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# ON A GENERALIZED BERGE STRONG EQUILIBRIUM

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Abstract. In this paper, we first introduce a generalized concept of Berge strong equilibrium for a generalized game $\mathcal{G} = (X_i; T_i, f_i)_{i \in I}$ of normal form, and using a fixed point theorem for compact acyclic maps in admissible convex sets, we establish the existence theorem of generalized Berge strong equilibrium for the game $\mathcal{G}$ with acyclic values. Also, we have demonstrated by examples that our new approach is useful to produce generalized Berge strong equilibria.

## 1. Introduction

In 1950, Nash [11] established a pioneering equilibrium existence theorem by using the Kakutani fixed point theorem, and next, by applying the Eilenberg-Montgomery fixed point theorem, Debreu [8] established a generalization of Nash's Theorem which assumes the best utility functions and response profile correspondences. Since then, the classical results of Nash and Debreu have served as basic references for the existence of generalized Nash equilibrium for a non-cooperative generalized game $\mathcal{G}$. In all of them, convexity of strategy spaces, continuity and concavity/quasiconcavity of payoff functions and constraint correspondences were assumed. Till now, there have been a number of generalizations, and also many applications of those theorems have been found in several areas, e.g., see [5, 6, 15] and references therein.

On the other hand, the notion of equilibrium for a coalition $R$ with respect to a coalition $S$ was introduced by Berge [6] in 1957, and next, Zhukovskii [16] introduced the Berge equilibrium in the sense of Zhukovskii. These equilibria can be used as an alternative solution when there is no Nash equilibrium, or when there are many. In this equilibrium, each player obtains his maximum payoff if the situation is favorable for him: by obligation or willingness, the other players choose strategies favorable for him. The concepts of Nash and Berge equilibria have been investigated by several authors as in [1-16], and existence theorems as well as relationship between Nash and Berge equilibria were explored in [4]. Indeed, in [1, 2, 3, 4], Abalo and Kostreva gave more

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general definitions of Berge equilibrium, and they provides theorems for the existence of Berge equilibrium which is based on an earlier theorem of Radjef [15]. However, their theorems are not sufficient for the existence of Berge equilibrium. Next, Nessah et al. [13] provide some sufficient conditions that overcome this problem, and give a valid proof for the existence theorem of Berge equilibrium. Also, in a recent paper [7], Daghdak and Florenzano introduce the best reply correspondence $\Gamma_{-i}$ for the complementary coalition $I-\{i\}$, and they prove the existence theorem of a Berge strong equilibrium for the non-cooperative game $\mathcal{G} = (X_i; f_i)_{i\in I}$ of normal form by applying the Kakutani-Fan fixed point theorem.

In this paper, we first introduce a generalized concept of Berge strong equilibrium for a generalized game $\mathcal{G} = (X_i; T_i, f_i)_{i\in I}$ of normal form, and using the fixed point theorem due to Park in [14] for compact acyclic maps in admissible convex sets, we establish the existence theorem of generalized Berge strong equilibrium for the game $\mathcal{G}$ with acyclic values. Our result is closely related to earlier works of Daghdak and Florenzano [7] in several aspects. Also, we have demonstrated by examples that our new approach is useful to produce general Berge strong equilibria. Indeed, two generalized games $\mathcal{G}_1$ and $\mathcal{G}_2$ are given such that $\mathcal{G}_1$ has both Nash equilibrium and generalized Berge strong equilibrium; however, $\mathcal{G}_2$ can not have a generalized Berge strong equilibrium but has a Nash equilibrium; on the other hand, the previous equilibrium existence theorems in [1-16] can not be suitable for these games.

## 2. Preliminaries

We begin with some notations and definitions. If $A$ is a nonempty set, we shall denote by $2^A$ the family of all subsets of $A$. If $A$ is a subset of a vector space, we shall denote by $co\,A$ the convex hull of $A$. Let $E$ be a topological vector space and $A$, $X$ be nonempty subsets of $E$. If $T : A \to 2^E$ and $S : A \to 2^X$ are multimaps (or correspondences), then $co\,T : A \to 2^E$ and $S \cap T : A \to 2^X$ are correspondences defined by $(co\,T)(x) = co\,T(x)$ $(S\cap T)(x) = S(x) \cap T(x)$ for each $x \in A$, respectively.

