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A RECURRENCE RELATION FOR THE JONES POLYNOMIAL

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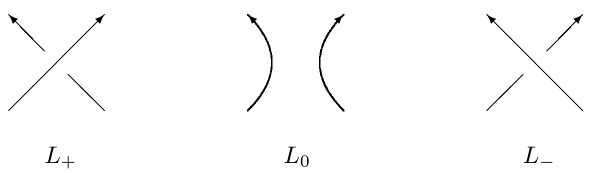
Abstract. Using a simple recurrence relation, we give a new method to compute the Jones polynomials of closed braids: we find a general expansion formula and a rational generating function for the Jones polynomials. The method is used to estimate the degree of the Jones polynomials for some families of braids and to obtain general qualitative results.

1. Introduction

The *Jones polynomial* $V_L(q)$ of an oriented link L is a Laurent polynomial in the variable $\sqrt{q}$ satisfying the skein relation

$$q^{-1}V_{L_+} - qV_{L_-} = (q^{1/2} - q^{-1/2})V_{L_0},$$

and such that the value of the unknot is 1 (see [10], [12], [14]). The relation holds for any oriented links having diagrams which are identical, except near one crossing where they differ as in the figure below:



Any link L can be obtained as a closure of a braid $\beta \in \mathcal{B}_n$ (for some n), $L = \widehat{\beta}$. We will use classical Artin presentation of braids ([1], [5]) with generators $x_1, \ldots, x_{n-1}$, where x_i is:

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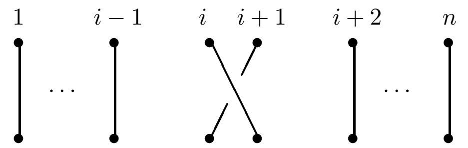
x_i



Fixing the sequence $(x_{i_1}, \dots, x_{i_k})$ of generators, $i_h \in \{1, 2, \dots, n-1\}$, and all the exponents $a_1, \dots, a_k$ ($a_h \in \mathbb{Z}$), but allowing the j-th exponent to vary, we have the braid $\beta(e) = x_{i_1}^{a_1} \cdots x_{i_j}^{e} \cdots x_{i_k}^{a_k} \in \mathcal{B}_n$ and its Jones polynomial $V_n(e) = V(\widehat{\beta(e)})$; in general $V_n(\beta)$ stands for $V(\widehat{\beta})$, where $\beta \in \mathcal{B}_n$. We will freely use Artin braid relations and Markov moves in proofs and some computations.

We also change the variable $s = q^{-1/2}$ in order to obtain, for large e, polynomials in s (and not Laurent polynomials in $\sqrt{q}$). Our first result is:

Theorem 1.1 (The recurrence relation)**.** *For any $e \in \mathbb{Z}$, we have*

$$V_n(e+2) = (s^3 - s)V_n(e+1) + s^4 V_n(e)\,.$$

This formula shows that in computations with the Jones polynomial of braids the exponents can be reduced to 0 and 1. This result is not new, as it follows from quadratic relations in Hecke algebras and the works V. F. R. Jones [11] and A. Ocneanu [18]. See also [15] for applications of these ideas to computations.

Systematic and elementary algebraic consequences of quadratic reduction give us a general expansion formula and the generating function for Jones polynomials.

Theorem 1.2 (The expansion formula)**.** *The following formula holds for braids in $\mathcal{B}_n$ ($a_1, \dots, a_k \in \mathbb{Z}$ and $J_* = (j_1, \dots, j_k)$) :*

$$V_n(x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k})(s) = \frac{1}{(s^2+1)^k} \sum_{J_* \in \{0,1\}^k} P_{j_1}^{[a_1]}(s) \cdots P_{j_k}^{[a_k]}(s) V_n(x_{i_1}^{j_1} \cdots x_{i_k}^{j_k})(s),$$

where

$$P_0^{[a]}(s) = s^{3a} + (-1)^a s^{a+2} \ \text{and}\ P_1^{[a]}(s) = s^{3a-1} + (-1)^{a+1} s^{a-1}.$$

We define the *generating function* for the Jones polynomials corresponding to the braids $\beta^{A_*} = x_{i_1}^{a_1} \cdots x_{i_k}^{a_k} \in \mathcal{B}_n$ with a fixed sequence $I_* = (i_1, \dots, i_k) \in \{1, \dots, n-1\}^k$, as a formal series in $t_1, \dots, t_k$

$$\mathcal{V}_{n,I_*}(t_1, \dots, t_k) = \sum_{A_* \in \mathbb{Z}^k} V_n(x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}) t_1^{a_1} \cdots t_k^{a_k}.$$

Theorem 1.3. *The generating function of the Jones polynomials of type (n, I_*) is a rational function in $t_1, \dots, t_k$ given by*

$$\mathcal{V}_{n,I_*}(t_1, \dots, t_k) = \frac{1}{q(t_1) \cdots q(t_k)} \sum_{J_* \in \{0,1\}^k} Q_{j_1}(t_1) \cdots Q_{j_k}(t_k) V_n(x_{i_1}^{j_1} \cdots x_{i_k}^{j_k}),$$

where

$$q(t) = (1 - s^3 t)(1 + st), Q_0(t) = 1 - (s^3 - s)t, \text{ and } Q_1(t) = t.$$

In Section 2 we give proofs for Theorems 1.1, 1.2 and 1.3. The necessary algebraic background is given in Appendix (for details see [17]).

In Section 3 we will use the recurrence relation to evaluate the degrees of the Jones polynomials. One result is the following:

Proposition 1.4. *If $e \gg 0$, then $V_n(e)$ is a polynomial in s and*

$$\lim_{e\to\infty} \deg V_n(e) = +\infty.$$

There are still open problems relating the Jones invariant with closed 3-braids (see the paper of J. Birman [6]). In Section 4 we compute the Jones polynomials of 2-braids, some families of 3-braids, and powers of Garside braid $\Delta_3 = x_1x_2x_1$ (see [5], [9]). We establish the following results:

Proposition 1.5. *For any $k \geq 0$, we have*

$$V_3(\Delta_3^{2k}) = 2s^{12k} + s^{6k+2} + s^{6k-2},$$

$$V_3(\Delta_3^{2k+1}) = -s^{6k+5} - s^{6k+1}.$$

Theorem 1.6. a) *For exponents $a_i \geq 1$ $(i = 1, \ldots, 2L)$, the Jones polynomial of the 3-braid $\beta^{A_*} = x_1^{a_1}x_2^{a_2}\cdots x_1^{a_{2L-1}}x_2^{a_{2L}}$ of total degree $A = \sum a_i$ satisfies*:

$$\deg V_3(x_1^{a_1}x_2^{a_2}\cdots x_1^{a_{2L-1}}x_2^{a_{2L}}) \leq 3A - 2L.$$

b) *For exponents $a_i \geq 2$ $(i = 1, \ldots, 2L)$ and $L \leq 4$ the equality holds*:

$$\deg V_3(x_1^{a_1}x_2^{a_2}\cdots x_1^{a_{2L-1}}x_2^{a_{2L}}) = 3A - 2L$$

and the leading coefficient of $V_3(\ \beta^{A_*})$ is 1.

