

## SAMPLING EXPANSION OF BANDLIMITED FUNCTIONS OF POLYNOMIAL GROWTH ON THE REAL LINE

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ABSTRACT. For a bandlimited function with polynomial growth on the real line, we derive a nonuniform sampling expansion using a special bandlimited function which has polynomial decay on the real line. The series converges uniformly on any compact subsets of the real line.

### 1. Introduction

The Shannon sampling theorem shows that if  $f$  is a function bandlimited to  $[-\pi, \pi]$  and square integrable, that is, the Fourier transform  $F \in L^2[-\pi, \pi]$  of  $f$  has a compact support in  $[-\pi, \pi]$  so that  $f$  can be written as

$$(1.1) \quad f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(w) e^{i w z} dw,$$

then  $f$  can be reconstructed with the sampling points  $t_n = n, n \in \mathbb{Z}$ , that is,

$$(1.2) \quad f(z) = \sum_{n \in \mathbb{Z}} f(t_n) \frac{\sin \pi(z - t_n)}{\pi(z - t_n)},$$

where  $\mathbb{Z}$  is the set of all integers. The right-side of (1.1) is called the inverse Fourier transform of  $F$  and denoted by  $\mathcal{F}^{-1}(F)$ . The Fourier transform  $F$  of a function  $f$  is denoted by  $\mathcal{F}(f)$  and defined by

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i w x} dx.$$

A function is called a bandlimited function with bandwidth  $\sigma > 0$  if the support of its Fourier transform on the real line is confined to  $[-\sigma, \sigma]$ .

For  $1 \leq p \leq \infty$  we denote by  $B_{\sigma}^p$  the class of all bandlimited functions  $f$  with bandwidth  $\sigma$  such that the restriction of  $f$  to the real line belongs to  $L^p(\mathbb{R}) := \{f : \|f\|_p = (\int_{\mathbb{R}} |f|^p)^{1/p} < \infty\}$ . The above sampling expansion (1.2) also holds for any function in  $B_{\pi}^p$ ,  $1 \leq p < \infty$  ([4], [16]).

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When sampling points are not equidistant, the sampling expansion is also possible from the theory of Riesz basis (an exact frame) ([3, 5, 11]). If a sequence  $\{t_n\}_{n \in \mathbb{Z}}$  of sampling points satisfies that

$$(1.3) \quad |t_n - n| \leq d < \frac{1}{4} \quad \text{for any } n \in \mathbb{Z},$$

then the set  $\{e^{it_n x}\}_{n \in \mathbb{Z}}$  forms a Riesz basis in  $L^2[-\pi, \pi]$  ([6], [8], [9]) and any function  $f$  in  $B_\pi^2$  can be written as

$$(1.4) \quad f(z) = \sum_{n \in \mathbb{Z}} f(t_n) \frac{G(z)}{G'(t_n)(z - t_n)},$$

where the series converges uniformly on each bounded horizontal strip of the complex plane and

$$(1.5) \quad G(z) = (z - t_0) \prod_{n=0}^{\infty} \left(1 - \frac{z}{t_n}\right) \left(1 - \frac{z}{t_{-n}}\right).$$

But sampling expansions (1.2) and (1.4) do not hold for functions in  $B_\pi^\infty$  ([16]).

Sampling theorems for bandlimited functions have been studied by many people. Campbell in [2] first gave the sampling expansions of bandlimited functions. Zakai in [15] introduced a class of bandlimited functions  $f$  satisfying

$$(1.6) \quad \int_{-\infty}^{\infty} \frac{|f(x)|^2}{1 + x^2} dx < \infty.$$

The set of all bandlimited functions satisfying (1.6) is called the Zakai class, and if a function  $f$  in the Zakai class has a bandwidth  $\sigma < \pi$ , then he showed that the sampling expansion (1.2) holds and the series does not converge absolutely. Any function in  $B_\pi^\infty$  lies in the Zakai class.

We denote by  $H_\sigma^k$  the class of all bandlimited functions with the bandwidth  $\sigma$  satisfying

$$(1.7) \quad \int_{-\infty}^{\infty} \frac{|f(x)|^2}{(1 + x^2)^k} dx < \infty.$$

The Fourier transform of a function in  $H_\sigma^k$  should be interpreted in the sense of distribution. Many literatures provide sampling expansions for functions in  $H_\sigma^k$  using the theory of distribution ([7, 10, 13, 14, 17]).

