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SEMICENTRAL IDEMPOTENTS IN A RING

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Abstract. Let R be a ring with identity 1, $I(R)$ be the set of all nonunit idempotents in R and $S_\ell(R)$ (resp. $S_r(R)$) be the set of all left (resp. right) semicentral idempotents in R. In this paper, the following are investigated: (1) $e \in S_\ell(R)$ (resp. $e \in S_r(R)$) if and only if $re = ere$ (resp. $er = ere$) for all nilpotent elements $r \in R$ if and only if $fe \in I(R)$ (resp. $ef \in I(R)$) for all $f \in I(R)$ if and only if $fe = efe$ (resp. $ef = efe$) for all $f \in I(R)$ if and only if $fe = efe$ (resp. $ef = efe$) for all $f \in I(R)$ which are isomorphic to e if and only if $(fe)^n = (efe)^n$ (resp. $(ef)^n = (efe)^n$) for all $f \in I(R)$ which are isomorphic to e where n is some positive integer; (2) For a ring R having a complete set of centrally primitive idempotents, every nonzero left (resp. right) semicentral idempotent is a finite sum of orthogonal left (resp. right) semicentral primitive idempotents, and eRe has also a complete set of primitive idempotents for any $0 \neq e \in S_\ell(R)$ (resp. $0 \neq e \in S_r(R)$).

1. Introduction and basic definitions

Throughout this paper, let R be a ring with identity 1, $J(R)$ denote the Jacobson radical of R and $I(R)$ be the set of all idempotents of R. An idempotent $e \in R$ is *left* (resp. *right*) *semicentral* in R if $Re = eRe$ (resp. $eR = eRe$) (refer [1]). It is easy to show that $e \in R$ is left (resp. right) semicentral in R if and only if $ae = eae$ (resp. $ea = eae$) for all $a \in R$. Two idempotents $e, f \in R$ are said to be *isomorphic* if there exist $a, b \in R$ such that $e = ab, f = ba$ (refer [2, 5]). In Section 2, the following equivalent conditions are obtained:

(1) $e \in R$ is left (resp. right) semicentral;
(2) $re = er$ for all units $r \in R$;
(3) $re = ere$ (resp. $er = ere$) for all nilpotent elements $r \in R$;
(4) fe (resp. ef) is an idempotent for all idempotents $f \in R$;
(5) $fe = efe$ (resp. $ef = efe$) for all idempotents $f \in R$;
(6) $fe = efe$ (resp. $ef = efe$) for all idempotents $f \in R$ which are isomorphic to e;

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(7) $(fe)^n = (efe)^n$ (resp. $(ef)^n = (efe)^n$) for all idempotents $f \in R$ which are isomorphic to e where n is some positive integer.

A subset S of a ring R is called *commuting* if $ef = fe$ for all $e, f \in S$. Recall that two idempotents $e, f \in R$ are said to be *orthogonal* if $ef = fe = 0$. Also recall that an idempotent $e \in R$ is said to be *primitive* if it can not be written as a sum of two nonzero orthogonal idempotents, or equivalently, eR (resp. Re) is indecomposable as a right (resp. left) R-module. Let $M(R)$ be the set of zero and all primitive idempotents of R, $S_\ell(R)$ (resp. $S_r(R)$) be the set of all left (resp. right) semicentral idempotents in R, and $M_\ell(R) = M(R) \cap S_\ell(R)$ (resp. $M_r(R) = M(R) \cap S_r(R)$). A subset S of $I(R)$ is also said to be *additive* in $I(R)$ if for all $e, f \in S$ ($e \neq f$), $e + f \in I(R)$ (refer [4]). For example, if R is a Boolean ring or a direct product of local rings, then $M(R)$ is additive in $I(R)$. In Section 2, it was also shown that (1) $M_\ell(R)$ (resp. $M_r(R)$) is additive in $I(R)$ if and only if $M_\ell(R)$ (resp. $M_r(R)$) is orthogonal; (2) Let $N \subseteq J(R)$ be an ideal of R such that idempotents in R/N can be lifted to R. (i) If $S_\ell(R)$ (resp. $S_r(R)$) is commuting, then $S_\ell(R/N)$ (resp. $S_r(R/N)$) is additive in $I(R/N)$ if and only if $S_\ell(R)$ (resp. $S_r(R)$) is additive in $I(R)$; (ii) If $M_\ell(R)$ (resp. $M_r(R)$) is commuting, then $M_\ell(R/N)$ (resp. $M_r(R/N)$) is additive in $I(R/N)$ if and only if $M_\ell(R)$ (resp. $M_r(R)$) is additive in $I(R)$.