The degree, coefficients, and breadth of the Jones polynomial are well understood for special classes of links ([16], [21]). See also [19] to study more about the subject. Research in this area (see [4], [7], [8]) is motivated by a natural question: are there nontrivial solutions for the equation $V(\widehat{\beta}) = 1$? In Section 5 we find that:

Theorem 1.7. *The sequence $(V_n(e))_{e\in\mathbb{Z}}$ could contain at most two polynomials $V_n(a)$ and $V_n(b)$ equal to 1, and in this case $|a - b| = 2$.*

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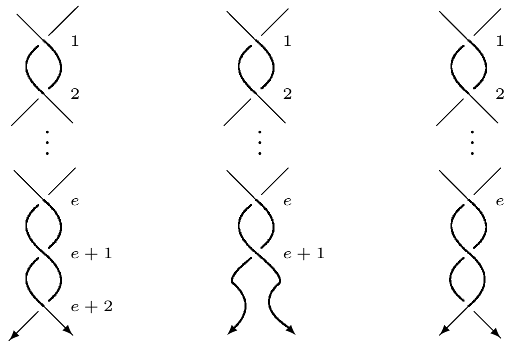
2. Proofs of the main results

In this section we fix $n \geq 2$, generators $x_{i_1}, \ldots, x_{i_k} \in \mathcal{B}_n$, the index j, and the exponents $(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k)$. First we translate the skein relation for the Jones polynomial into a recurrence relation of Fibonacci type:

$$V_n(e+2) = (s^3 - s)V_n(e+1) + s^4 V_n(e).$$

Proof of Theorem 1.1. Let $\gamma = \alpha x_{i_j}^e \beta$, where $\alpha = x_{i_1}^{a_1} \cdots x_{i_{j-1}}^{a_{j-1}}$ and $\beta = x_{i_{j+1}}^{a_{j+1}} \cdots x_{i_k}^{a_k}$ be fixed. Let us take e to be positive. Since the geometrical change appears only at the j-th position we draw a local picture (for the j-th factor only):



L_- L_0 L_+

Now it is clear from the figure that the relation $q^{-1}V_{L_+} - qV_{L_-} = (q^{1/2} - q^{-1/2})V_{L_0}$ becomes, after the changes $q \to s^{-2}$ and $L_- \to \widehat{\beta}(e+2)$, $L_0 \to \widehat{\beta}(e+1)$ and $L_+ \to \widehat{\beta}(e)$, simply $V_n(e+2) = (s^3 - s)V_n(e+1) + s^4 V_n(e)$. The negative case ($e < 0$) can be reduced to the positive case using the transformation $V_n(x_{i_1}^{a_1} \cdots x_{i_j}^{e} \cdots x_{i_k}^{a_k}) = V_n(x_{i_1}^{a_1} \cdots x_{i_j}^{m+e} x_{i_j}^{-m} \cdots x_{i_k}^{a_k})$ with m big enough. $\square$

Remark 2.1. One can find a similar recurrence relation for the Alexander-Conway polynomial and, in general, for the HOMFLY polynomial. It is interesting to remark that in the classical cases, Jones' and Alexander's, the roots of the characteristic equation are rational functions: $r_1 = -s$, $r_2 = s^3$ for the Jones polynomial and $r_1 = -s$, $r_2 = s^{-1}$ for the Alexander polynomial. Jones recurrence is nicer because the roots are polynomials. For the Alexander-Conway polynomial and more results on the roots of the characteristic equation for the HOMFLY polynomial see [2].

In Appendix (or see [17] for full details), multiple Fibonacci sequences are introduced. Our main example is the multiple Fibonacci sequence given by the Jones polynomials of closures of braids $V_{n,I_*}(x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k})$, where $a_1, \ldots, a_k \in \mathbb{Z}$; we fix n (all braids are in $\mathcal{B}_n$), k, and also $I_* = (i_1, \ldots, i_k)$ with indices $i_h \in \{1, \ldots, n-1\}$. Applying Theorem 6.1 we obtain:

Proof of Theorem 1.2. From the basic recurrence relation, we have $r^2 = (s^3 - s)r + s^4$, with the roots $r_1 = -s$ and $r_2 = s^3$. Hence $D = s^3 + s = s(s^2+1)$, $S_0^{[n]} = s^{3n+1} + (-1)^n s^{n+3} = s[s^{3n} + (-1)^n s^{n+2}]$, $S_1^{[n]} = s^{3n} + (-1)^{n+1}s^n = s[s^{3n-1} + (-1)^{n+1}s^{n-1}]$. Using the polynomials $P_0^{[a]}$ and $P_1^{[a]}$ (these are Laurent polynomials for $a < 0$) and also Theorem 6.1, we get

$$V_n(x_{i_1}^{a_1}x_{i_2}^{a_2}\cdots x_{i_k}^{a_k}) = \frac{1}{s^k(s^2+1)^k}\sum_{J_*\in\{0,1\}^k}(sP_{j_1}^{[a_1]})\cdots(sP_{j_k}^{[a_k]})V_n(x_{i_1}^{j_1}\cdots x_{i_k}^{j_k})$$
$$= \frac{1}{(s^2+1)^k}\sum_{J_*\in\{0,1\}^k}P_{j_1}^{[a_1]}\cdots P_{j_k}^{[a_k]}V_n(x_{i_1}^{j_1}\cdots x_{i_k}^{j_k})\,. \qquad \square$$

Proof of Theorem 1.3. In the second part of Theorem 6.1 take $r_1 = -s$ and $r_2 = s^3$. □

Remark 2.2. a) The expansion formula has 2^k terms, where k is the number of factors of the braid $\beta = x_{i_1}^{a_1}\cdots x_{i_k}^{a_k}$, less than $2^{|a_1|+\cdots+|a_k|}$, the number of terms in the Kauffman expansion (see [13], [14]).

b) This formula reduces the computation of the Jones polynomial of β to the computation of the Jones polynomial of (simpler) positive braids $\beta_{J_*} = x_{i_1}^{j_1}\cdots x_{i_k}^{j_k}$, with $J_* \in \{0,1\}^k$. If these braids contain exponents ≥ 2 (after possible concatenations the number of factors is less than k), another application of expansion formula will reduce degrees and the number of factors. Therefore, iterated application of the expansion formula reduces computations to the computation of the Jones polynomials of simple braids (see [3] for definition and proofs) $\beta_{I_*} = x_{i_1}x_{i_2}\cdots x_{i_k}$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq n-1$, and in this case $V_n(\beta_{I_*}) = (-s - s^{-1})^{n-k-1}$.

3. Degree of the Jones polynomial

First we study the behavior of the function $e \mapsto \deg V(\widehat{\beta(e)}) = \deg V(e)$ for an n-braid $\beta(e) = x_{i_1}^{a_1}\cdots x_{i_j}^{e}\cdots x_{i_k}^{a_k}$. More precise results are given for some families of 3-braids in Section 4. If the Laurent polynomial $f = a_q s^q + a_{q+1}s^{q+1} + \cdots + a_p s^p$ has coefficients $a_q, a_p \neq 0$, we denote the degree $\deg(f) = p$, the order $\mathrm{ord}(f) = q$, and the leading coefficient $\mathrm{coeff}(f) = a_p$; we also use the convention $\deg(0) \leq 0$.

Definition. a) $\beta(e)$, $\beta(e+1)$ is a *stable pair* if

$$\deg V(e+1) > 1 + \deg V(e).$$

b) $\beta(e)$, $\beta(e+1)$ is a *semistable pair* if

$$\deg V(e+1) \leq \deg V(e).$$

c) $\beta(e)$, $\beta(e+1)$ is a *critical pair* if

$$\deg V(e+1) = 1 + \deg V(e).$$

Proposition 3.1 (stable case)**.** *If $\beta(e)$, $\beta(e+1)$ is a stable pair, then for all $m \geq 1$ the pair $\beta(e+m)$, $\beta(e+m+1)$ is also stable,*

$$\deg V(e+m) = \deg V(e+1) + 3(m-1),$$

and $V(e+1)$ and $V(e+m)$ have the same leading coefficient.

Proof. We prove the statement by induction on m, starting with the stable pair $\beta(e)$, $\beta(e+1)$. If $\beta(e+m-1)$, $\beta(e+m)$ is a stable pair, then Theorem 1.1 implies

$$V(e+m+1) = (s^3 - s)V(e+m) + s^4 V(e+m-1) \tag{1}$$

and $\deg V(e+m) > 1 + \deg V(e+m-1)$ implies $\deg V(e+m+1) = \deg V(e+m) + 3 = \deg V(e+1) + 3m$ (hence $\beta(e+m)$, $\beta(e+m+1)$ is stable) and $\operatorname{coeff} V(e+m+1) = \operatorname{coeff} V(e+m)$. □

Proposition 3.2 (semistable case)**.** *If $\beta(e)$, $\beta(e+1)$ is a semistable pair, then for all $m \geq 1$ the pair $\beta(e+m)$, $\beta(e+m+1)$ is stable*; *for $m \geq 2$ we have*

$$\deg V(e+m) = \deg V(e) + 3m - 2$$

and also $V(e)$ and $V(e+m)$ have the same leading coefficient.