Let $I = \{1, 2, \ldots, n\}$ be a finite (or possibly countably infinite) set of players. For each $i \in I$, $X_i$ is a non-empty topological space as an action space, and denote $X_{-i} := \Pi_{j\in I-\{i\}} X_j$. For an action profile $x = (x_1, \ldots, x_n) \in X = \Pi_{i\in I} X_i$, we shall write $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in X_{-i}$, and we may simply write $x = (x_{-i}, x_i) \in X_{-i} \times X_i = X$.

Now we recall basic definitions of continuities concerned with multimaps. Let $X, Y$ be nonempty topological spaces and $T : X \to 2^Y$ be a multimap. A multimap $T : X \to 2^Y$ is said to be *lower semicontinuous* if for each $x \in X$ and each open set $V$ in $Y$ with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood $U$ of $x$ in $X$ such that $T(y) \cap V \neq \emptyset$ for each $y \in U$; and a multimap $T : X \to 2^Y$ is said to be *upper semicontinuous* if for each $x \in X$ and each open set $V$ in $Y$ with $T(x) \subset V$, there exists an open neighborhood $U$ of $x$ in $X$

such that $T(y) \subset V$ for each $y \in U$. And $T$ is said to be *continuous* if $T$ is both lower semicontinuous and upper semicontinuous. It is also known that $T : X \to 2^Y$ is lower semicontinuous if and only if for each closed set $V$ in $Y$, the set $\{x \in X \mid T(x) \subset V\}$ is closed in $X$. If a multimap $T : X \to 2^Y$ is upper semicontinuous with closed values, then $T$ has a closed graph. The converse is true whenever $Y$ is compact, e.g., see Aubin [5].

Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. For nonempty subsets in a topological vector space, we know the general implication that

$$\text{convex} \Rightarrow \text{star-shaped} \Rightarrow \text{contractible} \Rightarrow \text{acyclic} \Rightarrow \text{connected},$$

and not conversely. For topological spaces $X$ and $Y$, a multimap $F : X \to 2^Y$ is called an *acyclic map* whenever $F$ is upper semicontinuous with compact acyclic values. A nonempty subset $X$ of a topological vector space $E$ is said to be *admissible* (in the sense of Klee) [14], if for every compact subset $K$ of $X$ and every neighborhood $V$ of the origin $\mathbb{O}$ of $E$, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$, and $h(K)$ is contained in a finite dimensional subspace $L$ of $E$.

Note that every nonempty convex subset of a locally convex topological vector space is admissible. Other examples of admissible topological vector spaces are $l^p$ and $L^p(0,1)$ for $0 < p < 1$, the space $\mathfrak{M}(0,1)$ of equivalence classes of measurable functions on $[0,1]$, and others. Note also that every compact convex locally convex subset of a topological vector space is admissible. For details, see [14] and references therein.

Assuming the acyclicity of best reply correspondence, and the admissible (non-convex) pure strategy spaces, Park [14] proved an existence theorem of Nash equilibrium for a generalized game $\mathcal{G} = (X_i; T_i, f_i)_{i\in I}$ by using his generalization of the Eilenberg-Montgomery fixed point theorem.

The following is a basic tool for proving the existence of Berge strong equilibrium, which is a particular form of Theorem 1 in [14]:

**Lemma 2.1.** *Let $I$ be a finite index set. For each $i \in I$, $X_i$ is a nonempty compact convex subset of a Hausdorff topological vector space $E_i$, and $T_i : X = \Pi_{j\in I} X_j \to 2^{X_i}$ is an acyclic map. If $X$ is an admissible subset of $E = \Pi_{i\in I} E_i$, then there exists a fixed point $\bar{x} \in X$ for the multimap $T = \Pi_{i\in I} T_i$, i.e., $\bar{x}_i \in T_i(\bar{x})$ for each $i \in I$.*