Recall that a central idempotent c of a ring R is said to be *centrally primitive* in R if $c \neq 0$ and c cannot be written as a sum of two nonzero orthogonal central idempotents in R (equivalently, cR is indecomposable as a ring). Also, R is said to have a complete set of primitive (resp. centrally primitive) idempotents if there exists a finite set of orthogonal primitive (resp. centrally primitive) idempotents whose sum is the identity of R [5, Sects. 21 and 22]. It was shown that a ring R having a complete set of primitive idempotents has a complete set of centrally primitive idempotents [5, Theorem 22.5]. By [5, Proposition 22.1], it was also shown that if R has a complete set $\{c_1, c_2, \ldots, c_n\}$ of centrally primitive idempotents, then any central idempotent is a sum of a subset of $\{c_1, c_2, \ldots, c_n\}$. In Section 3, it was shown that (1) for a ring R having a complete set T of centrally primitive idempotents, any nonzero left (resp. right) semicentral idempotent of R is a sum of orthogonal left (resp. right) semicentral primitive idempotents of R and eRe has also a complete set of centrally primitive idempotents for any nonzero idempotent $e \in R$; (2) for a ring R having a complete set T of primitive idempotents, any complete set of centrally primitive idempotents is contained in T and it consists of all centrally primitive idempotents of R.

2. Properties of semicentral idempotents in a ring

In this section, we will find some properties of left (resp. right) semicentral idempotents of a ring R.

Proposition 2.1. *For an idempotent e of a ring R the following conditions are equivalent*:

(1) *$e \in R$ is left (resp. right) semicentral*;
(2) *$re = ere$ (resp. $er = ere$) for all units $r \in R$*;
(3) *$re = ere$ (resp. $er = ere$) for all nilpotent elements $r \in R$*;
(4) *fe (resp. ef) is an idempotent for all idempotents $f \in R$*;
(5) *$fe = efe$ (resp. $ef = efe$) for all idempotents $f \in R$*;
(6) *$fe = efe$ (resp. $ef = efe$) for all idempotents $f \in R$ which are isomorphic to e*;
(7) *$(fe)^n = (efe)^n$ (resp. $(ef)^n = (efe)^n$) for all idempotents $f \in R$ which are isomorphic to e where n is some positive integer.*

Proof. First, we will prove it in the left semicentral case. (1) $\Rightarrow$ (2), (3), (4) and (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are obvious.

(2) $\Rightarrow$ (3): Suppose that the condition (2) holds. Let r be an arbitrary nilpotent element of R. Then $1+r$ is a unit of R. By assumption (2), $(1+r)e = e(1+r)e$, and then $re = ere$. Hence (3) holds.

(3) $\Rightarrow$ (1): Suppose that the condition (3) holds. Let $a \in R$ be arbitrary. Consider the element $r = (1-e)ae \in R$. Then $r^2 = 0$, and so $re = ere$ and this yields $(1-e)ae = 0$. Thus $ae = eae$, and so e is left semicentral.

(4) $\Rightarrow$ (5): Suppose that the condition (4) holds. Since $1 - f \in R$ are idempotents for all idempotents $f \in R$, $(1-f)e = ((1-f)e)^2$ by assumption. Thus $e - fe = (1-f)e = ((1-f)e)^2 = e - fe - efe + (fe)^2 = e - efe$, so $fe = efe$ for all idempotents $f \in R$.

(7) $\Rightarrow$ (1): Suppose that the condition (6) holds and assume that e is not left semicentral. Then there is $a \in R$ such that $ae - eae \neq 0$. Consider $f = e + ae - eae$. Then $f^2 = f \neq e, fe = f$ and $ef = e$, so these are isomorphic idempotents. Therefore, $e = (efe)^n \neq (fe)^n = f$ for any positive integer n, which contradicts to the assumption (6). Hence e is left semicentral.