Proof. The pair $\beta(e)$, $\beta(e+1)$ is semistable and the recurrence

$$V(e+2) = (s^3 - s)V(e+1) + s^4 V(e)$$

implies $\deg V(e+2) = \deg V(e) + 4 > \deg V(e+1) + 1$, hence the pair $\beta(e+1)$, $\beta(e+2)$ is stable and we can apply Proposition 3.1. □

In the last (critical) case we cannot obtain complete results; see the Case 2*c*) below.

Proposition 3.3 (critical case)**.** *Let $\beta(e)$, $\beta(e+1)$ be a critical pair and C the sum of leading coefficients of $V(e)$ and $V(e+1)$.*

Case 1) *If $C \neq 0$, then for all $m \geq 1$ the pair $\beta(e+m)$, $\beta(e+m+1)$ is stable,*

$$\deg V(e+m) = \deg V(e+1) + 3(m-1)\,,$$

and also the leading coefficient of $V(e+m)$ is C.

Case 2) *If $C = 0$ and*

2a) $\deg V(e+2) = 2 + \deg V(e+1)$, *then for all $m \geq 1$ the pair $\beta(e+m)$, $\beta(e+m+1)$ is stable and for all $m \geq 2$ the following holds*

$$\deg V(e+m) = \deg V(e+1) + 3m - 4.$$

2b) $\deg V(e+2) \leq \deg V(e+1)$, *then $\beta(e+2)$, $\beta(e+3)$ is a stable pair and for all $m \geq 2$ the pair $\beta(e+m)$, $\beta(e+m+1)$ is stable. For all $m \geq 3$ we have the relation*

$$\deg V(e+m) = \deg V(e+1) + 3m - 5.$$

2c) $\deg V(e+2) = 1 + \deg V(e+1)$, *then the pair $\beta(e+1)$, $\beta(e+2)$ is critical.*

Proof. Case 1) We prove it by induction on m. For $m = 1$, Theorem 1.1 implies

$$V(e+2) = (s^3 - s)V(e+1) + s^4 V(e),$$

and condition $C \neq 0$ implies that $V(e+2)$ has degree $\deg V(e+1) + 3$ and leading coefficient C. We can apply Proposition 3.1 for the stable pair $\beta(e+1)$, $\beta(e+2)$.

Case 2) The hypotheses imply that $\beta(e+1)$, $\beta(e+2)$ is a) stable, b) semistable, and c) critical, respectively. □

Remark 3.4. In the critical case the degree of $V(e+2)$ can be arbitrarily small compared to degree of $V(e)$; see, in Section 4, the family $V_3(a_1, 1, 3, 1)$ and also the degrees in the sequence $V_\Delta(6k+1)$, $V_\Delta(6k+2)$, $V_\Delta(6k+3)$.

The order of $V(e)$ is increasing for large value of e.

Proposition 3.5. *With the same notations we have*:

a) $\operatorname{ord} V(e+2) \geq 1 + \min(\operatorname{ord} V(e), \operatorname{ord} V(e+1))$.

b) *For any* $m \geq 2$, $\operatorname{ord} V(e+m) \geq \min(\operatorname{ord} V(e), \operatorname{ord} V(e+1)) + (m-1)$. *In particular, if* $e \gg 0$, *then* $V(e)$ *is a polynomial in* s.

Proof. a) is a direct consequence of Theorem 1.1, and b) comes from a) by induction. □

Proof of Proposition 1.4. First choose $e_0 \geq 0$ such that for any $m \geq e_0$, $V(m)$ is a polynomial in s, as in the previous proposition. Let us suppose that in the sequence $\beta(e_0), \beta(e_0+1), \ldots$ there is a stable (or semistable) pair, say $\beta(e_1), \beta(e_1+1)$. Then according to Proposition 3.1 (or Proposition 3.2), all the consecutive pairs in the sequence $\beta(e_1+1), \beta(e_1+2), \ldots$ are stable. Therefore $\deg V(e_1+m) \to \infty$ as $m \to \infty$. If any pair of the sequence $\beta(e_0), \beta(e_0+1), \ldots$ is critical, then the sequence $\deg \beta(e_0), \deg \beta(e_0+1), \ldots$ is arithmetic and again we get the desired result.
Second Proof. We can use Proposition 3.5 along with the fact that the Jones polynomial cannot be 0. □

Similar results can be proved for negative exponents.

Proposition 3.6. *With* $e < 0$ *we have*:

a) $\deg V(e-2) \leq \max(\deg V(e-1), \deg V(e)) - 1$.

b) *For any* $m \geq 2$, $\deg V(e-m) \leq \max(\deg V(e-1), \deg V(e)) - (m-1)$. *In particular, for* $e \ll 0$, $V(e)$ *is a polynomial in* s^{-1} *and*

$$\lim_{e \to -\infty} \deg V(e) = -\infty.$$

Proof. Use the recurrence relation in the form

$$V(e-2) = (s^{-3} - s^{-1})V(e-1) + s^{-4} V(e).$$

□

4. Examples

Three types of examples are given: arbitrary braids in two-strand braid group $\mathcal{B}_2$, braids $x_1^{a_1}x_2^{a_2}\cdots x_2^{a_{2L}}$ in $\mathcal{B}_3$ (with complete results for $L \leq 2$), and powers of the Garside braid Δ_3. Some of the results are well known, especially those which are connected with torus links, some others (Propositions 4.4, 4.5, and part of Proposition 4.9) are new, and all of them show how to use the recurrence relation.

Proposition 4.1. *Let x_1^a, $a \in \mathbb{Z}$, be a braid in $\mathcal{B}_2$, and $\widehat{x_1^a}$ the corresponding link. Its Jones polynomial is given by: for $a \leq -2$*

$$V_2(a) = -s^{3a+1}+s^{3a+3}-s^{3a+5}+\cdots-(-1)^{a+1}s^{a-5}+(-1)^{a+1}s^{a-3}+(-1)^{a+1}s^{a+1},$$

for the next three values

$$V_2(-1) = 1,\ V_2(0) = -s - s^{-1},\ V_2(1) = 1,$$

and for $a \geq 2$

$$V_2(a)= -s^{3a-1}+s^{3a-3}-s^{3a-5}+\cdots-(-1)^{a+1}s^{a+5}+(-1)^{a+1}s^{a+3}+(-1)^{a+1}s^{a-1}.$$

Proof. The Jones polynomials of the trivial two-component link and trivial knot are given by

$$V_2(0) = -s - s^{-1}, V_2(\pm 1) = 1. \tag{2}$$

From the basic recurrence relation the general term is

$$V_2(a) = \left(\frac{-s}{1+s^2}\right)(s^3)^a + \left(\frac{-1-s^2-s^4}{s+s^3}\right)(-s)^a\,. \tag{3}$$

Elementary computations give the desired result.

For $a \leq -2$, the coefficients $\frac{-s}{1+s^2}$ and $\frac{-1-s^2-s^4}{s+s^3}$ are invariant under $s \to s^{-1}$, therefore

$$V_2(a)(s) = \left(\frac{-s}{1+s^2}\right)(s^3)^a + \left(\frac{-1-s^2-s^4}{s+s^3}\right)(-s)^a = V_2(-a)(s^{-1})\,, \tag{4}$$

and this proves the formula for negative exponents. □

Proposition 4.2. *Let $\alpha, \beta \in \mathcal{B}_n$, $\gamma = \alpha\beta$, and $\gamma_k = \alpha x_n^k \beta \in \mathcal{B}_{n+1}$ ($k \in \mathbb{Z}$), then*

$$V_{n+1}(\gamma_k) = V_n(\gamma)V_2(x_1^k).$$

Proof. The relation is a consequence of the multiplicative property of the Jones polynomial: $V(K_1\#K_2) = V(K_1)V(K_2)$.

A direct consequence of the recurrence relation provides a different proof: the polynomials $V_{n+1}(\gamma_k)$ and $V_n(\gamma)V_2(x_1^k)$ coincide for $k = 0, 1$ and satisfy the same recurrence relation. □

Corollary 4.3. $V_3(x_1^{a_1}x_2^{a_2}) = V_2(x_1^{a_1})V_2(x_1^{a_2})$.