## 3. A new model for a generalized Berge strong equilibrium

First, we recall some notions and terminologies on the generalized Berge strong equilibrium for pure strategic games as in [7]. Let $I = \{1, 2, \dots, n\}$ be a finite (or possibly countably infinite) set of players. A non-cooperative *generalized game* of normal form (also, a *social system* or an *abstract economy*) is an ordered $3n$-triple $\mathcal{G} = (X_i; T_i, f_i)_{i\in I}$ where for each player $i \in I$, $X_i$ is a pure strategy space for the player $i$, and the set $X = \Pi_{i=1}^n X_i$, *joint strategy space*, is the Cartesian product of the individual strategy spaces, and the element of $X_i$

is called a *strategy.* And, $f_i :\to \mathbb{R}$ is a *payoff function* (or *utility function*), and $T_i : X \to 2^{X_{-i}}$ is a *complementary constraint correspondence* for the player $i$.

For each coalition $K \subset I$, we denote by $-K$ the set $\{i \in I \mid i \notin K\}$ of the coalition $I - K$. Recall that in a non-cooperative game of normal form $\mathcal{G} = (X_i; f_i)_{i\in I}$, a strategy profile $\bar{x} \in X$ is a *Nash equilibrium* [11] for $\mathcal{G}$ if for each $i \in I$,

$$f_i(\bar{x}_{-i}, \bar{x}_i) \geq f_i(\bar{x}_{-i}, x_i) \quad \text{for all} \;\; x_i \in X_i;$$

and for a more general equilibrium definition due to Berge, let $R = \{R_i\}_{i\in M}$ be a partition of $I$ and $S = \{S_i\}_{i\in M}$ be a set of subsets of $I$. A feasible strategy $\bar{x} \in X$ is a *Berge equilibrium* [3], for $\mathcal{G}$ if for each $m \in M$ and for any $r_m \in R_m$,

$$f_{r_m}(\bar{x}_{-S_m}, \bar{x}_{S_m}) \;\geq\; f_{r_m}(\bar{x}_{-S_m}, x_{S_m}) \quad \text{for all} \;\; x_{S_m} \in X_{S_m}.$$

Let us assume that $R_i := \{i\}$ for all $i \in I$. Then it is clear that the family $R = \{R_i\}_{i\in I}$ is a partition of the set of players $I$. If we let $S_i := I - \{i\}$ for all $i \in I$, then the Berge equilibrium reduces to the Berge equilibrium in the sense of Zhukovskii [16]. Indeed, note that Nash equilibrium is clearly a special case of Berge equilibrium where $M = I$, $R_i = \{i\}$, and $S_i = I - \{i\}$ for all $i \in M$.

Next, we recall other equilibrium concepts which are related with the Nash equilibrium and the Berge equilibrium. A strategy profile $\bar{x} \in X$ is an *equilibrium for a coalition* $K$ due to Berge [6] if for each $x_K \in X_K$ and $i \in K$,

$$f_i(\bar{x}_{-K}, \bar{x}_K) \geq f_i(\bar{x}_{I-K}, x_K);$$

which means at the profile $\bar{x} \in X$ the coalition $K$ can not increase the payoffs of any of its members. Next, a strategy profile $\bar{x} \in X$ is called a *Berge strong equilibrium* [7] for $\mathcal{G}$ if for each $i \in I$ and for all $j \in I - \{i\}$,

$$f_j(\bar{x}_{-i}, \bar{x}_i) \;\geq\; f_j(x_{-i}, \bar{x}_i) \quad \text{for all} \;\; x_{-i} \in X_{-i}.$$

Thus, according to Berge [6], a Berge strong equilibrium is an equilibrium point for all the coalition $I - \{i\}$ for each $i \in I$, i.e., at a Berge strong equilibrium, no coalition of type $I - \{i\}$ can increase the payoffs of any of its members. Note that Nash equilibrium is an equilibrium point for all coalition of type $\{i\}$ only. This shows that the Berge strong equilibrium enjoys a very strong stability compared to the Nash equilibrium. For more details on the properties of the Berge strong equilibrium, see Larbani and Nessah [10].

