}) = I(\mathbf{0})$. For the converse, suppose $I(x) = 0$. Using (MET) and (C), we have

$$d(x, \mathbf{0}) = I(x \dot{-} \mathbf{0}) + I(\mathbf{0} \dot{-} x) = I(x) + I(\mathbf{0}) = I(x) = 0,$$

and thus $x = \mathbf{0}$.

(ii) By (i) and (C), we have $\mathbf{0} \dot{-} x = \mathbf{0}$ for all x. In particular, $\mathbf{0} \dot{-} \neg x = \mathbf{0}$, so using (N), we have $x \dot{-} \mathbf{1} = \neg \mathbf{1} \dot{-} \neg x = \mathbf{0} \dot{-} \neg x = \mathbf{0}$.

(iii) By (N) and (C), we have $\mathbf{1} \dot{-} x = \neg x \dot{-} \neg \mathbf{1} = \neg x \dot{-} \mathbf{0} = \neg x$. Using (L1), (ii) and (C), we get $\neg\neg x = \mathbf{1} \dot{-} (\mathbf{1} \dot{-} x) = \mathbf{1} \wedge x = x \wedge \mathbf{1} = x \dot{-} (x \dot{-} \mathbf{1}) = x$.

(iv) By (L1) and (ii), we have $x \wedge \mathbf{0} = \mathbf{0} \wedge x = \mathbf{0} \dot{-} (\mathbf{0} \dot{-} x) = \mathbf{0}$. By (ii) and (C), we get $x \wedge \mathbf{1} = x \dot{-} (x \dot{-} \mathbf{1}) = x \dot{-} \mathbf{0} = x$.

(v) Using (iv) and (iii), we have

$$x \vee \mathbf{0} = \neg(\neg x \wedge \neg \mathbf{0}) = \neg(\neg x \wedge \mathbf{1}) = \neg(\neg x) = x.$$

Using (N) and (iv), we have $x \vee \mathbf{1} = \neg(\neg x \wedge \neg \mathbf{1}) = \neg(\neg x \wedge \mathbf{0}) = \neg \mathbf{0} = \mathbf{1}$.

(vi) Setting $x = y$ in (MET), we have $I(x \dot{-} x) + I(x \dot{-} x) = 0$; this yields $x \dot{-} x = \mathbf{0}$ using (i).

(vii) By (C), (ADD), (iii), and (iv), we have

$$1 = I(\mathbf{1}) = I(\mathbf{1} \dot{-} x) + I(x \wedge \mathbf{1}) = I(\neg x) + I(x),$$

whereby $I(\neg x) = 1 - I(x)$.

(viii) By (H2), we have $x \dot{-} \frac{x}{2} = (\frac{x}{2} \dot{+} \frac{x}{2}) \dot{-} \frac{x}{2} = \frac{x}{2}$. Then by (ADD) and (L1), we have

$$\begin{aligned} I(x) &= I(x \dot{-} \frac{x}{2}) + I(\frac{x}{2} \wedge x) = I(\frac{x}{2}) + I(x \wedge \frac{x}{2}) = I(\frac{x}{2}) + I\big(x \dot{-} (x \dot{-} \frac{x}{2})\big) \\ &= I(\frac{x}{2}) + I(\frac{x}{2}). \end{aligned}$$

(ix) To show $x \dot{+} y = y \dot{+} x$, it suffices to show $\neg(\neg x \dot{-} y) = \neg(\neg y \dot{-} x)$, which follows from (N) and (iii).

(x) By (MET), (H1), and (viii), we have

$$\begin{aligned} d(\frac{x}{2}, \frac{y}{2}) &= I(\frac{x}{2} \dot{-} \frac{y}{2}) + I(\frac{y}{2} \dot{-} \frac{x}{2}) = I(\frac{x \dot{-} y}{2}) + I(\frac{y \dot{-} x}{2}) \\ &= \frac{I(x \dot{-} y)}{2} + \frac{I(y \dot{-} x)}{2} = \frac{d(x, y)}{2}. \end{aligned}$$

(xi) By (N) and (ii), we have $\mathbf{1} \dot{+} x = \neg(\neg \mathbf{1} \dot{-} x) = \neg(\mathbf{0} \dot{-} x) = \neg \mathbf{0} = \mathbf{1}$. Then by (ix), we have $x \dot{+} \mathbf{1} = \mathbf{1} \dot{+} x = \mathbf{1}$. By (iii), $\mathbf{0} \dot{+} x = \neg(\neg \mathbf{0} \dot{-} x) = \neg(\mathbf{1} \dot{-} x) = \neg\neg x = x$. By (ix), we have $x \dot{+} \mathbf{0} = \mathbf{0} \dot{+} x = x$.

(xii) By (H2), we have $\frac{\mathbf{1}}{2} \dot{-} \frac{x}{2} = (\frac{x}{2} \dot{+} \frac{y}{2}) \dot{-} \frac{x}{2} = \frac{y}{2}$. Then by (H1), we have $\mathbf{1} \dot{-} x = y$, whereby $x = \neg y$ by (iii). □

Proposition 2.3. *Let $\mathcal{M}$ be a model of* RV. *Let $D = \{x \in M \mid I(x \wedge \neg x)\} = 0$. For all $x, y \in D$, define $x^{\complement} := \neg x$, $x \cap y := x \wedge y$, $x \cup y := x \vee y$, and $\mu(x) := I(x)$. Then D is a uniformly definable set in M and $(D, \mathbf{0}, \mathbf{1}, \cdot^{\complement}, \cap, \cup, \mu)$ is a model of*

Pr. *Moreover, if M is of the form $L^1\big((\Omega,\mathcal{F},\mu),[0,1]\big)$ for the probability space $(\Omega,\mathcal{F},\mu)$, then D is $L^1\big((\Omega,\mathcal{F},\mu),\{0,1\}\big)$.*

Proof. Let $\mathcal{M}$ be a model of RV and let $D=\{y\in M \mid I(y\wedge\neg y)=0\}$. By Fact 2.2(i), we know $D=\{y\in M \mid y\wedge\neg y=\mathbf{0}\}$. By [3, Theorem 9.12], (E1), and (E2), we know that $I(x\wedge\neg x)=\operatorname{dist}(x,D)$. Hence, D is a uniformly definable set in M; that is, the defining formula for D does not depend on M.