Next, we can prove it in the right semicentral case by the similar argument used in the left semicentral case. □

Corollary 2.2. *For an idempotent e of a ring R the following conditions are equivalent*:

(1) *$e \in R$ is central*;
(2) *$re = er$ for all units $r \in R$*;
(3) *$re = er$ for all nilpotent elements $r \in R$*;
(4) *fe and ef are idempotents for all idempotents $f \in R$*;
(5) *$fe = ef$ for all idempotents $f \in R$*;
(6) *$fe = ef$ for all idempotents $f \in R$ which are isomorphic to e*;
(7) *$(fe)^n = (ef)^n$ for all idempotents $f \in R$ which are isomorphic to e where n is some positive integer.*

Proof. It follows from Proposition 2.1. □

Corollary 2.3. *For a ring R an idempotent e of R is left semicentral if and only if $1 - e$ is right semicentral.*

Proof. Let e be a left semicentral idempotent of R. Then $fe = efe$ for all idempotents $f \in R$ by Proposition 2.1. Therefore, $(1-e)f(1-e) = f-ef-fe+efe = f - ef = f(1-e)$, which implies that $1-e$ is a right semicentral idempotent of R by Proposition 2.1. The converse holds by the similar argument. □

Example 1. Let R be the 2 by 2 upper triangular matrix ring over $\mathbb{Z}_3$ where $\mathbb{Z}_3$ is a field of integers modulo 3. Consider two idempotents $e = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ of R. Since ef is not an idempotent of R, e is not right semicentral by Proposition 2.1. But we can checked that e is left semicentral. By Corollary 2.3, $1-e$ is right semicentral idempotent but not left semicentral idempotent of R.

Remark 1. Let $S_\ell(R)$ (resp. $S_r(R)$) be the set of all left (resp. right) semicentral idempotents of a ring R. Then we note the following:

(1) $S_\ell(R)$ (resp. $S_r(R)$) is closed under multiplication.

(2) $S_\ell(R)$ (resp. $S_r(R)$) is closed under conjugation, i.e., $ueu^{-1} \in S_\ell(R)$ (resp. $ufu^{-1} \in S_r(R)$) for all $e \in S_\ell(R)$ (resp. $f \in S_r(R)$) and all units $u \in R$.

(3) $e \in S_\ell(R)$ (resp. $f \in S_r(R)$) if and only if $e + ea(1-e) \in S_\ell(R)$ (resp. $f + fa(1-f) \in S_r(R)$) for each $a \in R$.

Note that if e and $e' = e + ea(1-e)$ $(a \in R)$ are idempotents of a ring R, then $e + ea(1-e) = ueu^{-1}$ for some unit $u \in R$ by [5, Exercise 21.4, page 333]. But the converse may not be true by the following example:

Example 2. Let R be the 2 by 2 matrix ring over $\mathbb{Z}_2$ where $\mathbb{Z}_2$ is a field of integers modulo 2. Consider the idempotent $e = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ of R. Then we check that

$$\{e + ea(1-e) | a \in R\} = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

Take $f^2 = f = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in R$. Then $f \notin \{e + ea(1-e) | a \in R\}$. On the other hand, e and f are conjugate since $e = ufu^{-1}$ for some unit $u = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in R$.

Now we raise the following question:

Question 1. Let e, e' be isomorphic idempotents of a ring R. If e is left (right) semicentral, then is e' left (right) semicentral?

Recall that [5, Exercise 21.16, page 334] if eRe is a semilocal ring, then e, e' are isomorphic if and only if $e' = ueu^{-1}$ for some unit $u \in R$. Hence if eRe is a semilocal ring, then the answer to the above question is true by Remark 1-(2).

Lemma 2.4. *Let R be a ring and S be a subset of R. Then S is additive in $I(R)$ if and only if S is commuting and $2ef = 0$ for all $e, f \in S$ $(e \neq f)$.*

Proof. Suppose that S is additive in $I(R)$. Let $e, f \in S_\ell(R)$ $(e \neq f)$ be arbitrary. Then $e + f = (e+f)^2 = e + ef + fe + f$, and so $ef = -fe$. Thus $ef = e(ef) = e(-fe) = (-ef)e = (fe)e = fe$. Hence S is commuting and also $2ef = 0$ for all $e, f \in S$ $(e \neq f)$. The converse is clear. □

Lemma 2.5. *For a ring R the following conditions are equivalent*:

(1) *$S_\ell(R)$ is commuting*;
(2) *$S_r(R)$ is commuting*;
(3) $S_\ell(R) = B(R)$;
(4) $S_r(R) = B(R)$.