In other words, while at a Nash equilibrium, none of the players of the game $\mathcal{G}$ have interest to modify his strategy, and at a strong Berge equilibrium, for each player $i$, it is the complementary coalition which has no interest to deviate. As we can see in Example 1 in [7], there exists a game $\mathcal{G}$ which has a Nash equilibrium; but no one has a Berge strong equilibrium at all. As remarked in [7], it is easily seen that a Berge strong equilibrium is a Nash equilibrium, and it is obvious that they coincide in two person games, e.g., see [7, 9, 10]. However, as we remarked, the Berge strong equilibrium enjoys a strong stability property compared to the Nash equilibrium so that it is meaningful to find the sufficient conditions for the existence of a Berge strong equilibrium. Now we have the following implication between these equilibria:

$$\text{Berge strong equilibrium} \Rightarrow \text{Nash equilibrium} \Rightarrow \text{Berge equilibrium.}$$

Next, we will introduce the following which generalizes the Berge strong equilibrium by adopting the complementary constraint correspondences:

**Definition 3.1.** Let $\mathcal{G} = (X_i; T_i, f_i)_{i\in I}$ be a non-cooperative generalized game of normal form. Then, a strategy $n$-tuple $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) \in X$ is called a *generalized Berge strong equilibrium* for the game $\mathcal{G}$ if for each $i \in I$ and for all $j \in I - \{i\}$, we have

$$\bar{x}_{-i} \in T_i(\bar{x}) \quad \text{and} \quad f_j(\bar{x}_{-i}, \bar{x}_i) \geq f_j(y_{-i}, \bar{x}_i) \quad \text{for all} \quad y_{-i} \in T_i(\bar{x}).$$

*Remarks.* (1) In Definition 3.1, if $T_i(x) := X_{-i}$ for each $x \in X$ and $i \in I$, then the condition "$\bar{x}_{-i} \in T_i(\bar{x})$" is clearly satisfied. In this case, the generalized Berge strong equilibrium for the game $\mathcal{G}$ reduces to the Berge strong equilibrium in [7], or Berge equilibrium in the sense of Zhukovskii [16]. Also, our definition is different from the Berge strong equilibrium for an abstract economy in [7]. Indeed, in Definition 2 of [7] or in Theorem 6 of [14], they assume that $T_i : X \to 2^{X_i}$ is a player's (not complementary) constraint correspondence.

(2) We now give some comments on the generalized Berge strong equilibrium. A generalized Berge strong equilibrium $\bar{x} \in X$ means that when a player $i$ plays his strategy $\bar{x}_i$ from the generalized Berge strong equilibrium $\bar{x}$, he cannot obtain a maximum payoff unless the remaining players $I - \{i\}$ willingly (or are obliged to) play the strategy $\bar{x}_{-i}$ from the complementary constraint strategic set $T_i(\bar{x})$. In other words, if at least one of the players of coalition $I - \{i\}$ deviates from his equilibrium strategy, payoff of the player $i$ in the resulting strategy profile would be at most equal to his payoff $f_i(\bar{x})$ on the complementary feasible constraint set $T_i(\bar{x})$.

Throughout this paper, all topological spaces are assumed to be Hausdorff, and for the other standard notations and terminologies, we shall refer to Aubin [5], Daghdak and Florenzano [7], and the references therein.

## 4. Existence of a generalized Berge strong equilibrium

The utility of the player $i$ resulting from the strategy $x$ of actions is the real number $f_i(x)$ where the function $f_i$ is assumed to be certain continuity and convexity conditions relative to $x$. For any $\bar{x} = (\bar{x}_{-i}, \bar{x}_i) \in X$ and $i \in I$, the player $i$ considers his fixed strategy $\bar{x}_i$, and the other players choose their strategies $x_{-i} \in X_{-i}$ so as to maximize his utility $f_j(x_{-i}, \bar{x}_i)$ in his complementary constraint strategy set $T_i(\bar{x}) \subseteq X_{-i}$ for all $j \in I - \{i\}$.