In this paper we provide a simpler way to obtain the nonuniform sampling expansion for a function in  $H_\sigma^k$  which converges uniformly on any compact subsets of the real line without using the theory of distribution. In Section 2, using an auxiliary bandlimited function having polynomial decay on the real line, we derive a sampling expansion of a function in  $H_\sigma^k$ . In Section 3, we provide an example of an auxiliary bandlimited function having polynomial decay on the real line. This auxiliary bandlimited function plays an important role in the scheme of reducing PAPR in OFDM signals ([12]).

## 2. Nonuniform sampling theorem on $H_\sigma^k$

In this section we consider a function in  $H_\sigma^k$  and derive its nonuniform sampling expansion.

We write by  $\mathcal{E}$  the space consisting of all functions which are infinitely many times differentiable in  $\mathbb{R}$  and denote by  $\mathcal{E}'$  the dual space of  $\mathcal{E}$ . To discuss the sampling expansion of a function in  $H_\sigma^k$ , we need the following Paley-Wiener-Schwarz Theorem ([1]).

**Theorem 2.1.** *If  $f$  is a bandlimited function with bandwidth  $\sigma > 0$  and the Fourier transform of  $f$  belongs to  $\mathcal{E}'$ , then there exist a positive constant  $M$  and a nonnegative integer  $N$  such that*

$$(2.8) \quad |f(z)| \leq M(1 + |z|)^N e^{\sigma|y|} \quad \text{for any } z = x + iy \in \mathbb{C}.$$

*Conversely, every entire function on  $\mathbb{C}$  satisfying (2.8) is a bandlimited function with bandwidth  $\sigma$ .*

If  $f$  is a bandlimited function with bandwidth  $\sigma$  and satisfies the growth condition (2.8), then the Fourier transform  $F \in \mathcal{E}'$  of  $f$  has the order  $N$  and  $f$  belongs to  $H_\sigma^{N+1}$ .

We note that for a function  $f \in B_\sigma^p, 1 \leq p \leq \infty$ , there exists a positive constant  $C$  such that

$$(2.9) \quad |f(z)| \leq C e^{\sigma|y|} \quad \text{for any } z = x + iy \in \mathbb{C}.$$

In (2.9), we may replace  $C$  by  $\sup_{x \in \mathbb{R}} |f(x)|$ .

By a simple transformation, we may assume that the bandwidth  $\sigma$  is less than  $\pi$ . To obtain the sampling expansion for a function in  $H_\sigma^k$  we first consider a function in  $B_\pi^2$  and its nonuniform sampling expansion.

The sampling expansions of functions in  $B_\pi^2$  with sampling points  $\{t_n\}$  satisfying (1.3) are well-known in [3], [4], [5] and [16]. In the following theorem, we can obtain the truncation error from Lemma 16.1 in [8].

**Theorem 2.2.** *Let  $\{t_n\}$  be a sequence satisfying (1.3) and  $G$  be the infinite product in (1.5). For any  $f \in B_\pi^2$ ,  $f$  can be expanded as a series*

$$f(z) = \sum_{n \in \mathbb{Z}} f(t_n) \frac{G(z)}{G'(t_n)(z - t_n)},$$

where the series converges uniformly on each bounded horizontal strip  $\{z \in \mathbb{C} : |\operatorname{Im} z| < a\}$ ,  $a > 0$ , of the complex plane and also converges in  $\|\cdot\|_2$ -norm. Moreover, there exists a constant  $C_f$  depending on  $\|f\|_2$  such that for any compact subset  $K$  of  $\mathbb{C}$  and a large integer  $N$ ,

$$(2.10) \quad \begin{aligned} & \sup_{z \in K} \left| f(z) - \sum_{|n| \leq N} f(t_n) \frac{G(z)}{G'(t_n)(z - t_n)} \right| \\ & \leq C_f N^{4d-1} \sup_{z \in K} (|z| + 1)^{4d} e^{\pi |\operatorname{Im} z|}. \end{aligned}$$

In order to consider a sampling expansion for a function in  $H_\sigma^k$ , we introduce an auxiliary bandlimited function  $\varphi_\alpha$  for a nonnegative integer  $\alpha$  satisfying the following properties:

- P1)  $\varphi_\alpha$  is a bandlimited function with bandwidth 1.
- P2)  $\varphi_\alpha(x) \neq 0$  for any  $x \in \mathbb{R}$ .
- P3) There exists  $C > 0$  such that  $|\varphi_\alpha(x)| \leq \frac{C}{(1+|x|)^\alpha}$  for any  $x \in \mathbb{R}$ .