First, we want to show that D is closed under $\neg,\wedge,\vee$ and also $\mathbf{0},\mathbf{1}\in D$. For all $x,y\in D$, we have $x\wedge\neg x=y\wedge\neg y=\mathbf{0}$. Since $\neg\neg x=x$ and $x\wedge\neg x=\neg x\wedge x$, we get $\neg x\wedge\neg(\neg x)=\mathbf{0}$, whence $\neg x\in D$. To show $x\wedge y\in D$, it suffices to show $(x\wedge y)\wedge\neg(x\wedge y)=(x\wedge y)\wedge(\neg x\vee\neg y)=\mathbf{0}$. By the fact that $\vee$ and $\wedge$ satisfy the distributive lattice axioms, we need only show that $\big((x\wedge y)\wedge\neg x\big)\vee\big((x\wedge y)\wedge\neg y\big)=\mathbf{0}$, which is true since $x\wedge\neg x=y\wedge\neg y=\mathbf{0}$. Then since $x\vee y=\neg(\neg x\wedge\neg y)$, we know $x\vee y\in D$ as well. By Fact 2.2(iv), we know $\mathbf{0}\wedge\neg\mathbf{0}=\mathbf{0}$, and thus $\mathbf{0}\in D$. Hence, $\mathbf{1}=\neg\mathbf{0}\in D$.

Second, for all $x,y\in D$, define $x\cap y:=x\wedge y$, $x\cup y:=x\vee y$, and $x^{\mathsf{C}}=\neg x$. We show that $(D,\mathbf{0},\mathbf{1},\cdot^{\mathsf{C}},\cap,\cup,\mu)$ is a model of Pr. For all $x\in D$, we have $x\wedge\neg x=\mathbf{0}$, and then $\neg(x\wedge\neg x)=\neg\mathbf{0}=\mathbf{1}$. Then by Fact 2.2(iii) and (L1), we have $\mathbf{1}=\neg(x\wedge\neg x)=\neg x\vee\neg\neg x=\neg x\vee x=x\vee\neg x$. Because $\wedge,\vee$ also satisfy the axioms for distributive lattices, we see that $(D,\mathbf{0},\mathbf{1},\cdot^{\mathsf{C}},\cup,\cap)$ satisfies all boolean algebra axioms in Pr.

For all $x\in D$, define $\mu(x):=I(x)$. By Fact 2.2(i) and (vii), we have $\mu(\mathbf{0})=0$ and $\mu(\mathbf{1})=1$. For all $x,y\in D$, we have $I(x\vee y)=I\big(\neg(\neg x\wedge\neg y)\big)=1-I(\neg x\wedge\neg y)$, by Fact 2.2(vii). By (ADD) and (N), we have

$$I(\neg y)=I(\neg y\dot{-}\neg x)+I(\neg x\wedge\neg y)=I(x\dot{-}y)+I(\neg x\wedge\neg y),$$

and thus $I(\neg x\wedge\neg y)=I(\neg y)-I(x\dot{-}y)$. Hence,

$$I(x\vee y)=1-I(\neg x\wedge\neg y)=1-(I(\neg y)-I(x\dot{-}y)),$$

whence $I(x\vee y)=I(y)+I(x\dot{-}y)$ by Fact 2.2(vii). By (ADD), we have

$$I(x)=I(x\dot{-}y)+I(y\wedge x).$$

Then by eliminating the term $I(x\dot{-}y)$, we get $I(x\vee y)+I(y\wedge x)=I(x)+I(y)$, whence $I(x\cup y)+I(x\cap y)=I(x)+I(y)$. Therefore $\mu(x\cup y)+\mu(x\cap y)=\mu(x)+\mu(y)$. Consequently, $(D,\mathbf{0},\mathbf{1},\cdot^{\mathsf{C}},\cap,\cup,\mu)$ satisfies the measure axioms in Pr.

Next, for all $x,y\in D$, by (P2) we know $d(x\dot{+}y,x\vee y)=0$, and thus $x\dot{+}y=x\vee y$. Since $(x\dot{+}y)=\neg(\neg x\dot{-}y)$, by Fact 2.2(iii) we have $x\dot{-}y=\neg(\neg x\dot{+}y)$. Then by Fact 2.2(iii), we have $x\dot{-}y=\neg(\neg x\dot{+}y)=\neg(\neg x\vee y)=\neg\neg x\wedge\neg y=x\wedge\neg y$. By (MET), we have

$$\begin{aligned}d(x,y)&=I(x\dot{-}y)+I(y\dot{-}x)=I(x\wedge\neg y)+I(y\wedge\neg x)=\mu(x\cap y^{\mathsf{C}})+\mu(y\cup x^{\mathsf{C}})\\&=\mu(x\triangle y).\end{aligned}$$

Hence, $(D, \mathbf{0}, \mathbf{1}, \cdot^{\complement}, \cap, \cup, \mu)$ satisfies Axiom (iii) in Pr. Since d is a complete metric on M and D is a zeroset (thus it is closed), the metric d is complete on D.

Since $x^{\complement} = \neg x$ for all $x \in D$ and $\neg$ is 1-Lipschitz, we get $\cdot^{\complement}$ is 1-Lipschitz. By (P2) and Fact 2.2(iii), for all $x, y \in D$, we have

$$x \cap y = x \wedge y = \neg\neg(\neg\neg x \wedge \neg\neg y) = \neg(\neg x \vee \neg y) = \neg(\neg x \dotplus \neg y) = \neg\neg x \dot{-} \neg y = x \dot{-} \neg y.$$

Since $\neg$ is 1-Lipschitz and $\dot{-}$ is 2-Lipschitz, we have that $\cap$ is 2-Lipschitz. Since $x \cup y = (x \cap y)^{\complement}$ for all $x, y \in D$, we know that $\cup$ is 2-Lipschitz. Since $\mu(x) = I(x)$ for all $x \in D$ and I is 1-Lipschitz, we know that μ is 1-Lipschitz. Hence, $(D, \mathbf{0}, \neg, \cdot^{\complement}, \cap, \cup, \mu)$ is an L_{Pr}-structure. Therefore, $(D, \mathbf{0}, \neg, \cdot^{\complement}, \cap, \cup, \mu)$ is a model of Pr.

Suppose M is of the form $L^1\big((\Omega, \mathcal{F}, \mu), [0,1]\big)$, where $(\Omega, \mathcal{F}, \mu)$ is a probability space. Then for every $f \in L^1\big((\Omega, \mathcal{F}, \mu), \{0,1\}\big)$, there is $A \in \mathcal{F}$, such that $f = [\chi_A]_{a.s.}$. Thus $f \wedge \neg f = \mathbf{0}$, whereby $f \in D$. For the converse, take $x \in D$ with $I(x \wedge \neg x) = 0$. Suppose $x = [f]_{a.s.}$ for an $\mathcal{F}$-measurable $f \colon \Omega \to [0,1]$. Then $\int_\Omega \min(f, 1-f) d\mu = 0$, whereby f is *a.s.* a characteristic function. Hence $x \in L^1\big((\Omega, \mathcal{F}, \mu), \{0,1\}\big)$, and thus $D = L^1\big((\Omega, \mathcal{F}, \mu), \{0,1\}\big)$. □

The following lemmas are used in the proofs of Proposition 2.7 and Theorem 2.10.