For each $i \in I$ and $x = (x_{-i}, x_i) \in X$, the best complementary response strategy profile $M_i(x)$ for the player $i$ is defined by

(∗)
$$M_i(x) := \bigcap_{j\in I-\{i\}} \big\{(y_{-i}, \cdot) \in X \;\big|\; y_{-i} \in T_i(x),\ f_j(y_{-i}, x_i) = \max_{z\in T_i(x)} f_j(z, x_i)\big\}.$$

Then, for example, we should check the well-definedness of $M_n(x)$ for the fixed player $n$. For each $x = (x_{-1}, x_1) = (x_{-2}, x_2) = \cdots = (x_{-n}, x_n) \in X$, we have the following $(n-1)$ best complementary response profiles

$$\big\{(y_{-n}, \cdot) \in X \;\big|\; y_{-n} \in T_n(x),\ f_1(y_{-n}, x_n) = \max_{z \in T_n(x)} f_1(z, x_n)\big\};$$

$$\big\{(y_{-n}, \cdot) \in X \;\big|\; y_{-n} \in T_n(x),\ f_2(y_{-n}, x_n) = \max_{z \in T_n(x)} f_2(z, x_n)\big\};$$

$$\cdots$$

$$\big\{(y_{-n}, \cdot) \in X \;\big|\; y_{-n} \in T_n(x),\ f_{n-1}(y_{-n}, x_n) = \max_{z \in T_n(x)} f_{n-1}(z, x_n)\big\};$$

and then, $M_n(x)$ is equal to the intersection of the above $(n-1)$ best complementary response profiles. Also, we note that each of the first best complementary response profiles is exactly the intersection of two strategy profiles

$$\big\{(y_{-n}, \cdot) \in X \big|\; y_{-n} \in T_n(x)\big\} \bigcap \big\{(y_{-n}, \cdot) \in X \big|\; f_1(y_{-n}, x_n) = \max_{z \in T_n(x)} f_1(z, x_n)\big\},$$

where the $n$-th component of strategy profile $(y_{-n}, \cdot) \in X$ is dummy so that the intersection of two strategy sets in $X_n$ is meaningless in some sense for the player $n$ to determine his/her best response strategies.

If we define a best complementary response correspondence $M : X \to 2^X$ by $M(x) := \bigcap_{i \in I} M_i(x)$ for each $x \in X$, then we have:

**Lemma 4.1.** *Let $\mathcal{G} = (X_i; T_i, f_i)_{i \in I}$ be a non-cooperative generalized game of normal form, and a best complementary response correspondence $M : X \to 2^X$ be given by $M(x) = \bigcap_{i \in I} M_i(x)$ for each $x \in X$. Assume that $M(x)$ is nonempty for each $x \in X$. If $\bar{x} \in X$ is a fixed point for $M$, then $\bar{x}$ is a generalized Berge strong equilibrium for the game $\mathcal{G}$.*

*Proof.* Let $\bar{x} \in X$ be a fixed point for $M$. Then for each $i \in I$, $\bar{x} \in M_i(\bar{x})$ so that for all $j \in I - \{i\}$, we have

$$\bar{x}_{-i} \in T_i(\bar{x}) \quad \text{and} \quad f_j(\bar{x}_{-i}, \bar{x}_i) = \max_{z \in T_i(\bar{x})} f_j(z, \bar{x}_i);$$

which means that $\bar{x} \in X$ is exactly a generalized Berge strong equilibrium for the game $\mathcal{G}$. □

Now we are ready to prove the existence theorem of generalized Berge strong equilibrium as follow:

**Theorem 4.2.** *Let $I = \{1, 2, \dots, n\}$ be a finite set of players, and let $\mathcal{G} = (X_i; T_i, f_i)_{i \in I}$ be a non-cooperative generalized game of normal form where $X_i$ is a nonempty compact convex strategy subset of the player $i$ in a topological vector space $E_i$. Assume that the joint strategy space $X = \Pi_{i \in I} X_i = X_{-i} \times X_i$ is an admissible subset of $E = \Pi_{i \in I} E_i$, and suppose that for each $i \in I$,*

*(1) $f_i : X_{-i} \times X_i \to \mathbb{R}$ is (jointly) continuous in $X$;*

*(2) $T_i : X_{-i} \times X_i \to 2^{X_{-i}}$ is a closed correspondence in $X$ such that each $T_i(x)$ is a nonempty subset of $X_{-i}$;*

(3) $T_i : X \to 2^{X_{-i}}$ *is lower semicontinuous in* $X$;