Proof. (1) $\Leftrightarrow$ (2) follows from Corollary 2.3. (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1) are obvious.

(1) $\Rightarrow$ (3): Assume that $S_\ell(R)$ is commuting and let $e \in S_\ell(R)$ and $a \in R$ be arbitrary. Write $f = e + ea(1-e)$. Then $f \in S_\ell(R)$ by Remark 1-(3). Since $S_\ell(R)$ is commuting, $e = fe = ef = f = e + ea(1-e)$, and so $ea = eae = ae$. Hence e is central, and thus (1) implies (3). Similarly, we have (2) $\Rightarrow$ (4). □

Proposition 2.6. *For a ring R the following conditions are equivalent*:

(1) *$S_\ell(R)$ (resp. $S_r(R)$) is additive in $I(R)$*;
(2) *$S_\ell(R)$ (resp. $S_r(R)$) is commuting and $2e = 0$ for all $e \in S_\ell(R)$ (resp. $e \in S_r(R)$)*;
(3) *$S_\ell(R)$ (resp. $S_r(R)$) is commuting and the characteristic of R is equal to 2.*

Proof. First, we will prove it in the left semicentral case.

(1) $\Rightarrow$ (2): Suppose that $S_\ell(R)$ is additive in $I(R)$. Then $S_\ell(R)$ is commuting by Lemma 2.4. Let $e \in S_\ell(R)(e \neq 1)$ be arbitrary. Since $S_\ell(R)$ is additive in $I(R)$ and $1, e \in S_\ell(R)$, $1 + e \in I(R)$, and then $2e = 0$.

(2) $\Rightarrow$ (3): Suppose that $S_\ell(R)$ is commuting and $2e = 0$ for all $e \in S_\ell(R)$. Since $1 - e \in S_r(R)$ by Corollary 2.3 and $S_\ell(R) = S_r(R)$ by Lemma 2.5, we have $2(1-e) = 0$ by assumption, and so $2 \cdot 1 = 2e = 0$. Hence the characteristic of R is equal to 2.

(3) $\Rightarrow$ (1): Obvious.

Next, we can prove it in the right semicentral case by the similar argument used in the left semicentral case. □

Corollary 2.7. *Let R be a ring. Then $B(R)$ is additive in $I(R)$ if and only if $B(R)$ forms a Boolean ring.*

Proof. It follows from Lemma 2.5 and Proposition 2.6. □

Note that [5, Exercise 21.13, page 334] if e, f are commuting idempotents of a ring R such that $\bar{e} = \bar{f} \in R/N$ where N is a nil ideal of R, then $e = f$. It is well known that if N is a nil ideal of a ring R, then $N \subseteq J(R)$. In general, we have the following:

Proposition 2.8. *Let $N \subseteq J(R)$ be an ideal of a ring R. If $e, f \in R$ are commuting idempotents such that $\bar{e} = \bar{f} \in R/N$, then $e = f$.*

Proof. Since $\bar{e} = \bar{f} \in R/N$, $e - f \in N$. Since $ef = fe$, we have $(e-f)^2 = e - 2ef + f = (e-f)^4$, and so $(e-f)^2 \in I(R)$. Thus $(e-f)^2 \in I(R) \cap N \subseteq I(R) \cap J(R)$. Since $I(R) \cap J(R) = \{0\}$, $(e-f)^2 = e - 2ef + f = 0$. Hence

$e + f = 2ef$ $(*)$. By multiplying with e (resp. f) from the both sides of $(*)$, we have $e = ef$ (resp. $f = ef$). Hence $e - f = ef - ef = 0$. □

Recall $M_\ell(R)$ (resp. $M_r(R)$) is the set of all left (resp. right) semicentral primitive idempotents of a ring R.