Lemma 2.4. *Let $\mathcal{M} \models \mathsf{RV}$. Then:*

(i) *For all $m, n \in \mathbb{N}$ and all $x_1, \dots, x_n \in M$, we have*

$$\frac{1}{2}\Big(\frac{x_1}{2^m} \dotplus \cdots \dotplus \frac{x_n}{2^m}\Big) = \frac{x_1}{2^{m+1}} \dotplus \cdots \dotplus \frac{x_n}{2^{m+1}}.$$

(ii) *For all $m, n \in \mathbb{N}$ and all $x_1, \dots, x_m, y_1, \dots, y_n \in M$, we have*

$$(x_1 \dotplus \cdots \dotplus x_m) \dotplus (y_1 \dotplus \cdots \dotplus y_n) = x_1 \dotplus \cdots \dotplus x_m \dotplus y_1 \dotplus \cdots \dotplus y_n.$$

(iii) *For all $n \in \mathbb{N}$ and all $x \in M$, we have*

$$\underbrace{\frac{x}{2^n} \dotplus \cdots \dotplus \frac{x}{2^n}}_{2^n \ times} = x.$$

(iv) *For all $m, n \in \mathbb{N}$ and all $x_1, \dots, x_n \in M$, we have*

$$\frac{1}{2^m}(x_1 \vee \cdots \vee x_n) = \frac{x_1}{2^m} \vee \cdots \vee \frac{x_n}{2^m}.$$

Proof. (i): Use induction on n and (H5).

(ii): Use induction on n and (P3).

(iii): Use induction on n, (ii), and (H2).

(iv): Use induction on n and (H3). □

Lemma 2.5. *Let $\mathcal{M} \models \mathsf{RV}$ and let $x, y, z \in M$ be such that $x \wedge y = y \wedge z = z \wedge x = 0$. Then for all $n \in \mathbb{N}$, all $x_0, x_1, \dots, x_n \in M$ such that $x_i \wedge x_j = \mathbf{0}$ if $i \neq j$, and all $r, s, t, r_0, r_1, \dots, r_n \in \mathbb{D}$, we have:*

(i) $rx \wedge sy = \mathbf{0}$.
(ii) $rx \dot{-} sy = rx$.
(iii) $rx \dot{+} sy = rx \vee sy$.
(iv) $(r_0x_0 \dot{+} \cdots \dot{+} r_{n-1}x_{n-1}) \wedge r_nx_n = \mathbf{0}$.
(v) $r_1x_1 \dot{+} \cdots \dot{+} r_nx_n = r_1x_1 \vee \cdots \vee r_nx_n$.
(vi) $I(r_1x_1 \dot{+} \cdots \dot{+} r_nx_n) = I(r_1x_1) + \cdots + I(r_nx_n)$.

Proof. We leave the proofs of (i), (ii), and (iii) to the readers.

(iv): We use induction on n.

(v): We use induction on n.

(vi): We use induction on n. □

Lemma 2.6. *Let $\mathcal{M} \models \mathsf{RV}$ and let $D = \{x \in M \mid I(x \wedge \neg x) = 0\}$. Then for all $r, s \in \mathbb{D}$, all $x \in M$, and all $a, a_1, \ldots, a_k \in D$, where $k \in \mathbb{N}$ and $a_i \wedge a_j = \mathbf{0}$ if $i \neq j$, we have*

(i) $\frac{rx}{2} = \frac{r}{2}x$.
(ii) $ra \dot{+} sa = (r \dot{+} s)a$.
(iii) $\neg(ra) = (\neg r)a \dot{+} \neg a$.
(iv) $ra \dot{-} sa = (r \dot{-} s)a$.
(v) $r(a_1 \dot{+} \cdots \dot{+} a_k) = ra_1 \dot{+} \cdots \dot{+} ra_k$.
(vi) $ra \wedge sa = (r \wedge s)a$.
(vii) $I(ra) = rI(a)$, *and thus* $ra = \mathbf{0}$ *if and only if* $r = 0$ *or* $a = \mathbf{0}$.

Proof. We assume familiarity with Fact 2.2. Suppose r or s is neither 0 nor 1, otherwise this is trivial.

(i): This follows from (H5) and Lemma 2.4(i).

(ii): Suppose $r = \frac{m_1}{2^{n_1}}$, $s = \frac{m_2}{2^{n_2}}$, where $n_1, n_2 \in \mathbb{N}$, $0 < m_1 < 2^{n_1}$, $0 < m_2 < 2^{n_2}$, and $n_1 \leq n_2$. By Lemma 2.4(iii), we have

$$\frac{a}{2^{n_1}} = \underbrace{\frac{a}{2^{n_2}} \dot{+} \cdots \dot{+} \frac{a}{2^{n_2}}}_{2^{n_2-n_1} \text{ times}}.$$

Then by Lemma 2.4(ii) and induction, we have $ra = \underbrace{\frac{a}{2^{n_2}} \dot{+} \cdots \dot{+} \frac{a}{2^{n_2}}}_{m_1 2^{n_2-n_1}}$, and thus $ra \dot{+} sa = \underbrace{\frac{a}{2^{n_2}} \dot{+} \cdots \dot{+} \frac{a}{2^{n_2}}}_{m_1 2^{n_2-n_1}+m_2}$. Suppose $r \dot{+} s < 1$. If $2 \nmid m_1 2^{n_2-n_1} + m_2$, then

$$\underbrace{\frac{a}{2^{n_2}} \dot{+} \cdots \dot{+} \frac{a}{2^{n_2}}}_{m_1 2^{n_2-n_1}+m_2} = \frac{m_1 2^{n_2-n_1} + m_2}{2^{n_2}} a = (r \dot{+} s)a.$$

Otherwise, say $\frac{m_1 2^{n_2-n_1}+m_2}{2^{n_2}} = \frac{m_3}{2^{n_3}}$, where $n_3 \in \mathbb{N}$ and $0 < m_3 < 2^{n_3}$. Then by Lemma 2.4(ii, iii) and induction, we have

$$\underbrace{\frac{a}{2^{n_2}} \dotplus \cdots \dotplus \frac{a}{2^{n_2}}}_{m_1 2^{n_2-n_1}+m_2} = \underbrace{\frac{a}{2^{n_3}} \dotplus \cdots \dotplus \frac{a}{2^{n_3}}}_{m_3 \text{ times}} = (r \dotplus s)a.$$

Hence $ra \dotplus sa = (r \dotplus s)a$ if $r \dotplus s < 1$.