(4) $M_i : X \to 2^X$, *defined by the equation* $(*)$, *is such that for each* $x \in X$, $\bigcap_{i\in I} M_i(x)$ *is nonempty acyclic.*

*Then there exists a generalized Berge strong equilibrium* $\bar{x} \in X$ *for the game* $\mathcal{G}$, *i.e., for each* $i \in I$ *and for all* $j \in I - \{i\}$,

$$\bar{x}_{-i} \in T_i(\bar{x}) \quad \text{and} \quad f_j(\bar{x}_{-i}, \bar{x}_i) \geq f_j(y_{-i}, \bar{x}_i) \quad \text{for all } y_{-i} \in T_i(\bar{x}).$$

*Proof.* For each $i \in I$, we define the best complementary response correspondence $M_i : X \to 2^X$ of the player $i$ by for each $x \in X$,

$$M_i(x) := \bigcap_{j\in I-\{i\}} \big\{(y_{-i}, \cdot) \in X \bigm| y_{-i} \in T_i(x),\ f_j(y_{-i}, x_i) = \max_{z\in T_i(x)} f_j(z, x_i)\big\}.$$

Then, by the assumption (4), the best complementary response profile $M_i(x)$ is well-defined and nonempty for each $x \in X$.

Next, we define the best complementary response correspondence $M : X \to 2^X$ by

$$M(x) := \bigcap_{i\in I} M_i(x) \quad \text{for each } x \in X.$$

Since each $M_i(x)$ is exactly equal to the intersection of the $(n-1)$ best complementary response profiles which is given by for $j \in I - \{i\}$,

$$\big\{(y_{-i}, \cdot) \in X \big| \ y_{-i} \in T_i(x)\big\} \bigcap \big\{(y_{-i}, \cdot) \in X \big| \ f_j(y_{-i}, x_i) = \max_{z_{-i}\in T_i(x)} f_j(z_{-i}, x_i)\big\},$$

and the $i$-th component of strategy profile $(y_{-i}, \cdot) \in X$ is dummy, we have that the first set $\big\{(y_{-i}, \cdot) \in X \bigm| y_{-i} \in T_i(x)\big\}$ is equal to the compact set $T_i(x) \times X_i$. Since $X_{-i} = \Pi_{j\in I-\{i\}} X_j$ is compact, and $T_i : X \to 2^{X_{-i}}$ is a closed multimap, $T_i$ is upper semicontinuous in $X$ such that each $T_i(x)$ is a nonempty compact subset of $X_{-i}$. By the assumptions (1)-(4), since $M_i$ is the intersection of the upper semicontinuous complementary response profiles, $M_i$ is also upper semicontinuous in $X$ such that each $M_i(x)$ is nonempty compact. Indeed, since $T_i$ is a continuous correspondence by the assumptions (2) and (3), for each $j \in I - \{i\}$, the correspondence

$$\phi_j : x \mapsto \big\{(y_{-i}, \cdot) \in X \bigm| f_j(y_{-i}, x_i) = \max_{z\in T_i(x)} f_j(z, x_i)\big\}$$

has a closed graph in $X \times X$ by Theorem 3 in [5]. Thus, $\phi_j$ is upper semicontinuous. Since $M_i(x) = \bigcap_{j\in I-\{i\}} \big(\big[T_j(x) \times X_j\big] \cap \phi_j(x)\big)$ for each $x \in X$, $M_i$ is the intersection of upper semicontinuous correspondences so that $M_i$ is also upper semicontinuous in $X$. Hence, the best complementary response correspondence $M(x) := \bigcap_{i\in I} M_i(x)$ is also upper semicontinuous in $X$. By the assumption (4) again, $M(x)$ is nonempty compact acyclic for each $x \in X$ so that $M$ is an acyclic map in $X$. Therefore, by applying Lemma 2.1 to the multimap $M : X \to 2^X$, there exists a fixed point $\bar{x} \in X$ for $M$, i.e., $\bar{x} \in M(\bar{x}) = \bigcap_{i\in I} M_i(\bar{x})$ for each $i \in I$. By Lemma 4.1, $\bar{x} \in X$ is exactly the generalized Berge strong equilibrium for the game $\mathcal{G}$, i.e., for each $i \in I$, and for all $j \in I - \{i\}$, we have