Using the above function  $\varphi_\alpha$  satisfying the properties P1)-P3), we obtain the following sampling expansion for a function in  $H_\sigma^k$ . We denote by  $\lceil a \rceil$  the smallest integer not less than  $a$ .

**Theorem 2.3.** *Let  $\varphi_\alpha$  be a function satisfying P1)-P3), the sampling points  $\{t_n\}$  satisfy (1.3) and  $G$  be the infinite product in (1.5). For a nonnegative integer  $k$ , let  $\ell = \lceil k/\alpha \rceil$ ,  $0 < \sigma < \pi$ ,  $\delta = \pi - \sigma$ , and  $\psi(z) = [\varphi_\alpha(\delta z/\ell)]^\ell$ . Then any function  $f$  in  $H_\sigma^k$  can be written as*

$$(2.11) \quad f(x) = \sum_{n \in \mathbb{Z}} f(t_n) \frac{\psi(t_n)}{\psi(x)} \frac{G(x)}{G'(t_n)(x - t_n)} \quad \text{for any } x \in \mathbb{R},$$

where the series converges uniformly on each compact subset of  $\mathbb{R}$ . Moreover, there exists a positive constant  $C$  depending on  $\|f \cdot \psi\|_2$  such that for any compact subset  $K$  of  $\mathbb{R}$  and a large integer  $N$ ,

$$(2.12) \quad \sup_{x \in K} \left| f(x) - \sum_{|n| \leq N} f(t_n) \frac{\psi(t_n)}{\psi(x)} \frac{G(x)}{G'(t_n)(x - t_n)} \right| \leq CN^{4d-1} \sup_{x \in K} (|x| + 1)^{4d} / \inf_{x \in K} \psi(x).$$

*Proof.* Since  $\varphi_\alpha \in B_1^\infty$ , there exists a constant  $C$  such that

$$(2.13) \quad |\varphi_\alpha(z)| \leq Ce^{|y|} \quad \text{for any } z = x + iy \in \mathbb{C}$$

and

$$(2.14) \quad \left| \left( \varphi_\alpha \left( \frac{\delta z}{\ell} \right) \right)^\ell \right| \leq C^\ell e^{\delta|y|} \quad \text{for any } z = x + iy \in \mathbb{C}.$$

By Theorem 2.1,  $\psi$  is a bandlimited function with bandwidth  $\delta$  and the multiplication  $f \cdot \psi$  is a bandlimited function with bandwidth  $\pi$ . Observing that there exists a constant  $C'$  such that

$$(2.15) \quad \left| \left( \varphi_\alpha \left( \frac{\delta x}{\ell} \right) \right)^\ell \right| \leq \frac{C'}{(1+|x|)^k} \quad \text{for any } x \in \mathbb{R},$$

$f \cdot \psi \in L^2(\mathbb{R})$ . Hence  $f \cdot \psi \in B_\pi^2$ .

By Theorem 2.2, the multiplication function  $f \cdot \psi$  can be written as

$$(2.16) \quad f(x) \cdot \psi(x) = \sum_{n=-\infty}^{\infty} f(t_n) \psi(t_n) \frac{G(x)}{G'(t_n)(x - t_n)},$$

where the series converges uniformly on each bounded horizontal strip of the complex plane. Since  $\psi$  has no zero on the real line, we can divide the both

sides of (2.16) by  $\psi$  to obtain the following sampling expansion for  $f$  on the real line

$$f(x) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{\psi(t_n)}{\psi(x)} \frac{G(x)}{G'(t_n)(x-t_n)} \quad \text{for any } x \in \mathbb{R},$$

where the series converges uniformly on each compact subsets of  $\mathbb{R}$ . By Theorem 2.2, there exists a positive constant  $C$  depending on  $\|f \cdot \psi\|_2$  such that for each compact subset  $K$  of  $\mathbb{R}$  and a large integer  $N$ , the truncation error (2.12) holds.  $\square$