Proposition 2.9. *Let R be a ring R. Then $M_\ell(R)$ (resp. $M_r(R)$) is additive in $I(R)$ if and only if $M_\ell(R)$ (resp. $M_r(R)$) is orthogonal.*

Proof. Suppose that $M_\ell(R)$ (resp. $M_r(R)$) is additive in $I(R)$ and assume that there exist $e, f \in M_\ell(R)$ (resp. $e, f \in M_r(R)$) such that $ef \neq 0$. Since $M_\ell(R)$ (resp. $M_r(R)$) is additive in $I(R)$, $M_\ell(R)$ (resp. $M_r(R)$) is commuting by Lemma 2.4, and so $ef = fe$. Note that $e = ef + (e - ef)$ and $ef(e - ef) = (e - ef)ef = 0$. Since e is primitive and $ef \neq 0$, $e = ef$. By the similar argument, we have $f = fe$ $(= ef)$. Thus $e = f$, a contradiction. Therefore, $ef = 0$, and so $M_\ell(R)$ (resp. $M_r(R)$) is orthogonal. The converse is clear. □

Proposition 2.10. *Let $N \subseteq J(R)$ be an ideal of R such that idempotents in R/N can be lifted to R. Then we have the following*:

(1) *If $S_\ell(R)$ (resp. $S_r(R)$) is commuting, then $S_\ell(R/N)$ (resp. $S_r(R/N)$) is orthogonal if and only if $S_\ell(R)$ (resp. $S_r(R)$) is orthogonal*;

(2) *If $M_\ell(R)$ (resp. $M_r(R)$) is commuting, then $M_\ell(R/N)$ (resp. $M_r(R/N)$) is orthogonal if and only if $M_\ell(R)$ (resp. $M_r(R)$) is orthogonal.*

Proof. (1) First, we will prove it in the left semicentral case. Suppose that $S_\ell(R/N)$ is orthogonal. Let $e, f \in S_\ell(R)$ $(e \neq f)$ be arbitrary. Clearly, $\bar{e}, \bar{f} \in S_\ell(R/N)$. Assume that $e, f \neq 0$. If $\bar{e} = \bar{f}$, then $e = f$ by Proposition 2.8, which is a contradiction. Thus $\bar{e} \neq \bar{f}$. Since $S_\ell(R/N)$ is orthogonal, $\bar{e}\bar{f} = \bar{f}\bar{e} = \bar{0}$, and so $ef, fe \in N$. By Proposition 2.1, $ef, fe \in I(R)$, and then $ef, fe \in I(R) \cap N \subseteq I(R) \cap J(R) = \{0\}$. Hence $S_\ell(R)$ is orthogonal. The converse is clear. Similarly, we can prove it in the right semicentral case.

(2) Note that if $e \in R$ is a primitive idempotent, then $\bar{e} \in R/N$ is also a primitive idempotent by [5, Proposition 21.22]. Hence it follows from the similar argument given in the proof of (1). □

Remark 2. Let $N \subseteq J(R)$ be an ideal of a ring R such that idempotents in R/N can be lifted to R. By Proposition 2.8, we note that if $S_\ell(R)$ (resp. $M_\ell(R)$, $M_r(R)$) is commuting, then $|S_\ell(R)| = |S_\ell(R/N)|$ (resp. $|M_\ell(R)| = |M_\ell(R/N)|$, $|M_r(R)| = |M_r(R/N)|$) where $|S|$ is the cardinality of a set S.

Corollary 2.11. *Let $N \subseteq J(R)$ be a nil ideal of a ring R in which every idempotent is central. Then $I(R)$ is orthogonal if and only if $I(R/N)$ is orthogonal.*

Proof. It follows from Lemma 2.5 and Proposition 2.10. □

Proposition 2.12. *For an idempotent e of a ring R the following conditions are equivalent*:

(1) *Every $e \in M_\ell(R)$ (resp. $e \in M_r(R)$) is central*;

(2) *$re = er$ for all $e \in M_\ell(R)$ (resp. $e \in M_r(R)$) and all units $r \in R$;*
(3) *$re = er$ for all $e \in M_\ell(R)$ (resp. $e \in M_r(R)$) and all nilpotent elements $r \in R$;*
(4) *$M_\ell(R)$ (resp. $M_r(R)$) is commuting;*
(5) *$ef = fe$ for all $f \in M_\ell(R)$ (resp. $f \in M_r(R)$) which are isomorphic to e;*
(6) *$(ef)^n = (fe)^n$ for all $f \in M_\ell(R)$ (resp. $f \in M_r(R)$) which are isomorphic to e where n is some positive integer.*

Proof. We will prove it in the left semicentral case. It is enough to show that (6) $\Rightarrow$ (1). Suppose that the condition (6) holds and assume that there exists $e \in M_\ell(R)$ such that e is not central. Then $ea \neq ae$ for some $a \in R$. Consider $f = e + ea(1-e)$. Clearly $e \neq f$, and $f \in S_\ell(R)$ by Remark 1. Since $f = ef$ and $e = fe$, f is isomorphic to e. We note that f is a primitive idempotent of R. Indeed, since $eR = efeR \subseteq efR \subseteq eR$, $eR = efR = fR$, and so f is a primitive idempotent of R. Therefore, $e = (fe)^n \neq (ef)^n = f$ for any positive integer n, which contradicts to the assumption (6). Hence $e \in M_\ell(R)$ is central.