Now we compute the degree and the leading coefficient for the Jones polynomial $V_3(a_1, a_2, a_3, a_4) = V_3(x_1^{a_1} x_2^{a_2} x_1^{a_3} x_2^{a_4})$, where $a_i \geq 1$. We denote by A the total degree $a_1 + a_2 + a_3 + a_4$.

Proposition 4.4 (generic case)**.** *If $a_1, a_2, a_3, a_4 \geq 2$, then the leading term of the Jones polynomial $V_3(a_1, a_2, a_3, a_4)$ is $+s^{3A-4}$.*

Proof. Using the general expansion formula we find

$$V_3(a_1, a_2, a_3, a_4) = \frac{1}{(s^2+1)^4} \sum_{J_* \in \{0,1\}^4} P_{j_1}^{[a_1]} P_{j_2}^{[a_2]} P_{j_3}^{[a_3]} P_{j_4}^{[a_4]} V_3(j_1, j_2, j_3, j_4)\,.$$

The degree and leading coefficient of $V_3(j_1, j_2, j_3, j_4)$ are in Table 2 (proof of Theorem 1.6 contains a user guide for the table). In the above formula maximal degree is obtained from $V_3(1,0,1,0) = V_3(0,1,0,1)$ (coefficient $1+1$) and from $V_3(1,1,1,1)$ (coefficient -1). □

Proposition 4.5. *The leading terms of the Jones polynomials $V_3(a_1, a_2, a_3, a_4)$ for positive exponents are given by the next table*:

TABLE 1. $V_3(a_1, a_2, a_3, a_4)(s)$

(a_1, a_2, a_3, a_4)	*Critical cases*	*Stable cases*
$(1, 1, a_3, a_4)$	$-$	$a_3, a_4 \geq 1 : -s^{3A-4} + \cdots$
$(a_1, 1, 2, 1)$	$a_1 = 2 : 2s^{3A-6} + \cdots$	$a_1 \geq 3 : s^{3A-6} + \cdots$
$(a_1, 1, 3, 1)$	$a_1 = 3 : -s^{3A-8} + \cdots$ $a_1 = 4 : -s^{3A-16} + \cdots$	$a_1 \geq 5 : -s^{3A-10} + \cdots$
$(a_1, 1, a_3, 1)$	$-$	$a_1, a_3 \geq 4 : s^{3A-8} + \cdots$
$(a_1, 1, 3, 2)$	$-$	$a_1 \geq 3 : -s^{3A-6} + \cdots$
$(a_1, 1, a_3, a_4)$	$-$	$a_1, a_3, a_4 \geq 3 : -s^{3A-6} + \cdots$
(a_1, a_2, a_3, a_4)	$-$	$a_1, a_2, a_3, a_4 \geq 2 : s^{3A-4} + \cdots$

Proof. For the first line we use Markov moves and Proposition 4.1:

$$V_3(x_1 x_2 x_1^{a_3} x_2^{a_4}) = V_3(x_2^{a_3} x_1 x_2^{a_4+1}) = V_3(x_1^{a_3+a_4+1} x_2) = V_2(x_1^{a_3+a_4+1}).$$

In the second line we start the recurrence with $V_3(1,1,2,1) = V_2(4) = -s^{11} + s^9 - s^7 - s^3$ and $V_3(2,1,2,1) = 2s^{12} + s^8 + s^4$, a critical pair with constant $C = 1$; we obtain, for $a_1 \geq 3$, $V_3(a_1, 1, 2, 1) = s^{3A-6} + \cdots$ using Proposition 3.3 case 1.

For the family $V_3(a_1, 1, 3, 1)$ the recurrence starts with $V_3(1,1,3,1) = V_2(5) = -s^{14} + s^{12} - s^{10} + s^8 + s^4$ and $V_3(2,1,3,1) = s^{15} - s^{13} - s^9 - s^5$, and we obtain $V_3(3,1,3,1) = -s^{16} + s^{10} + s^6$ (two consecutive critical pairs with $C = 0$) and also $V_3(4,1,3,1) = -s^{11} - s^7$; the last pair is semistable and we use Proposition 3.2.

In the generic case $V_3(a_1,1,a_3,1), a_1,a_3\geq 4$, the expansion formula has only four J_* blocks

$$\frac{1}{(s^2+1)^2}\Big[P_0^{[a_1]}P_0^{[a_3]}(s^6+s^4+s^2+\cdots)\\ +\big(P_1^{[a_1]}P_0^{[a_3]}+P_0^{[a_1]}P_1^{[a_3]}\big)(-s^5-s)+P_1^{[a_1]}P_1^{[a_3]}(-s^8+s^6+s^2)\Big];$$

the leading terms of the polynomials $P_0^{[a_1]}P_0^{[a_3]}$, $P_1^{[a_1]}P_0^{[a_3]}$ (and $P_0^{[a_1]}P_1^{[a_3]}$), and $P_1^{[a_1]}P_1^{[a_3]}$ are respectively $s^{3(a_1+a_3)}$, $s^{3(a_1+a_3)-1}$, and $s^{3(a_1+a_3)-2}$. For $a\geq 4$ the difference between the highest degree of $P_j^{[a]}$ $(j=0,1)$ and the degree of the second term is at least 6, therefore in the square bracket above, after cancelation of terms of degrees $3(a_1+a_3)+6$ and $3(a_1+a_3)+4$, the leading term has degree $3(a_1+a_3)+2$ with leading coefficient 1.

For the line $(a_1,1,3,2)$ the recurrence starts with the semistable pair

$$V_3(1,1,3,2)=V_2(6)=-s^{17}+\cdots \text{ and}$$
$$V_3(2,1,3,2)=(s^3-s)V_3(2,1,3,1)+s^4V_3(2,1,3,0)=-s^{16}+\cdots;$$

we obtain $V_3(3,1,3,2)=-s^{21}+\cdots$, and the last pair is stable.

In the line $(a_1,1,a_3,a_4)$, the generic case $a_1,a_3,a_4\geq 3$ is given by the expansion formula with eight nonzero terms

$$\frac{1}{(s^2+1)^3}\Big[P_0^{[a_1]}P_0^{[a_3]}P_0^{[a_4]}(-s-\cdots)+P_0^{[a_1]}P_0^{[a_3]}P_1^{[a_4]}(s^6+s^4+\cdots)\\ +\big(P_1^{[a_1]}P_0^{[a_3]}P_0^{[a_4]}+P_0^{[a_1]}P_1^{[a_3]}P_0^{[a_4]}\big)\cdot 1\\ +\big(P_1^{[a_1]}P_1^{[a_3]}P_0^{[a_4]}+P_1^{[a_1]}P_0^{[a_3]}P_1^{[a_4]}+P_0^{[a_1]}P_1^{[a_3]}P_1^{[a_4]}\big)(-s^5+\cdots)\\ +P_1^{[a_1]}P_1^{[a_3]}P_1^{[a_4]}(-s^8+s^6+\cdots)\Big];$$

the leading term of the product $P_{j_1}^{[a_1]}P_{j_3}^{[a_3]}P_{j_4}^{[a_4]}$ is $s^{3(a_1+a_3+a_4)-j_1-j_3-j_4}$, and the next terms have degrees less than or equal to $3(a_1+a_3+a_4)-j_1-j_3-j_4-4$ (here we use $a_1,a_3,a_4\geq 3$). In the square bracket, after the cancelation of terms of degree $3(a_1+a_3+a_4)+5$, the leading term is $-s^{3(a_1+a_3+a_4)+3}$.