### 3. Example

In the last section, we provided the concrete sampling expansion of a function in  $H_\sigma^k$ ,  $0 < \sigma < \pi$ , in terms of the bandlimited function  $\varphi_\alpha$  satisfying P1)-P3). In this section, we show an example of  $\varphi_\alpha$  and provide the sampling expansion of a function in  $H_\sigma^k$ . We define a function  $\varphi$  by

$$(3.17) \quad \varphi(z) = \begin{cases} \frac{1 - \operatorname{sinc}(z/\pi)}{z^2}, & z \neq 0, \\ \frac{1}{6}, & z = 0. \end{cases}$$

where

$$\operatorname{sinc} z = \begin{cases} \frac{\sin \pi z}{\pi z}, & z \neq 0, \\ 1, & z = 0. \end{cases}$$

The function  $\varphi$  is a bandlimited function with bandwidth 1 and  $\varphi$  has no zero on the real line. The bandlimited function  $\varphi$  also satisfies P3) with  $\alpha = 2$ .

Let  $k$  be a nonnegative integer,  $0 < \sigma < \pi$ ,  $\ell = \lceil k/2 \rceil$  and  $\delta = \pi - \sigma$ . Let  $\{t_n\}_{n \in \mathbb{Z}}$  be a sequence satisfying (1.3) and let  $G$  be the infinite product in (1.5). To reconstruct a function in  $H_\sigma^k$ , we define a function  $\psi$  by

$$(3.18) \quad \psi(z) = [\varphi(\frac{\delta z}{\ell})]^\ell.$$

It follows from Theorem 2.3 that for a function  $f$  in  $H_\sigma^k$  we obtain the sampling expansion of  $f$  on the real line

$$f(x) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{\psi(t_n)}{\psi(x)} \frac{G(x)}{G'(t_n)(x-t_n)} \quad \text{for any } x \in \mathbb{R},$$

where

$$\psi(x) = \begin{cases} \left(\frac{1}{6}\right)^\ell, & x = 0, \\ \left(\frac{\ell^2 \delta x - \ell^3 \sin \frac{\delta x}{\ell}}{(\delta x)^3}\right)^\ell, & x \neq 0, \end{cases}$$

and the series converges uniformly on any compact subsets of the real line.

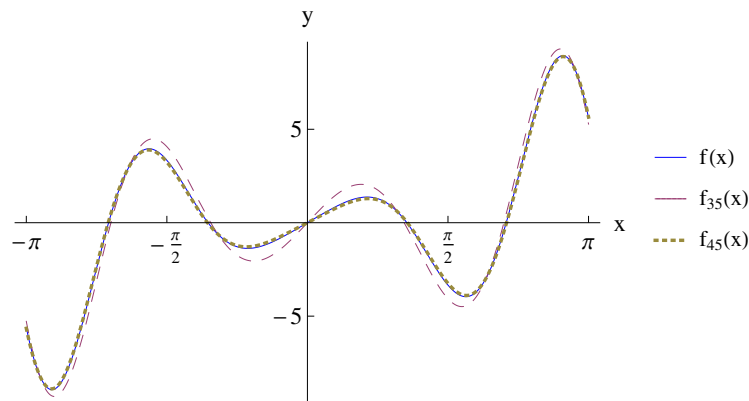


FIGURE 1. The graphs of the original function  $f(x)$  and the approximations  $f_{35}(x)$ ,  $f_{45}(x)$ .

#### 4. Simulation

In this section, we consider the function  $f(x) = (1 + x^2) \sin \frac{9\pi}{10}x$  in  $H_{\frac{9\pi}{10}}^3$  and the approximation function  $f_N(x) = \sum_{n=-N}^N f(t_n) \frac{\psi(t_n)}{\psi(x)} \frac{G(x)}{G'(t_n)(x-t_n)}$  of  $f$  for a positive integer  $N$ . Here  $t_n = n$  for  $n \in \mathbb{Z}$ . In Figure 1, the graphs of  $y = f(x)$ ,  $y = f_{35}(x)$  and  $y = f_{45}(x)$  are depicted, and the values of  $y = f_{45}(x)$  on the interval  $[-\pi, \pi]$  are very close to those of  $y = f(x)$ , so  $f_N(x)$  approximates  $f(x)$  for large  $N$ .

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