Similarly, we can also prove it in the right semicentral case. □

Remark 3. It is clear that if $M(R)$ (resp. $M_\ell(R)$, $M_r(R)$) is commuting, then $M(R)$ (resp. $M_\ell(R)$, $M_r(R)$) is multiplicative. But the converse may not hold. Indeed, let R be the 2 by 2 matrix ring over $\mathbb{Z}_2$. Then we check that

$$M_\ell(R) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

$$(\text{resp. } M_r(R) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\})$$

and so $M_\ell(R)$ (resp. $M_r(R)$) is multiplicative but not commuting.

3. Some rings having a complete set of centrally primitive idempotents

Proposition 3.1. *If a ring R has a complete set of left (or right) semicentrally primitive idempotents, then c_i is central for all $i = 1, \dots, n$.*

Proof. Let $\{c_1, c_2, \dots, c_n\}$ be a complete set $\{c_1, c_2, \dots, c_n\}$ of left semicentrally primitive idempotents. Then $1 = c_1 + c_2 + \cdots + c_n$, and so $r = rc_1 + rc_2 + \cdots + rc_n = c_1rc_1 + c_2rc_2 + \cdots + c_nrc_n$ for all $r \in R$. Thus $c_ir = c_irc_i = rc_i$ for all $i = 1, \dots, n$, and so c_i is central for all $i = 1, \dots, n$. If $\{c_1, c_2, \dots, c_n\}$ is a complete set $\{c_1, c_2, \dots, c_n\}$ of right semicentrally primitive idempotents, then c_i is central for all $i = 1, \dots, n$ by the similar argument. □

Proposition 3.1 tells us that a ring R has a complete set of left (or right) semicentrally primitive idempotents if and only if a ring R has a complete set of centrally primitive idempotents. In [5, Proposition 22.1], it was shown that if R has a complete set $\{c_1, c_2, \dots, c_n\}$ of centrally primitive idempotents, then any central idempotents is a sum of a subset of $\{c_1, c_2, \ldots, c_n\}$. On the other hand, we have the following:

Proposition 3.2. *If a ring R has a complete set of centrally primitive idempotents, then any nonzero left* (*resp. right*) *semicentral idempotent of R is a sum of orthogonal left* (*resp. right*) *semicentral idempotents of R.*

Proof. **Case 1.** Left case.

Let $e \in R$ be any nonzero left semicentral idempotent and $\{c_1, c_2, \ldots, c_n\}$ be a complete set of centrally primitive idempotents of R. Since $1 = c_1 + c_2 + \cdots + c_n$, $e = ec_1 + ec_2 + \cdots + ec_n$. If $ec_i \neq 0$ for some i, then ec_i is a primitive idempotent of R by [3, Theorem 2.10]. On the other hand, for each i $(ec_i)r(ec_i) = e(rc_i)e = r(ec_i)$ for all r, and so each ec_i is a left semicentral idempotent of R. Thus if $ec_i \neq 0$ for some i, then ec_i is a left semicentral primitive idempotent of R, so $e = \sum_{ec_i \neq 0} ec_i$, which is a sum of left semicentral primitive idempotents of R. Clearly, $\{ec_i : ec_i \neq 0\}$ is orthogonal.

Case 2. Right case.

It follows from the similar argument given in the proof of Case 1. □

Corollary 3.3. *If a ring R has a complete set $\{c_1, c_2, \ldots, c_n\}$ of centrally primitive idempotents, then any central idempotent is a sum of a subset of $\{c_1, c_2, \ldots, c_n\}$.*

Proof. Let $e \in R$ be any central idempotent. Then $e = \sum_{ec_i \neq 0} ec_i$, which is a sum of primitive left semicentral idempotents of R as in the proof of Proposition 3.2. Note that if $ec_i \neq 0$ for some i, then $ec_i = c_i$. Therefore, we have $e = \sum_{ec_i \neq 0} ec_i = \sum_{ec_i \neq 0} c_i$. □

Proposition 3.4. *Let R be a ring which has a complete set of primitive idempotents. Then eRe has also a complete set of primitive idempotents for all nonzero left* (*resp. right*) *semicentral idempotent $e \in R$.*

Proof. **Case 1.** Left case.