Suppose $r \dotplus s \geq 1$. By Lemma 2.4(ii, iii) and induction, it suffices to prove $a \dotplus ta = a$ for all $t \in \mathbb{D}$, which follows from

$$a \dotplus ta = \neg(\neg a \dot{-} ta) = \neg((\mathbf{0} \vee \neg a) \dot{-} ta) = \neg((\mathbf{0} \dot{-} ta) \vee \neg a) = \neg(\neg a) = a.$$

(iii): Since $a \in D$, by Proposition 2.3 we have $a \wedge \neg a = \mathbf{0}$ and $a \vee \neg a = \mathbf{1}$. It is easy to verify that

$$\frac{ra}{2} \dotplus \frac{(\neg r)a \dotplus \neg a}{2} = \frac{1}{2}(a \vee \neg a) = \frac{\mathbf{1}}{2}.$$

Then by Fact 2.2(xii), we have $\neg(ra) = (\neg r)a \dotplus \neg a$.

(iv): Note that $\neg(t_1 \dot{-} t_2) = \neg t_1 \dotplus t_2$ for all $t_1, t_2 \in [0,1]$. Then by (iii), (ii), (P3), and Fact 2.2, we have

$$\neg\big((r \dot{-} s)a\big) = ((\neg r)a \dotplus sa) \dotplus \neg a = \neg(ra \dot{-} sa).$$

Hence, $ra \dot{-} sa = (r \dot{-} s)a$.

(v): Suppose $r = \frac{m}{2^n}$, where $n \in \mathbb{N}$, $0 < m < 2^n$, and $2 \nmid m$. Then by Lemma 2.5(v), Lemma 2.4(iv), (P3), Fact 2.2(ix), and induction, we have

$$r(a_1 \dotplus \cdots \dotplus a_k) = ra_1 \dotplus \cdots \dotplus ra_k.$$

(vi): Since $ra \wedge sa = ra \dot{-} (ra \dot{-} sa)$, this follows from (iv).

(vii): Suppose $r = \frac{m}{2^n}$, where $m, n \in \mathbb{N}$, $0 < m < 2^n$, and $2 \nmid m$.

Using induction on k, we have that $I(\frac{k}{2^n}a) = \frac{k}{2^n}I(a)$ for all $1 \leq k \leq 2^n$, and thus $I(ra) = rI(a)$. □

Proposition 2.7. *Let $\mathcal{M}$ be a model of* RV *and let $D = \{x \in M \mid x \wedge \neg x = \mathbf{0}\}$. Let $\mathcal{M}_0 \subseteq \mathcal{M}$ be the smallest L_{RV}-prestructure containing D. Then M_0 is the set $\{r_1a_1 \dotplus \cdots \dotplus r_ka_k \mid k \in \mathbb{N}, r_1, \dots, r_k \in \mathbb{D}, a_1, \dots, a_k \in D,$ and $a_i \wedge a_j = \mathbf{0}$ if $i \neq j\}$. Moreover, every nonzero element in M_0 has a unique decomposition $r_1a_1 \dotplus \cdots \dotplus r_ka_k$, where*

1. *$k \in \mathbb{N}$;*
2. *$r_1, \dots, r_k \in \mathbb{D}$ with $0 < r_1 < \cdots < r_k$;*
3. *$a_1, \dots, a_k \in D$ such that $a_i \neq \mathbf{0}$ for each i, and $a_i \wedge a_j = \mathbf{0}$ whenever $i \neq j$.*

Proof. By Proposition 2.3, $(D, \mathbf{0}, \mathbf{1}, \neg, \wedge, \vee)$ is a boolean algebra. Let S denote the set $\{r_1a_1 \dotplus \cdots \dotplus r_ka_k \mid k \in \mathbb{N}, r_1, \dots, r_k \in \mathbb{D}, a_1, \dots, a_k \in D,$ and $a_i \wedge a_j = \mathbf{0}$ if $i \neq j\}$. Clearly, $S \subseteq M_0$. We will show that $(S, \mathbf{0}, \neg, \dot{-}, \frac{1}{2}, I, d)$ is an L_{RV}-prestructure. Taking $k = 1$, $r_1 = 0$, and $a_1 = \mathbf{0}$ in the definition of membership shows that $\mathbf{0} \in S$. By Fact 2.2(iii), for all $x, y \in M$ we have $x \dot{-} y = \neg(\neg x \dot{+} y)$. Hence, we need only show that S is closed under $\neg$, $\dot{+}$, and $\frac{1}{2}$.

Take $x = r_1 a_1 \dot{+} \cdots \dot{+} r_k a_k \in S$, where $r_1, \ldots, r_k \in \mathbb{D}$, $a_1, \ldots, a_k \in D$, and $a_i \wedge a_j = \mathbf{0}$ if $i \neq j$. By Lemma 2.5(v), we have $x = r_1 a_1 \vee \cdots \vee r_k a_k$. Then by Lemma 2.4(iv), Lemma 2.6(i), and Lemma 2.5(v), we have

$$\frac{x}{2} = \frac{r_1 a_1}{2} \vee \cdots \vee \frac{r_k a_k}{2} = \frac{r_1}{2} a_1 \vee \cdots \vee \frac{r_k}{2} a_k = \frac{r_1}{2} a_1 \dot{+} \cdots \dot{+} \frac{r_k}{2} a_k \in S.$$

Hence, S is closed under $\frac{1}{2}$. Let $y = (\neg r_1) a_1 \dot{+} \cdots \dot{+} (\neg r_k) a_k \dot{+} \neg(a_1 \vee \cdots \vee a_k)$. Because D is a boolean algebra and $a_i \wedge a_j = \mathbf{0}$ if $i \neq j$, we know $y \in S$. We will show $y = \neg x$. Similar to the calculation of $\frac{x}{2}$, we have $\frac{y}{2} = \frac{\neg r_1}{2} a_1 \dot{+} \cdots \dot{+} \frac{\neg r_k}{2} a_k \dot{+} \frac{\neg(a_1 \vee \cdots \vee a_k)}{2}$. Then by (P3), Fact 2.2(ix), induction, Lemma 2.6(ii), Lemma 2.5(v), Lemma 2.4(iv), and the fact that $(D, \mathbf{0}, \mathbf{1}, \neg, \wedge, \vee)$ is a boolean algebra, we have

$$\begin{aligned}
\frac{x}{2} \dot{+} \frac{y}{2} &= (\frac{r_1}{2} a_1 \dot{+} \cdots \dot{+} \frac{r_k}{2} a_k) \dot{+} (\frac{\neg r_1}{2} a_1 \dot{+} \cdots \dot{+} \frac{\neg r_k}{2} a_k \dot{+} \frac{\neg(a_1 \vee \cdots \vee a_k)}{2}) \\
&= (\frac{r_1}{2} \dot{+} \frac{\neg r_1}{2}) a_1 \dot{+} \cdots \dot{+} (\frac{r_k}{2} \dot{+} \frac{\neg r_k}{2}) a_k \dot{+} \frac{\neg(a_1 \vee \cdots \vee a_k)}{2} \\
&= \frac{1}{2} a_1 \dot{+} \cdots \dot{+} \frac{1}{2} a_k \dot{+} \frac{\neg(a_1 \vee \cdots \vee a_k)}{2} \\
&= \frac{1}{2}(a_1 \vee \cdots \vee a_k \vee \neg(a_1 \vee \cdots \vee a_k)) = \frac{1}{2}\mathbf{1} = \frac{\mathbf{1}}{2}.
\end{aligned}$$

Hence by Fact 2.2(xii), we have $\neg x = y \in S$. That is, S is closed under $\neg$.