The generic case, $a_1,a_2,a_3,a_4\geq 4$, was analyzed in Proposition 4.4. □

The missing cases $(2,1,a_3,a_4)$ and $(a_1,1,2,a_4)$ can be reduced to the previous list:

$$x_1^2x_2x_1^{a_3}x_2^{a_4}=x_1x_2^{a_3}x_1x_2^{a_4+1}\sim x_1^{a_3}x_2x_1^{a_4+1}x_2,$$
$$x_1^{a_1}x_2x_1^2x_2^{a_4}=x_2x_1x_2^{a_1}x_1x_2^{a_4}\sim x_1^{a_1}x_2x_1^{a_4+1}x_2.$$

If $a_1\geq 3$, $a_3\geq 4$, then $(a_1,1,a_3,2)$ can be reduced, too:

$$x_1^{a_1}x_2x_1^{a_3}x_2^2=x_2x_1x_2^{a_1}x_1^{a_3-1}x_2^2\sim x_1^3x_2x_1^{a_1}x_2^{a_3-1}.$$

Now our purpose is to compute the Jones polynomial of the braid $\alpha(n)=x_1x_2x_1x_2\cdots$ (n factors); this sequence contains the powers of $\Delta_3=\Delta$: $\alpha(3k)=\Delta^k$. We will use the next table where X is the canonical form of $\alpha(n)$ (i.e., the smallest word in the length-lexicografic order with $x_1 < x_2$) and Y is a conjugate of X, suitable for computations. The number of factors x_1^a and x_2^b in each Y is $2k+2$.

$\alpha(n)$	X	Y
Δ^{2k}	$x_1^{2k}x_2x_1^2x_2^2\cdots x_1^2x_2$	$x_1^{2k}x_2x_1^2x_2^2\cdots x_1^2x_2$
$\Delta^{2k}x_1$	$x_1^{2k+1}x_2x_1^2x_2^2\cdots x_1^2x_2$	$x_1^{2k+1}x_2x_1^2x_2^2\cdots x_1^2x_2$
$\Delta^{2k}x_1x_2$	$x_1^{2k+1}x_2x_1^2x_2^2\cdots x_1^2x_2^2$	$x_1^{2k+1}x_2x_1^2x_2^2\cdots x_1^2x_2^2$
Δ^{2k+1}	$x_1^{2k+1}x_2x_1^2x_2^2\cdots x_2^2x_1$	$x_1^{2k+2}x_2x_1^2x_2^2\cdots x_1^2x_2^2$
$\Delta^{2k+1}x_2$	$x_1^{2k+2}x_2x_1^2x_2^2\cdots x_2^2x_1$	$x_1^{2k+3}x_2x_1^2x_2^2\cdots x_1^2x_2^2$
$\Delta^{2k+1}x_2x_1$	$x_1^{2k+2}x_2x_1^2x_2^2\cdots x_2^2x_1^2$	$x_1^{2k+4}x_2x_1^2x_2^2\cdots x_1^2x_2^2$

In order to simplify the notation we denote by $V_\Delta(n) = V_3(\alpha(n))$ the Jones polynomial of the closure of 3-braid $x_1x_2x_1x_2\cdots$ (n factors).

Proposition 4.6. *The Jones polynomials $V_\Delta(n)$ satisfy the following recurrence relations*:

$$\begin{aligned}
V_\Delta(2k+1) &= (s^3-s)V_\Delta(2k) + s^4V_\Delta(2k-1)\\
V_\Delta(6k+4) &= (s^3-s)V_\Delta(6k+3) + s^4V_\Delta(6k+2)\\
V_\Delta(6k+2) &= (s^3-s)V_\Delta(6k+1) + (s^7-s^5)V_\Delta(6k-1) + s^8V_\Delta(6k-2)\\
V_\Delta(6k) &= (s^3-s)[V_\Delta(6k-1) + s^4V_\Delta(6k-3) + s^8V_\Delta(6k-5)]\\
&\quad + s^{12}V_\Delta(6k-6).
\end{aligned}$$

Proof. The proof is by induction. For the first relation, the case $6k+5$ is given by the basic recurrence relation and the table. We give a general proof, using a different idea: $\alpha(2k-1) = x_1x_2\cdots x_1$ ($2k-1$ factors), $\alpha(2k) = x_1x_2\cdots x_2$ ($2k$ factors), $\alpha(2k+1) = x_1x_2\cdots x_1x_2x_1 \sim x_2x_1\cdots x_1x_2 \sim x_1x_2\cdots x_1x_2^2$ and now we can apply the basic recurrence relation. $V_\Delta(6k+4)$ is also given by the basic recurrence relation with $V_3(x_1^{2k+3}x_2x_1^2\cdots)$, $V_3(x_1^{2k+2}x_2x_1^2\cdots)$ and $V_3(x_1^{2k+1}x_2x_1^2\cdots)$. For the last two relations we have to apply Theorem 1.1 two or three times.

To compute $V_\Delta(6k+2)$ we use the basic recurrence relation twice: first, the recurrence relation among $\alpha(6k+2) = x_1^{2k+1}x_2x_1^2\cdots x_1^2x_2^2$, $\alpha(6k+1) = x_1^{2k+1}x_2x_1^2\cdots x_1^2x_2^1$ and $x_1^{2k+1}x_2x_1^2\cdots x_2^2x_1^2x_2^0$ which is conjugate to the braid $\beta = x_1^{2k+3}x_2x_1^2\cdots x_2^2$. The new braids β, $\alpha(6k-1) \sim x_1^{2k+2}x_2x_1^2\cdots x_2^2$ and $\alpha(6k-2) \sim x_1^{2k+1}x_2x_1^2\cdots x_2^2$ are related by the basic recurrence relation.

For the last formula we use the recurrence relation for the braid $\alpha(6k) \sim x_1^{2k}x_2x_1^2\cdots x_1^2x_2^1$ on the second last position; we get the braids $x_1^{2k}x_2x_1^2x_2^2\cdots x_1^1x_2^1 \sim x_1^{2k+2}x_2x_1^2x_2^2\cdots x_1^2x_2^2 = \alpha(6k-1)$ and $\gamma = x_1^{2k}x_2x_1^2x_2^2\cdots x_1^2x_2^3$. Using the recurrence relation for γ on the last position we obtain $\alpha(6k-3)$ and the braid $\delta = x_1^{2k}x_2x_1^2x_2^2\cdots x_1^2x_2$. A new application of the recurrence for δ at the first position gives the braids $\alpha(6k-5)$ and $\alpha(6k-6)$. Therefore, we obtain

$$V_\Delta(6k) = (s^3-s)V_\Delta(6k-1) + s^4V_\Delta(\gamma)$$

$$\begin{aligned}&= (s^3 - s)V_\Delta(6k-1) + s^4(s^3-s)V_\Delta(6k-3) + s^8 V_\Delta(\delta)\\ &= (s^3-s)\big[V_\Delta(6k-1) + s^4 V_\Delta(6k-3) + s^8 V_\Delta(6k-5)\big]\\ &\quad + s^{12}V_\Delta(6k-6).\end{aligned}$$ □

Lemma 4.7. *The first Jones polynomials are given by:*

$$\begin{aligned}V_\Delta(0) &= s^2 + 2 + s^{-2}\\ V_\Delta(1) &= -s - s^{-1}\\ V_\Delta(2) &= 1\\ V_\Delta(3) &= -s^5 - s\\ V_\Delta(4) &= -s^8 + s^6 + s^2\\ V_\Delta(5) &= -s^{11} + s^9 - s^7 - s^3.\end{aligned}$$

Proof. First three are obvious; next use Markov moves:

$$V_\Delta(3) = V_2(2), V_\Delta(4) = V_2(3), V_\Delta(5) = V_2(4).$$ □

Remark 4.8. The recurrence relations of Proposition 4.6 give the following recurrence matrix:

$$\begin{pmatrix} V_\Delta(6k) \\ V_\Delta(6k+1) \\ \vdots \\ V_\Delta(6k+5)\end{pmatrix} = A(k)\begin{pmatrix} V_\Delta(6k-6) \\ V_\Delta(6k-5) \\ \vdots \\ V_\Delta(6k-1)\end{pmatrix},$$

where the Jordan normal form of $A(k)$ has a nice structure:

$$A(k) \sim \begin{pmatrix} J_3 & 0 & 0 & 0 \\ 0 & s^6 & 0 & 0 \\ 0 & 0 & s^{12} & 0 \\ 0 & 0 & 0 & s^{18}\end{pmatrix}, \qquad J_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0\end{pmatrix}.$$

This fact is reflected in the following very simple general formulae of the Jones polynomials $V_\Delta(n)$.