$$\bar{x}_{-i} \in T_i(\bar{x}) \quad \text{and} \quad f_j(\bar{x}_{-i}, \bar{x}_i) \geq f_j(y_{-i}, \bar{x}_i) \quad \text{for all } \; y_{-i} \in T_i(\bar{x});$$

which completes the proof. □

*Remarks.* (1) In Theorem 4.2, if $T_i(x) := X_{-i}$ for each $x \in X$, then the assumptions (2) and (3) are automatically satisfied. In this case, the best complementary response correspondence $M_i$ for the player $i$ is defined by

$$M_i(x) := \bigcap_{j \in I - \{i\}} \big\{(y_{-i}, \cdot) \in X \;\big|\; f_j(y_{-i}, x_i) = \max_{z \in X_{-i}} f_j(z, x_i)\big\} \quad \text{for each } \; x \in X.$$

Therefore, Theorem 4.2 is very different from Theorem 1 in [7] in the following aspects:

(a) we do not need the locally convex assumption on $X_i$;

(b) we do not need the quasiconvex assumption on $f_i$;

(c) we do need the assumption that "$\bigcap_{i \in I} M_i(x)$ is nonempty acyclic"

(2) In Theorem 4.2, if we assume that each $T_i(x)$ is a convex subset of $X_{-i}$, and for each $i \in I$ and for all $j \in I - \{i\}$, the function $y_{-i} \mapsto f_j(y_{-i}, x_i)$ is quasiconcave for any fixed $x_i \in X_i$, then we can see that the set

$$\big\{(y_{-i}, \cdot) \in X \;\big|\; y_{-i} \in T_i(x)\big\} \bigcap \big\{(y_{-i}, \cdot) \in X \;\big|\; f_j(y_{-i}, x_i) = \max_{z \in T_i(x)} f_j(z, x_i)\big\}$$

is convex for each $x \in X$ so that $M_i(x)$ is convex. Therefore, the acyclic assumption (4) on $M_i(x)$ in Theorem 4.2 can be deleted without affecting the conclusion.

(3) By defining the best complementary response correspondence $M_i$ for the player $i$ as in Theorem 4.2, we can generalize Theorem 1 in [7] in a non-cooperative generalized game $\mathcal{G} = (X_i; T_i, f_i)_{i \in I}$ of normal form by modifying the proof of Theorem 1.

## 5. Two examples of non-cooperative generalized games

First, we will give an example of generalized game which has a generalized Berge strong equilibrium, and hence the game has also a Nash equilibrium as follow:

**Example 5.1.** Let $I = \{1, 2\}$ be a set of two players. Let $\mathcal{G}_1 = (X_i; T_i, f_i)_{i \in I}$ be a generalized game such that for each $i \in I$, $X_i := [0, 1]$ is a nonempty compact convex subset of $\mathbb{R}$ and the payoff function $f_i : X = X_1 \times X_2 \to \mathbb{R}$, and a continuous complementary constraint correspondence $T_i : X \to 2^{X_{-i}}$ are defined as follow: For each $(x_1, x_2) \in X$,

$$f_1(x_1, x_2) := x_1^2 + x_2, \quad f_2(x_1, x_2) := x_1 \, x_2^2,$$
$$T_1(x_1, x_2) := X_2, \qquad T_2(x_1, x_2) := X_1.$$

Then the action sets $X_i$ and $X_{-i}$ are compact convex, and equal to $[0, 1]$, and every payoff functions $f_i$ are clearly continuous. Also it is easy to see that

$f_1(x_1, x_2)$ is linear (quasiconcave) in the variable $x_2$ but not quasiconcave in the variable $x_1$, and $f_2(x_1, x_2)$ is linear (quasiconcave) in the variable $x_1$ but not quasiconcave in the variable $x_2$ so that we can not directly apply the existence theorem of Nash equilibrium for the game $\mathcal{G}_1$. However, we can apply Theorem 4.2 to $\mathcal{G}_1$ so that we can obtain a generalized Berge strong equilibrium for the game $\mathcal{G}_1$. Indeed, for each $x = (x_1, x_2) \in X_1 \times X_2$,