Take $x = r_1 a_1 \dot{+} \cdots \dot{+} r_k a_k$ and $y = s_1 b_1 \dot{+} \cdots \dot{+} s_l b_l \in S$, where $r_1, \ldots, r_k$, $s_1, \ldots, s_l \in \mathbb{D}$, $a_1, \ldots, a_k$, $b_1, \ldots, b_l \in D$, $a_i \wedge a_{i'} = \mathbf{0}$ if $1 \leq i \neq i' \leq k$, and $b_j \wedge b_{j'} = \mathbf{0}$ if $1 \leq j \neq j' \leq l$. Let a_0 be $\neg(a_1 \vee \cdots \vee a_k)$ and let b_0 be $\neg(b_1 \vee \cdots \vee b_l)$. Since $(D, \mathbf{0}, \mathbf{1}, \neg, \wedge, \vee)$ is a boolean algebra, we have that $\{a_0, \ldots, a_k\}$ and $\{b_0, \ldots, b_l\}$ are two partitions of $\mathbf{1}$. Let $\{c_1, \ldots, c_m\}$ is the partition generated by partitions $\{a_0, \ldots, a_k\}$ and $\{b_0, \ldots, b_l\}$. Then by Lemma 2.6(v), (P3), Fact 2.2(ix), and induction, we may assume that $x = r'_1 c_1 \dot{+} \cdots \dot{+} r'_m c_m$ and $y = s'_1 c_1 \dot{+} \cdots \dot{+} s'_m c_m$, where $r'_1, \ldots, r'_m, s'_1, \ldots, s'_m \in \mathbb{D}$ (could be 0), and $\{c_1, \ldots, c_m\}$ is a partition of $\mathbf{1}$. Then by Lemma 2.6, we have $x \dot{+} y = (r'_1 \dot{+} s'_1) c_1 \dot{+} \cdots \dot{+} (r'_m \dot{+} s'_m) c_m \in S$. Thus S is closed under $\dot{+}$. Therefore, S is an L_{RV}-prestructure, and thus $S \supseteq M_0$. Hence, $S = M_0$, the smallest L_{RV}-prestructure containing D in M.

Consider a nonzero element x in M_0. Suppose x is of the form $x = t_1 c_1 \dot{+} \cdots \dot{+} t_k c_k$, where $t_1, \ldots, t_k \in \mathbb{D}$, $c_1, \ldots, c_k \in D$, and $c_i \wedge c_j = \mathbf{0}$ if $i \neq j$. Suppose k is the smallest integer for such decomposition. By (P3), Fact 2.2(ix), and induction, we may reorder those terms such that $x = r_1 a_1 \dot{+} \cdots \dot{+} r_k a_k$, where $r_1 \leq r_2 \leq \cdots \leq r_k \in \mathbb{D}$ and $a_1, \ldots, a_k \in D$, and $a_i \wedge a_j = \mathbf{0}$ if $i \neq j$. By the fact that k is chosen to be smallest, we have $0 < r_1 < \cdots < r_k$ and $a_i \neq \mathbf{0}$ for each $1 \leq i \leq k$. Then a standard manipulation of lattices yields that this decomposition is unique. We leave it to the readers. $\square$

Proposition 2.8. *Let $\mathcal{M}$ be a model of* RV *and let $D = \{x \in M \mid x \wedge \neg x = \mathbf{0}\}$. Let $\mathcal{M}_0 \subseteq \mathcal{M}$ be the smallest L_{RV}-prestructure containing D. Then M_0 is the set $\{r_1 a_1 \dotplus \cdots \dotplus r_n a_n \mid n \in \mathbb{N}, r_1, \ldots, r_n \in \mathbb{D}, a_1, \ldots, a_n \in D,$ and $a_i \wedge a_j = \mathbf{0}$ if $i \neq j\}$. Moreover, every L_{Pr}-isomorphism $\phi\colon D \to L^1\big((\Omega, \mathcal{F}, \mu), \{0,1\}\big)$, where $(\Omega, \mathcal{F}, \mu)$ is a probability space, will be uniquely extended to an L_{RV}-embedding*

$$\Phi\colon M_0 \to L^1\big((\Omega, \mathcal{F}, \mu), \mathbb{D}\big)$$

which is defined by $\Phi(r_1 a_1 \dotplus \cdots \dotplus r_n a_n) = r_1 \phi(a_1) \dotplus \cdots \dotplus r_n \phi(a_n)$, where $n \in \mathbb{N}$, $r_i \in \mathbb{D}$ and $a_i \in D$ for all $1 \leq i \leq n$, and $a_i \wedge a_j = \mathbf{0}$ whenever $i \neq j$.

Proof. Suppose ϕ can be extended to an L_{RV}-embedding

$$\Phi\colon M_0 \to L^1\big((\Omega, \mathcal{F}, \mu), \mathbb{D}\big).$$

Then $\Phi(r_1 a_1 \dotplus \cdots \dotplus r_n a_n) = r_1 \Phi(a_1) \dotplus \cdots \dotplus r_n \Phi(a_n)$, where $n \in \mathbb{N}$, $r_i \in \mathbb{D}$ and $a_i \in D$ for all $1 \leq i \leq n$, and $a_i \wedge a_j = \mathbf{0}$ for all $1 \leq i \neq j \leq n$. Since Φ is an extension of ϕ, we have $\Phi(r_1 a_1 \dotplus \cdots \dotplus r_n a_n) = r_1 \phi(a_1) \dotplus \cdots \dotplus r_n \phi(a_n)$. Hence such an extension Φ is uniquely determined by ϕ.