Proposition 4.9. *For any $k \geq 0$ the Jones polynomials of $\alpha(n) = x_1x_2x_1\cdots$ (n-times) are given by*:

$$\begin{aligned}V_\Delta(6k) &= 2s^{12k} + s^{6k+2} + s^{6k-2},\\ V_\Delta(6k+1) &= s^{12k+3} - s^{12k+1} - s^{6k+3} - s^{6k-1},\\ V_\Delta(6k+2) &= -s^{12k+4} + s^{6k+4} + s^{6k},\\ V_\Delta(6k+3) &= -s^{6k+5} - s^{6k+1},\\ V_\Delta(6k+4) &= -s^{12k+8} + s^{6k+6} + s^{6k+2},\\ V_\Delta(6k+5) &= -s^{12k+11} + s^{12k+9} - s^{6k+7} - s^{6k+3}.\end{aligned}$$

Proof. The proof is by induction: the case $k = 0$ is covered by Lemma 4.7 and the induction step can be checked with Proposition 4.6. For instance,

$$\begin{aligned} V(6k+6) &= (s^3 - s)\Big[(-s^{12k+11} + s^{12k+9} - s^{6k+7} - s^{6k+3}) \\ &\quad + s^4(-s^{6k+5} - s^{6k+1}) + s^8(s^{12k+3} - s^{12k+1} - s^{6k+3} - s^{6k-1})\Big] \\ &\quad + s^{12}(2s^{12k} + s^{6k+2} + s^{6k-2}) \\ &= (s^3 - s)(-s^{6k+11} - s^{6k+9} - 2s^{6k+7} - s^{6k+5} - s^{6k+3}) \\ &\quad + 2s^{12k+12} + s^{6k+14} + s^{6k+10} \\ &= 2s^{12(k+1)} + s^{6(k+1)+2} + s^{6(k+1)-2} \end{aligned}$$

and

$$\begin{aligned} V(6k+11) &= (s^3 - s)(-s^{12k+20} + s^{6k+12} + s^{6k+8}) \\ &\quad + s^4(-s^{6k+11} - s^{6k+7}) \\ &= -s^{12(k+1)+11} + s^{12(k+1)+9} - s^{6(k+1)+7} - s^{6(k+1)+3}. \end{aligned}$$

□

Now we analyze the degrees of the Jones polynomials of general positive 3-braids $\beta^{A_*} = x_1^{a_1} x_2^{a_2} \cdots x_2^{a_{2L}}$, where $A_* = (a_1, \ldots, a_{2L})$, $a_i \geq 0$; we use the short notations $A = \sum_{i=1}^{2L} a_i$, the total degree of β^{A_*}, and $Z = \operatorname{card}\{i | 1 \leq i \leq 2L, a_i = 0\}$. As an example, $J_*^{1,0} = (1, 0, \ldots, 1, 0)$ and $J_*^{0,1} = (0, 1, \ldots, 0, 1)$ have the same total sum and number of zeros $J^{1,0} = J^{1,0} = L = Z$. For a positive sequence A_* and $J_* \in \{0,1\}^{2L}$, the J_*-block is the product of two Laurent polynomials in s, $P_{J_*}^{A_*} = P_{j_1}^{[a_1]} P_{j_2}^{[a_2]} \cdots P_{j_{2L}}^{[a_{2L}]}$ and $V_{J_*} = V_3(\beta^{J_*})$.

Lemma 4.10. *If all a_i are positive, the inequality holds*:

$$\deg P_{J_*}^{A_*} \leq 3A - J,$$

with equality if and only if $j_i = 0$ implies $a_i \geq 2$.

Proof. By definition, $\deg P_0^{[1]} \leq 0$, $\deg P_0^{[a]} = 3a$ if $a \geq 2$, and $\deg P_1^{[a]} = 3a - 1$ if $a \geq 1$. □

We start the proof of Theorem 1.6 with a more general result:

Theorem 4.11. *If the positive 3-braid $\beta^{A_*} = x_1^{a_1} x_2^{a_2} \cdots x_1^{a_{2L-1}} x_2^{a_{2L}}$ has degree A and the number of zero exponents is Z, then the inequality holds*:

$$\deg V_3(\beta^{A_*}) \leq 3A - 2L + 2Z.$$

Lemma 4.12. *The inequality in Theorem 4.11 for $L-1$ factors and the inequality in Theorem 1.6 for L factors imply the inequality in Theorem 4.11 for L factors.*

Proof. If the braid $\beta^{A_*} = x_1^{a_1}x_2^{a_2}\cdots x_2^{a_{2L}}$ has all exponents ≥ 1, then Theorem 1.6 (L) gives the result. If at least one exponent is zero, then $\beta^{A_*} = \beta^{A'_*}$, where A'_* is a sequence of length $2(L-1)$ obtained from A_* by deletion of a zero exponent and concatenation of its neighbors: as an example, if $A_* = (2,1,0,3,4,0)$, then A'_* can be $(2,4,4,0)$ or $(6,1,0,3)$ (any choice gives the braid $x_1^6x_2^4$). The new ingredients of A'_* are $A' = A$, $L' = L-1$, and $Z' = Z-1$ or $Z-2$ (if we delete one zero and concatenate two others), therefore

$$\deg V_3(\beta^{A_*}) = \deg V_3(\beta^{A'_*}) \leq 3A' - 2L' + 2Z' \leq 3A - 2L + 2Z. \qquad \square$$

Proof of Theorem 1.6. a) We start induction on $L \geq 1$. If $L = 1$, $V_3(x_1^{a_1}x_2^{a_2}) = V_2(x_1^{a_1})V_2(x_1^{a_2})$, and we have, up to a symmetry, the next cases:

(a_1, a_2)	$\deg V_3(x_1^{a_1}x_2^{a_2})$	$3A - 2L + 2Z$
$(0,0)$	2	2
$(1,0)$	1	3
$(1,1)$	≤ 0	4
$(\geq 2, 0)$	$3a_1$	$3a_1$
$(\geq 2, 1)$	$3a_1 - 1$	$3a_1 + 1$
$(\geq 2, \geq 2)$	$3a_1 + 3a_2 - 2$	$3a_1 + 3a_2 - 2$

Suppose that $L \geq 2$. According to Lemma 4.12 it is enough to show that any J_*-block of the expansion formula for $V_3(\beta^{A_*})$ $(a_i \geq 1)$

$$(s^2+1)^{2L}V_3(\beta^{A_*}) = \sum_{J_* \in \{0,1\}^{2L}} P_{J_*}^{A_*} V_{J_*}$$

has deg $\leq 3A + 2L$; because there is no factor $P_j^{[0]}$, we have $\deg P_{J_*}^{A_*} \leq 3A - J$, and it is enough to show that $\deg V_{J_*} \leq J + 2L$. We begin with terms having a small contribution:

Case 1: $J \leq L-1$. In this case the $0,1$ sequence J_* contains at least two zeros which are neighbors (j_{2L} and j_1 are neighbors), we delete both and obtain a new sequence J'_* of length $2(L-1)$ with $J' = J$ and number of zeros $Z' = 2(L-1) - J$. Theorem 4.11 $(L-1)$ gives $\deg V_{J_*} \leq 3J' - 2L' + 2Z' = J + 2L - 2$.

Case 2: $J = L$ but J_* is different from $J_*^{1,0}$ and $J_*^{0,1}$. This is similar to case 1 because one can find two zero neighbors.

Now we are looking for the main contributors:

Case 3: $J_*^{1,0}$ and $J_*^{0,1}$. Their Jones polynomials coincide with $V_3(x_1^L)$ of degree $3L = J + 2L$.

Case 4: $L+1 \leq J \leq 2L-1$. Consider the new sequence J'_* obtained after deletion of all zeros and concatenation of (nonzero) neighbors, if necessary. For this new sequence we have $J' = J$, $Z' = 0$ and deletion of i consecutive zeros reduces the length of J_* by $i^+ = 2\left(i - \lfloor \frac{i}{2} \rfloor\right) = i$ (for i even) and $i+1$ (for i odd). Denote by Z_i the number of sequences of i consecutive zeros; for example, if $J_* = (1,0,1,0,0,1,1,1,1,0)$, then $J'_* = (3,1,1,1)$ and $Z_1 = 2$, $Z_2 = 1$, $Z_{\geq 3} = 0$.