Let $e \in R$ be an arbitrary nonzero left semicentral idempotent and $\{e_1, e_2, \ldots, e_n\}$ be a complete set of primitive idempotents. Then $1 = e_1 + e_2 + \cdots + e_n$, and so $e = e_1e + e_2e + \cdots + e_ne$. Since $e \in R$ is a left semicentral idempotent, $e_ie = ee_ie$ for all i. If $ee_ie \neq 0$ for some i, then ee_ie is a primitive idempotent of eRe by [1, Lemma 1.5]. Note that $\{ee_ie : ee_ie \neq 0\}$ is orthogonal and $e = \sum_{ee_ie \neq 0} ee_ie$. Therefore, $\{ee_ie : ee_ie \neq 0\}$ is a complete set of primitive idempotents of eRe.

Case 2. Right case.

It follows from the similar argument given in the proof of Case 1. □

Proposition 3.5. *If R is a ring which has a complete set T of primitive idempotents, then we have the following*:

(1) *If there exists a primitive idempotent $e \in R$ such that $ef = fe$ for all $f \in T$, then $e \in T$*;

(2) *All centrally primitive idempotents of R are contained in T;*
(3) *The set of all centrally primitive idempotents of R forms a complete set of centrally primitive idempotents of R.*

Proof. (1) Let $T = \{e_1, e_2, \ldots, e_n\}$. Then $1 = e_1 + e_2 + \cdots + e_n$, and so $e = e_1e + e_2e + \cdots + e_ne$. Note that if $e_ie \neq 0$ for some i, then $e = e_ie + (e - e_ie)$ such that $e_ie(e - e_ie) = (e - e_ie)e_ie = 0$, i.e., e is a sum of two orthogonal idempotents $e_ie, e - e_ie$ of R. Since e is a primitive idempotent of R, $e = e_ie$. Similarly, if $e_ie \neq 0$ for some i, then $e_i = e_ie + (e_i - e_ie)$ such that $e_ie(e_i - e_ie) = (e_i - e_ie)e_ie = 0$, i.e., e_i is a sum of orthogonal idempotents $e_ie, e_i - e_ie$ of R. Since e_i is a primitive idempotent of R, $e_i = e_ie$. Hence $e = e_ie = e_i \in T$.

(2) It follows from (1).

(3) Since R has a complete set of primitive idempotents, R has also a complete set T_1 of centrally primitive idempotents of R. Assume that there exists a centrally primitive idempotent $e \in R$ such that $e \notin T_1$. Let $T_1 = \{c_1, c_2, \ldots, c_n\}$. Then $1 = c_1 + c_2 + \cdots + c_n$, and so $e = c_1e + c_2e + \cdots + c_ne$. Note that if $c_ie \neq 0$ for some i, then $e = c_ie + (e - c_ie)$ such that $c_ie(e - c_ie) = (e - c_ie)c_ie = 0$, i.e., e is a sum of two orthogonal central idempotents $c_ie, e - c_ie$ of R. Since e is a centrally primitive idempotent of R, $e = c_ie \in R$. Similarly, if $c_ie \neq 0$ for some i, then $c_i = c_ie + (c_i - c_ie)$ such that $c_ie(c_i - c_ie) = (c_i - c_ie)c_ie = 0$, i.e., c_i is a sum of orthogonal central idempotents $c_ie, c_i - c_ie$ of R. Since c_i is a centrally primitive idempotent of R, $c_i = c_ie$. Hence $e = c_ie = c_i \in T_1$, a contradiction. Hence T_1 consists of all centrally primitive idempotents of R. □

Remark 4. Let R be a ring which has a complete set of primitive idempotents. By Proposition 3.5, we note that (1) there exist a finite number of centrally primitive idempotents in R which forms a complete set of centrally primitive idempotents; (2) in particular, if R is an abelian ring (a ring in which every idempotent is central), then all primitive idempotents of R forms a complete set of primitive idempotents.

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