Let $D = \{x \in M \mid I(x \wedge \neg x)\} = 0$. By Proposition 2.3, we know that the L_{Pr}-structure $(D, \mathbf{0}, \neg, \cdot^{\mathsf{C}}, \cap, \cup, \mu)$ is a model of Pr. By [5, Theorem 5.2], there is a probability space $(\Omega, \mathcal{F}, \mu)$ such that D as an L_{Pr}-structure is isomorphic to $\widehat{\mathcal{F}}$. Let $\mathcal{N}$ denote the L_{RV}-structure $\big(L^1((\Omega, \mathcal{F}, \mu), [0,1]), \mathbf{0}, \neg, \dot{-}, \frac{1}{2}, I, d\big)$. By Proposition 2.1, we have $\mathcal{N} \models \mathsf{RV}$. Let $\mathcal{X}$ denote $L^1((\Omega, \mathcal{F}, \mu), \{0,1\})$. By Proposition 2.3, we have $(\mathcal{X}, 0, 1, \neg, \wedge, \vee, \bar{\mu}) \models \mathsf{Pr}$ and it is isomorphic to $\widehat{\mathcal{F}}$. Hence, D is L_{Pr}-isomorphic to $\mathcal{X}$. We call this isomorphism $\phi\colon D \to \mathcal{X}$. Then for all $x, y \in D$, we have that $\phi(x \dot{-} y) = \phi(x) \dot{-} \phi(y)$, $\phi(\mathbf{0}) = \mathbf{0}$, $\phi(\mathbf{1}) = \mathbf{1}$, $\int_\Omega \phi(x) d\mu = I^{\mathcal{N}}\big(\phi(x)\big) = I(x)$, and $d(x,y) = d^{\mathcal{N}}\big(\phi(x), \phi(y)\big)$. Let $\mathcal{M}_0$ be the smallest L_{RV}-prestructure containing D. By Proposition 2.7, we know that every nonzero element $x \in M_0$ has a unique decomposition of the form $x = r_1 a_1 \dotplus \cdots \dotplus r_n a_n$, where $n \in \mathbb{N}$, $0 < r_1 < \cdots < r_n \in \mathbb{D}$, $a_1, \ldots, a_n \in D$, $a_i \neq \mathbf{0}$ for each i, and $a_i \wedge a_j = \mathbf{0}$ whenever $i \neq j$. We extend $\phi\colon D \to \mathcal{X}$ to a mapping $\Phi\colon M_0 \to N$, by defining

$$\Phi(r_1 a_1 \dotplus \cdots \dotplus r_n a_n) := r_1 \phi(a_1) \dotplus \cdots \dotplus r_n \phi(a_n),$$

where $n \in \mathbb{N}$, $0 < r_1 < \cdots < r_n \in \mathbb{D}$, $a_1, \ldots, a_n \in D$, $a_i \neq \mathbf{0}$ for each i, and $a_i \wedge a_j = \mathbf{0}$ whenever $i \neq j$. Clearly, Φ is uniquely determined by ϕ.

Next, we will check that Φ preserves $\mathbf{0}, \mathbf{1}, \neg, \frac{1}{2}, \dotplus, I$ and d. We already know that $\Phi(\mathbf{0}) = \mathbf{0}$ and $\Phi(\mathbf{1}) = \mathbf{1}$. To show that Φ preserves $\neg, \frac{1}{2}, \dotplus$, we need the following claim:

Claim 2.9. *Take a nonzero $x \in M_0$. Suppose x has the form $r_1 a_1 \dotplus \cdots \dotplus r_n a_n$, where $n \in \mathbb{N}$, $0 < r_1 < \cdots < r_n \in \mathbb{D}$, $a_1, \ldots, a_n \in D$, $a_i \neq \mathbf{0}$ for each i, and $a_i \wedge a_j = \mathbf{0}$ whenever $i \neq j$. Suppose x has another form $s_1 b_1 \dotplus \cdots \dotplus s_m b_m$, where $m \in \mathbb{N}$, $s_1, \ldots, s_m \in \mathbb{D}$, $b_1, \ldots, b_m \in D$, and $b_k \wedge b_l = \mathbf{0}$ whenever $k \neq l$. Then*

$$\Phi(x) = r_1 \Phi(a_1) + \cdots + r_n \Phi(a_n) = s_1 \Phi(b_1) + \cdots + s_m \Phi(b_m).$$

Proof of Claim 2.9: Suppose x has the form $s_1b_1 \dot{+} \cdots \dot{+} s_mb_m$, where $m \in \mathbb{N}$, $s_1, \ldots, s_m \in \mathbb{D}$, $b_1, \ldots, b_m \in D$, and $b_k \wedge b_l = \mathbf{0}$ whenever $k \neq l$. Then after a standard procedure to reorder terms, delete 0 terms, and combine the terms with the same coefficients, the form of x becomes $r_1a_1 \dot{+} \cdots \dot{+} r_na_n$, where $n \in \mathbb{N}$, $0 < r_1 < \cdots < r_n \in \mathbb{D}$, $a_1, \ldots, a_n \in D$, $a_i \neq \mathbf{0}$ for each i, and $a_i \wedge a_j = \mathbf{0}$ whenever $i \neq j$, which is the unique decomposition shown in Proposition 2.7. It is easy to verify that during the reordering, deleting, and combining processes, although the form of x has changed, the sum $s_1\Phi(b_1) + \cdots + s_m\Phi(b_m)$ remains the same. This completes the proof of Claim 2.9.

Next, we will show that Φ preserves $\frac{1}{2}$, $\neg$, and $\dot{+}$. Take a nonzero $x \in M_0$. Suppose x has the form $r_1a_1 \dot{+} \cdots \dot{+} r_na_n$, where $n \in \mathbb{N}$, $0 < r_1 < \cdots < r_n \in \mathbb{D}$, $a_1, \ldots, a_n \in D$, $a_i \neq \mathbf{0}$ for each i, and $a_i \wedge a_j = \mathbf{0}$ whenever $i \neq j$. As shown in the proof of Proposition 2.7, we know that $\frac{x}{2} = \frac{r_1}{2}a_1 \dot{+} \cdots \dot{+} \frac{r_n}{2}a_n$, and

$$\neg x = \neg r_1a_1 \dot{+} \cdots \dot{+} \neg r_na_n \dot{+} \neg(a_1 \vee \cdots \vee a_n).$$

By Claim 2.9, we have

$$\Phi(\frac{x}{2}) = \frac{r_1}{2}\Phi(a_1) \dot{+} \cdots \dot{+} \frac{r_n}{2}\Phi(a_n) = \frac{1}{2}(r_1\Phi(a_1) \dot{+} \cdots \dot{+} r_n\Phi(a_n)) = \frac{1}{2}\Phi(x).$$

Hence, Φ preserves $\frac{1}{2}$. Also

$$\begin{aligned}\Phi(\neg x) &= \Phi(\neg r_1a_1 \dot{+} \cdots \dot{+} \neg r_na_n \dot{+} \neg(a_1 \vee \cdots \vee a_n)) \\ &= \neg r_1\Phi(a_1) \dot{+} \cdots \dot{+} \neg r_n\Phi(a_n) \dot{+} \Phi(\neg(a_1 \vee \cdots \vee a_n)).\end{aligned}$$

Then because $\Phi(a_n), \ldots, \Phi(a_1), \Phi(\neg(a_1 \vee \cdots \vee a_n))$ are in $L^1(\mu, [0,1])$, $a_i \wedge a_j = \mathbf{0}$ if $i \neq j$, and $\Phi\colon D \to \mathcal{X}$ is an L_{Pr}-isomorphism, we have that

$$\begin{aligned}&\neg r_1\Phi(a_1) \dot{+} \cdots \dot{+} \neg r_n\Phi(a_n) \dot{+} \Phi(\neg(a_1 \vee \cdots \vee a_n)) \\ &= (1 - r_1)\Phi(a_1) + \cdots + (1 - r_n)\Phi(a_n) + (1 - \Phi(a_1) - \cdots - \Phi(a_n)) \\ &= 1 - r_1\Phi(a_1) - \cdots - r_n\Phi(a_n) = \neg(r_1\Phi(a_1) + \cdots + r_n\Phi(a_n)) \\ &= \neg(r_1\Phi(a_1) \dot{+} \cdots \dot{+} r_n\Phi(a_n)).\end{aligned}$$

Hence, $\Phi(\neg x) = \neg\Phi(x)$; that is, Φ preserves $\neg$.