Therefore the total number of zeros in J_* is $Z = \sum_{i\geq 1} iZ_i = 2L - J$ and the length of J'_* is $2L' = 2L - \sum_{i\geq 1} i^+ Z_i$, so we can evaluate the degrees:

$$\begin{aligned}\deg V_{J_*} &= \deg V_{J'_*} \leq 3J' - 2L' \\ &= 3J - 2L + \sum_{i\geq 1} i^+ Z_i \\ &= 3J - 2L + 2\sum_{i\geq 1} iZ_i - 2\sum_{i\geq 2}\left\lfloor \frac{i}{2}\right\rfloor Z_i \\ &\leq 3J - 2L + 2(2L - J) \\ &= J + 2L.\end{aligned}$$

Case 5: $J_* = (1, 1, \dots, 1)$. This is the example studied in Proposition 4.9, with $2L$ factors, and we found the general formula

$$\deg V_\Delta(2L) = 4L = J + 2L. \qquad \square$$

Proof of Theorem 1.6. b) The case $L = 1$ is a consequence of Proposition 4.1 and Corollary 4.3.

The next cases, $L = 2, 3, 4$, are given by the following three tables, in which, for a given $J_* = (j_1, j_2 \dots, j_{2L})$, we use the notation:

$\delta = card\{1 \leq i \leq 2L \mid j_i = 0\}$; if all the exponents satisfy $a_i \geq 2$, this δ equals the difference $\deg(P^{A_*}_{J_*}) - \deg(P^{A_*}_{111\dots 1})$,

w is a word conjugate to $x_1^{j_1} x_2^{j_2} \cdots x_1^{j_{2L-1}} x_2^{j_{2L}}$ suitable for computations,

N is the number of braids $x_1^{h_1} x_2^{h_2} \cdots x_1^{h_{2L-1}} x_2^{h_{2L}}$ ($h_i \in \{0, 1\}$) in the same conjugacy class with $x_1^{j_1} x_2^{j_2} \cdots x_1^{j_{2L-1}} x_2^{j_{2L}}$,

T is the leading term of $V_3(x_1^{j_1} x_2^{j_2} \cdots x_1^{j_{2L-1}} x_2^{j_{2L}})$, and $\deg = \delta + \deg(T)$ (the top degrees are in bold characters).

The "simple" conjugate w is obtained using braid relations and conjugations.

To compute the value of N, for fixed L and δ, take $\binom{2L}{\delta}$ choices of sequences J_* containing 1, $(2L - \delta)$ times, and 0, δ times, and find the conjugacy classes of $x_1^{j_1} x_2^{j_2} \cdots x_1^{j_{2L-1}} x_2^{j_{2L}}$. The leading term T is obtained in most cases from Proposition 4.1 and Corollary 4.3. For $w = x_1^2 x_2 x_1^2 x_2 = \Delta_3^2$ one can use Proposition 4.9, and for the last two cases in Table 4 one can use Proposition 4.5, Table 1. The last column is easy to find: $\deg = \delta + \deg(T)$.

For instance, let us make the computations for $L = 4, \delta = 2$. There are $\binom{8}{2}$ sequences with two zeros: eight of them have adjacent zeros (in 01111110 the zeros are adjacent) and they are conjugate with x_*^J, $J_* = (11111100)$. The corresponding w is

$$x_1 x_2 x_1 x_2 x_1 x_2 x_1^0 x_2^0 = x_1^2 x_2 x_1^2 x_2 (= \Delta_3^2).$$

TABLE 2. $L = 2$, $V_{J_*} = V(x_1^{j_1} x_2^{j_2} x_1^{j_3} x_2^{j_4})$

δ	$J_* = (j_1, j_2, j_3, j_4)$	w	N	T	deg
4	0000	1	1	s^2	6
3	1000	x_1	4	$-s$	4
2	1100	x_1x_2	4	1	2
2	1010	x_1^2	2	s^6	**8**
1	1110	$x_1^2x_2$	4	$-s^5$	6
0	1111	$x_1^3x_2$	1	$-s^8$	**8**

TABLE 3. $L = 3$, $V_{J_*} = V(x_1^{j_1} x_2^{j_2} x_1^{j_3} x_2^{j_4} x_1^{j_5} x_2^{j_6})$

δ	$J_* = (j_1, j_2, j_3, j_4, j_5, j_6)$	w	N	T	deg
6	000000	1	1	s^2	8
5	100000	x_1	6	$-s$	6
4	110000	x_1x_2	9	1	4
4	101000	x_1^2	6	s^6	10
3	111000	$x_1^2x_2$	18	$-s^5$	8
3	101010	x_1^3	2	s^9	**12**
2	111100	$x_1^3x_2$	12	$-s^8$	10
2	110110	$x_1^2x_2^2$	3	s^{10}	**12**
1	111110	$x_1^4x_2$	6	$-s^{11}$	**12**
0	111111	$x_1^2x_2x_1^2x_2$	1	$2s^{12}$	**12**

There are eight sequences in which the distance between zeros is one, all of them are conjugate to

$$x_1x_2x_1x_2x_1x_2^0x_1x_2^0 = x_1^2x_2x_1^3 \sim x_1^5x_2,$$

and eight sequences in which the distance between the zeros is two, all conjugate to

$$x_1x_2x_1x_2x_1^0x_2x_1x_2^0 = x_1^3x_2x_1^2 \sim x_1^5x_2,$$

and the last four sequences in which the zeros are at distance three:

$$x_1x_2x_1x_2^0x_1x_2x_1x_2^0 = x_1x_2x_1x_1x_2x_1 = \Delta_3^2 = x_1^2x_2x_1^2x_2.$$

This combinatorics gives multiplicity 12 for (11111100) and multiplicity 16 for (11110110).

From these tables, one can compute the degree and the leading term of the block $P_{J_*}^{A_*}V_{J_*}$ using the formula

$$\delta + \deg(T) + \sum_{i=1}^{2L}(3a_i - 1) = \deg + 3A - 2L.$$

TABLE 4. $L = 4$, $V_{J_*} = V(x_1^{j_1} x_2^{j_2} x_1^{j_3} x_2^{j_4} x_1^{j_5} x_2^{j_6} x_1^{j_7} x_2^{j_8})$

δ	$J_* = (j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8)$	w	N	T	deg
8	00000000	1	1	s^2	10
7	10000000	x_1	8	$-s$	8
6	11000000	x_1x_2	16	1	6
6	10100000	x_1^2	12	s^6	12
5	11100000	$x_1^2x_2$	48	$-s^5$	10
5	10101000	x_1^3	8	s^9	14
4	11110000	$x_1^3x_2$	55	$-s^8$	12
4	11011000	$x_1^2x_2^2$	13	s^{10}	14
4	10101010	x_1^4	2	s^{12}	**16**
3	11111000	$x_1^4x_2$	48	$-s^{11}$	14
3	10101101	$x_1^3x_2^2$	8	s^{13}	**16**
2	11111100	$x_1^2x_2x_1^2x_2$	12	$2s^{12}$	14
2	11110110	$x_1^5x_2$	16	$-s^{14}$	**16**
1	11111110	$x_1^3x_2x_1^2x_2$	8	s^{15}	**16**
0	11111111	$x_1^3x_2x_1^3x_2$	1	$-s^{16}$	**16**

Finally, using the top degrees (in bold) with the corresponding coefficient of T and multiplicity N and subtracting the degree of the denominator $(s^2+1)^{2L}$, we get the result.

For instance, using Table 4, the leading coefficient of the Jones polynomial $V_3(x_1^{a_1}x_2^{a_2}x_1^{a_3}x_2^{a_4}x_1^{a_5}x_2^{a_6}x_1^{a_7}x_2^{a_8}), a_i \geq 2$, is

$$2 + 8 - 16 + 8 - 1 = 1$$

and its degree is

$$16 + 3A - 2 \cdot 4 - 2 \cdot 8 = 3A - 8 = 3A - 2L. \qquad \square$$

As we can see from the tables, there are braids β^{A_*} with positive exponents having maximal degree $\deg V_3(\beta^{A_*}) = 3A - 2L$, but not all their exponents are ≥ 2; other examples with maximal degree are given by the families $V_3(a_1, 1, \ldots, 1)$ and $V_3(a_1, a_2, 1, \ldots, 1)$, $a_1, a_2 \geq 2$, $2L$ indices, with leading terms s^{3A-2L} if $L \equiv 0 \pmod 3$ and $-s^{3A-2L}$ if $L \equiv 1 \pmod 3$. These examples show that combinatorics of J_*-blocks of maximal degree in the expansion formula is not so simple.