Take nonzero $x, y \in M_0$. Suppose they have the form $x = r_1a_1 \dot{+} \cdots \dot{+} r_ka_k$ and $y = s_1b_1 \dot{+} \cdots \dot{+} s_lb_l$, where $k, l \in \mathbb{N}$, $0 < r_1 < \cdots < r_k \in \mathbb{D}$, $0 < s_1 < \cdots < s_l \in \mathbb{D}$, $a_1, \ldots, a_k, b_1, \ldots, b_l \in D$, $a_i \neq \mathbf{0}$ for all $1 \leq i \leq k$, $b_j \neq \mathbf{0}$ for all $1 \leq j \leq l$, $a_i \wedge a_{i'} = \mathbf{0}$ whenever $1 \leq i \neq i' \leq k$, and $b_j \wedge b_{j'} = \mathbf{0}$ whenever $1 \leq j \neq j' \leq l$. Let a_0 be $\neg(a_1 \vee \cdots \vee a_k)$ and let b_0 be $\neg(b_1 \vee \cdots \vee b_l)$. Since $(D, \mathbf{0}, \mathbf{1}, \neg, \wedge, \vee)$ is a boolean algebra, we have $a_0 \vee a_1 \vee \cdots \vee a_k = b_0 \vee b_1 \vee \cdots \vee b_l = \mathbf{1}$, $a_i \wedge a_{i'} = \mathbf{0}$ for all $0 \leq i \neq i' \leq k$, and $b_j \wedge b_{j'} = \mathbf{0}$ for all $0 \leq j \neq j' \leq l$. That is, $\{a_0, \ldots, a_k\}$ and $\{b_0, \ldots, b_l\}$ are two partitions of $\mathbf{1}$. Then let $\{c_1, \ldots, c_m\}$ be the partition generated by partitions $\{a_0, \ldots, a_k\}$ and $\{b_0, \ldots, b_l\}$. Then by Lemma 2.6(v), (P3), Fact 2.2(ix), and induction, we may assume that $x = r'_1c_1 \dot{+} \cdots \dot{+} r'_mc_m$ and $y = s'_1c_1 \dot{+} \cdots \dot{+} s'_mc_m$, where

$r_1', \dots, r_m', s_1', \dots, s_m' \in \mathbb{D}$ (could be 0), and $\{c_1, \dots, c_m\}$ is a partition of $\mathbf{1}$. Then by Lemma 2.6, we have $x \dotplus y = (r_1' \dotplus s_1')c_1 \dotplus \dots \dotplus (r_m' \dotplus s_m')c_m$. Hence

$$\begin{aligned}\Phi(x \dotplus y) &= (r_1' \dotplus s_1')\Phi(c_1) + \dots + (r_m' \dotplus s_m')\Phi(c_m) \\ &= (r_1'\Phi(c_1) + \dots + r_m'\Phi(c_m)) \dotplus (s_1'\Phi(c_1) + \dots + s_m'\Phi(c_m)) \\ &= \Phi(x) \dotplus \Phi(y).\end{aligned}$$

Thus, Φ preserves $\dotplus$.

Now, we will prove Φ preserves I and d. Take a nonzero $x \in M_0$. Suppose x has the form $r_1a_1 \dotplus \dots \dotplus r_na_n$, where $n \in \mathbb{N}$, $0 < r_1 < \dots < r_n \in \mathbb{D}$, $a_1, \dots, a_n \in D$, $a_i \neq \mathbf{0}$ for each i, and $a_i \wedge a_j = \mathbf{0}$ whenever $i \neq j$. Since $\Phi\colon D \to \mathcal{X}$ is an isomorphism, we have $\Phi(a_i) \wedge \Phi(a_j) = \mathbf{0}$ whenever $i \neq j$. For all $f \in N$, we have $I^{\mathcal{N}}(f) = \int_\Omega f d\mu$. Therefore,

$$\begin{aligned}I^{\mathcal{N}}(\Phi(x)) = \int_\Omega \Phi(x) d\mu &= \int_\Omega \big(r_1\Phi(a_1) \dotplus \dots \dotplus r_n\Phi(a_n)\big) d\mu \\ &= r_1 \int_\Omega \Phi(a_1)d\mu + \dots + r_n \int_\Omega \Phi(a_n) d\mu \\ &= r_1 I^{\mathcal{N}}(\Phi(a_1)) + \dots + r_n I^{\mathcal{N}}(\Phi(a_n)) \\ &= r_1 I(a_1) + \dots + r_n I(a_n).\end{aligned}$$

By Lemma 2.6(vii), we have $I^{\mathcal{N}}(\Phi(x)) = I(r_1a_1) + \dots + I(r_na_n)$. Then by Lemma 2.5, we have $I(r_1a_1) + \dots + I(r_na_n) = I(r_1a_1 \dotplus \dots \dotplus r_na_n) = I(x)$, and thus $I(x) = I^{\mathcal{N}}(\Phi(x))$. That is, Φ preserves I. Since $d(x,y) = I(x \dot- y) + I(y \dot- x)$ for all $x, y \in M$ and Φ preserves I and $\dot-$, it follows that Φ preserves d.

Therefore, Φ is an L_{RV}-embedding from $\mathcal{M}_0$ to $\mathcal{N}$. □

Theorem 2.10. *Let $\mathcal{M}$ be a model of* RV. *Then $\mathcal{M}$ is isomorphic to the L_{RV}-structure $\Big(L^1\big((\Omega, \mathcal{F}, \mu), [0,1]\big), \mathbf{0}, \neg, \dot-, \frac{1}{2}, I, d\Big)$ for some probability space $(\Omega, \mathcal{F}, \mu)$.*

Proof. Let $D = \{x \in M \mid I(x \wedge \neg x)\} = 0$. By Proposition 2.3, we know that $(D, \mathbf{0}, \neg, \cdot^{\complement}, \cap, \cup, \mu)$ is a model of Pr. By [5, Theorem 5.2], we know that there is a probability space $(\Omega, \mathcal{F}, \mu)$ such that D as an L_{Pr}-structure is isomorphic to $\widehat{\mathcal{F}}$. Let $\mathcal{N} = \big(L^1((\Omega, \mathcal{F}, \mu), [0,1]), \mathbf{0}, \neg, \dot-, \frac{1}{2}, I, d\big)$. By Proposition 2.1, we have $\mathcal{N} \models \mathsf{RV}$. Let $\mathcal{X}$ denote $L^1((\Omega, \mathcal{F}, \mu), \{0,1\})$. By Proposition 2.3, we have $(\mathcal{X}, 0, 1, \neg, \wedge, \vee, \bar{\mu}) \models \mathsf{Pr}$ and it is isomorphic to $\widehat{\mathcal{F}}$. Hence, D is L_{Pr}-isomorphic to $\mathcal{X}$. We call this isomorphism $\phi\colon D \to \mathcal{X}$. Then by Proposition 2.8, we extend $\phi\colon D \to \mathcal{X}$ to an L_{RV}-embedding $\Phi\colon M_0 \to N$.

Let (M', d) be the completion of (M_0, d) in M. Because Φ is isometric, we know that Φ is extended uniquely to an embedding $\overline{\Phi}$ from $\mathcal{M}'$ to $\mathcal{N}$. Note that dyadic number valued simple functions are dense in N. Hence $\overline{\Phi}$ is a surjective embedding; that is, $\overline{\Phi}$ is an isomorphism between L_{RV}-structures $\mathcal{M}'$ and $\mathcal{N}$. Then we will show $\mathcal{M}'$ is $\mathcal{M}$.

For every $x \in M$ and $n \in \mathbb{N}$, using (APPR), there are elements $y_1, \ldots, y_{2^n} \in M$ such that
$\max_i I(y_i \wedge \neg y_i) \leq \frac{1}{2^n}$ and $d(x, \frac{1}{2^n} y_1 \dot{+} \cdots \dot{+} \frac{1}{2^n} y_{2^n}) \leq \frac{1}{2^n}$. For every $1 \leq i \leq 2^n$, since $I(y_i \wedge \neg y_i) = \operatorname{dist}(y_i, D)$, there is $z_i \in D$ such that $d(y_i, z_i) \leq \frac{1}{2^n}$. Then by (P1),

$$\begin{aligned}
& d(\frac{1}{2^n} y_1 \dot{+} \cdots \dot{+} \frac{1}{2^n} y_{2^n}, \frac{1}{2^n} z_1 \dot{+} \cdots \dot{+} \frac{1}{2^n} z_{2^n}) \\
\leq\ & d(\frac{1}{2^n} y_1, \frac{1}{2^n} z_1) + \cdots + d(\frac{1}{2^n} y_{2^n}, \frac{1}{2^n} z_{2^n}) \\
=\ & \frac{1}{2^n}\big(d(y_1, z_1) + \cdots + d(y_{2^n}, z_{2^n})\big) \\
\leq\ & \max_{1 \leq i \leq 2^n} d(y_i, z_i) \leq \frac{1}{2^n}
\end{aligned}$$

where the equality follows from Fact 2.2(x). Then

$$\begin{aligned}
& d(x, \frac{1}{2^n} z_1 \dot{+} \cdots \dot{+} \frac{1}{2^n} z_{2^n}) \\
\leq\ & d(x, \frac{1}{2^n} y_1 \dot{+} \cdots \dot{+} \frac{1}{2^n} y_{2^n}) + d(\frac{1}{2^n} y_1 \dot{+} \cdots \dot{+} \frac{1}{2^n} y_{2^n}, \frac{1}{2^n} z_1 \dot{+} \cdots \dot{+} \frac{1}{2^n} z_{2^n}) \\
\leq\ & \frac{1}{2^n} + \frac{1}{2^n} = \frac{1}{2^{n-1}}.
\end{aligned}$$

Since $\frac{1}{2^n} z_1 \dot{+} \cdots \dot{+} \frac{1}{2^n} z_{2^n} \in M_0$, it follows that M_0 is dense in M, whereby $M' = M$. Therefore $\overline{\Phi}$ is an isomorphism from $\mathcal{M}$ to $\mathcal{N}$. □

Corollary 2.11. *Let $\mathcal{M}$ be an L_{RV}-structure. Then $\mathcal{M}$ is a model of* ARV *if and only if $\mathcal{M}$ is isomorphic to $\Big(L^1\big((\Omega, \mathcal{F}, \mu), [0,1]\big), \mathbf{0}, \neg, \dot{-}, \frac{1}{2}, I, d\Big)$ for some atomless probability space $(\Omega, \mathcal{F}, \mu)$.*

Proof. By Proposition 2.1, we know that $\Big(L^1\big((\Omega, \mathcal{F}, \mu), [0,1]\big), \mathbf{0}, \neg, \dot{-}, \frac{1}{2}, I, d\Big)$, where $(\Omega, \mathcal{F}, \mu)$ is an atomless probability space, is a model of ARV.

For the other direction, by Theorem 2.10, $\mathcal{M}$ as a model of RV is isomorphic to the L_{RV}-structure $\Big(L^1\big((\Omega, \mathcal{F}, \mu), [0,1]\big), \mathbf{0}, \neg, \dot{-}, \frac{1}{2}, I, d\Big)$ for a probability space $(\Omega, \mathcal{F}, \mu)$. Let D denote $L^1\big((\Omega, \mathcal{F}, \mu), \{0,1\}\big)$. By Proposition 2.3, D is a model of Pr and D is L_{Pr}-isomorphic to $\widehat{\mathcal{F}}$. For all $x \in D$, using (NA) we know that for every $\epsilon > 0$, there is y_ϵ such that $\operatorname{dist}(y_\epsilon, D) \leq \epsilon$ and $|I(y_\epsilon \wedge x) - \frac{I(x)}{2}| \leq \epsilon$. Then there is $y \in D$ such that $|I(y \wedge x) - \frac{I(x)}{2}| \leq 2\epsilon$. Thus D as a L_{Pr}-structure satisfies Axiom (iv) in APr, whereby D is a model of APr. By [5, Corollary 6.1], we have that $(\Omega, \mathcal{F}, \mu)$ is atomless. □

Remark 2.12. In [2], Ben Yaacov showed that the class $\mathcal{RV}$ of random variable structures and the class $\mathcal{PR}$ of probability algebras are bi-interpretable. Then by [1, Theorem A.9], the class $\mathcal{RV}$ is elementary if and only if the class $\mathcal{PR}$ is, which is clear. In the proof of Theorem A.9 there is a way to give axioms for RV, albeit not in a very intuitive form.

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