Conjecture 4.13. If $a_i \geq 2$, the leading term of $V_3(x_1^{a_1}x_2^{a_2}\cdots x_1^{a_{2L-1}}x_2^{a_{2L}})$ is s^{3A-2L}.

5. There are few unit polynomials in a row

Now we apply the recurrence relation to evaluate the number of solutions of the equation $V(e) = 1$, where $V(e) = V_n(x_{i_1}^{a_1} \cdots x_{i_j}^{e} \cdots x_{i_k}^{a_k})$, with the same conventions: the sequence $x_{i_1}, \dots, x_{i_k}$ of generators of $\mathcal{B}_n$ is fixed, the exponents $a_1, \dots, \widehat{a_j}, \dots a_k$ are fixed, and e is an arbitrary integer.

Proposition 5.1. a) *If $V(e)$ and $V(e+1)$ are polynomials in s and $k \geq 2$, then $V(e+k)$ is a polynomial different from* 1.

b) *If $V(e)$ and $V(e-1)$ are polynomials in s^{-1} and $k \geq 2$, then $V(e-k)$ is a polynomial in s^{-1} different from* 1.

Proof. a) From the recurrence relation, $V(e+k)$ is a polynomial in s for $k \geq 2$. If $V(e+k) = 1$ for some $k \geq 2$, then $1 = (s^3 - s)V(e+k-1) + s^4 V(e+k-2)$, and this is impossible with polynomials. Similarly for part b). □

An obvious consequence of Propositions 3.5 and 3.6 is the fact that in any sequence $\big(V(e)\big)_{e\in\mathbb{Z}}$ there are only finitely many terms equal to 1. Now we prove a stronger statement: in such a sequence there are at most two polynomials equal to 1, and in this case they are very close:

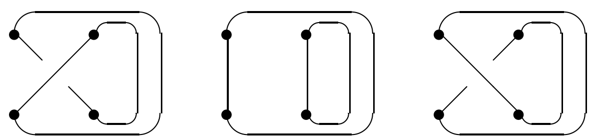
Lemma 5.2. *The sequence $(V(e))_{e\in\mathbb{Z}}$ could contain at most two terms equal to* 1. *If $V(a) = V(b) = 1$, $a \neq b$, then $|a-b| = 1$ or* 2.

Proof. Suppose $V(a) = 1$ and $V(b) = 1$ $(a < b)$. We can take $Q_n = V(n+a)$, where $n \in \mathbb{Z}$. Hence $Q_0 = 1$. From the recurrence relation we have

$$Q_n = \frac{1}{s^3+s}[(Q_1+s)s^{3n} + (s^3 - Q_1)(-s)^n].$$

Let us suppose that $Q_n = 1$ for a positive n; in our case $b-a$ is such an example. If n is even, we have $s^3 + s = (s^{3n} - s^n)Q_1 + s^{3n+1} + s^{n+3}$. This implies $Q_1 = \frac{1}{s^n(s^{2n}-1)}[s^3 + s - s^{3n+1} - s^{n+3}] = -s - \frac{1}{s^{n-1}(s^n+1)}(s^2+1)$. This gives Laurent polynomials only for $n = 0$ and $n = 2$. In the last case, $Q_1 = -s - \frac{1}{s}$, hence only Q_0 and Q_2 are equal to 1. If n is odd, we have $s^3 + s = (s^{3n} + s^n)Q_1 + s^{3n+1} - s^{n+3}$. This implies $Q_1 = \frac{1}{s^n(s^{2n}+1)}[s^3 + s - s^{3n+1} + s^{n+3}]$ $= -s + \frac{1}{s^n(s^{2n}+1)} s(s^2+1)(s^n+1)$. Laurent polynomials are obtained only for $n = 0$ and $n = 1$. In this last case, $Q_0 = Q_1 = 1$ and $Q_2 \neq 1$. □

Example 5.3. The closures of the braids x_1^{-1}, x_1^0, and x_1^1 in $\mathcal{B}_2$ give the links in the following figure:



$\widehat{x_1^{-1}}$ $\widehat{1}$ $\widehat{x_1}$

and the corresponding Jones polynomials are: $V_2(-1) = 1$, $V_2(0) = -s - s^{-1}$, and $V_2(1) = 1$. This shows that the case $V(a) = V(a+2) = 1$ is possible.

Lemma 5.4. *Let $\beta(e) = \alpha x_i^e \gamma$ and $\beta(e+1) = \alpha x_i^{e+1}\gamma$ be two braids in $\mathcal{B}_n$. Then $\widehat{\beta(e)}$ and $\widehat{\beta(e+1)}$ cannot be knots simultaneously. In particular, $V(e)$ and $V(e+1)$ evaluated at 1 cannot be 1 at the same time.*

Proof. If $\widehat{\beta(e)}$ is a knot, then the associated permutation $\pi(\beta(e))$ is an n-cycle. The permutation associated with $\beta(e+1)$ is $\pi(\alpha x_i^{e+1}\gamma) = \pi(\alpha x_i^e \gamma)\pi(\gamma^{-1}x_i\gamma)$, and this cannot be an n-cycle because the signature of the last factor is -1. □

Proof of Theorem 1.7. This is a consequence of Lemmas 5.2 and 5.4. □

6. Appendix

In [17] multiple Fibonacci sequences (and multiple Fibonacci modules) are introduced. A multiple sequence $(x_{n_1,\dots,n_p})_{n_1,\dots,n_p\in\mathbb{Z}}$ of elements in a ring $\mathcal{R}$ is called a *multiple Fibonacci sequence* of type $(\beta,\gamma)\in\mathcal{R}^2$ if for any $i\in\{1,\dots,p\}$ and any $k\in\mathbb{Z}$ we have

$$a_{n_1,\dots,n_{i-1},k+2,n_{i+1},\dots,n_p} = \beta a_{n_1,\dots,n_{i-1},k+1,n_{i+1},\dots,n_p} + \gamma a_{n_1,\dots,n_{i-1},k,n_{i+1},\dots,n_p}.$$

The $\mathcal{R}$-module of such multiple sequences is denoted by $\mathcal{F}^{[p]}(\beta,\gamma)$ and it is isomorphic with the tensor product $\mathcal{F}^{[1]}(\beta,\gamma)^{\otimes p}$.

Theorem 6.1. *Let $(x_{n_1,\dots,n_p})_{\geq 0}$ be an element in $\mathcal{F}^{[p]}(r_1+r_2,-r_1r_2)$.*

a) *The general term is given by*

$$x_{n_1,\dots,n_p} = D^{-p}\sum_{0\leq j_1,\dots,j_p\leq 1} S_{j_1}^{[n_1]}(r_1,r_2)\cdots S_{j_p}^{[n_p]}(r_1,r_2)x_{j_1,\dots,j_p}\,,$$

where $D = r_2 - r_1$, $S_0^{[n]}(r_1,r_2) = r_1^n r_2 - r_1 r_2^n$, $S_1^{[n]}(r_1,r_2) = r_2^n - r_1^n$.

b) *The generating function of $(x_{n_1,\dots,n_p})$ is given by*

$$G(t_1,\dots,t_p) = q(t_1)^{-1}\cdots q(t_p)^{-1}\sum_{0\leq j_1,\dots,j_p\leq 1} Q_{j_1}(t_1)\cdots Q_{j_p}(t_p)x_{j_1,\dots,j_p}\,,$$

where $q(t) = (1-r_1t)(1-r_2t)$, $Q_0(t) = 1-(r_1+r_2)t$ and $Q_1(t) = t$.

Proof. These require only elementary computations; one can find all the details in [17]. □

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