$$\begin{aligned}
M_1(x_1, x_2) &= \left\{(\cdot, v) \in X \;\middle|\; v \in T_1(x),\ f_2(x_1, v) = \max_{y' \in T_1(x)} f_2(x_1, y') = x_1\right\} \\
&= \left\{(a, 1) \;\middle|\; a \in [0, 1]\right\}; \\
M_2(x_1, x_2) &= \left\{(u, \cdot) \in X \;\middle|\; u \in T_2(x),\ f_1(u, x_2) = \max_{x' \in T_2(x)} f_1(x', x_2) = 1 + x_2\right\} \\
&= \left\{(1, b) \;\middle|\; b \in [0, 1]\right\},
\end{aligned}$$

so that for each $x \in X_1 \times X_2$, $M(x) := M_1(x) \cap M_2(x) = \{(1, 1)\}$ is nonempty acyclic. Therefore, all the assumptions of Theorem 4.2 are satisfied so that we obtain a generalized Berge strong equilibrium $\{(1, 1)\}$ for the game $\mathcal{G}_1$. In fact, $1 \in T_1(1, 1) = T_2(1, 1)$, and

$$\begin{aligned}
2 = f_1(1, 1) \ &\geq\ f_1(1, x_2) = 1 + x_2 \quad \text{for all} \quad x_2 \in X_2; \\
1 = f_2(1, 1) \ &\geq\ f_2(x_1, 1) = x_1 \qquad \text{for all} \quad x_1 \in X_1.
\end{aligned}$$

Furthermore, the generalized Berge strong equilibrium $\{(1, 1)\}$ is also a Nash equilibrium for the game $\mathcal{G}_1$. Indeed, $1 \in T_1(1, 1) = T_2(1, 1)$, and

$$\begin{aligned}
2 = f_1(1, 1) \ &\geq\ f_1(x_1, 1) = x_1^2 + 1 \quad \text{for all} \quad x_1 \in X_1; \\
1 = f_2(1, 1) \ &\geq\ f_2(1, x_2) = x_2^2 \qquad \text{for all} \quad x_2 \in X_2.
\end{aligned}$$

Indeed, Example 5.1 shows that our Theorem 4.2 is useful tool for finding a Nash equilbrium whenever a given game $\mathcal{G} = (X_i; f_i)_{i \in I}$ is not satisfied the assumptions on the existence theorem of Nash equilibrium for the game $\mathcal{G}$. However, we can show that there must exist a Nash equilibrium for $\mathcal{G}$ by taking the generalized Berge strong equilibrium for the game $\mathcal{G}$.

Next, we will give an example which show that a game $\mathcal{G}_2$ has a Nash equilibrium, but the game can not have a generalized Berge strong equilibrium as follow:

**Example 5.2.** Let $I = \{1, 2, 3\}$ be a set of three players. Let $\mathcal{G}_2 = (X_i; T_i, f_i)_{i \in I}$ be a generalized game such that for each $i \in I$, $X_i := [0, 1]$ is a nonempty compact convex subset of $\mathbb{R}$ and the payoff function $f_i : X = X_1 \times X_2 \times X_3 \to \mathbb{R}$, and a continuous complementary constraint correspondence $T_i : X \to 2^{X_i}$ are defined as follow: For each $(x, y, z) \in X$,

$$\begin{aligned}
&f_1(x, y, z) := x - z^2, \qquad f_2(x, y, z) := x^3 + yz, \qquad f_3(x, y, z) := y + z; \\
&T_1(x, y, z) := X_1, \qquad T_2(x, y, z) := X_2, \qquad T_3(x, y, z) := X_3.
\end{aligned}$$

(for a generalized Berge strong equilibrium case, we may assume

$$T_1(x, y, z) := X_2 \times X_3, \quad T_2(x, y, z) := X_1 \times X_3, \quad T_3(x, y, z) := X_1 \times X_2.)$$

Then the action sets $X_i$ are compact and convex, and every payoff functions $f_i$ are clearly continuous. Also it is easy to see that $f_1(x,y,z)$ is linear (concave) in the variable $x$, and quasiconcave in the variable $(y,z)$, $f_2(x,y,z)$ is linear (concave) in the variable $y$, but not quasiconcave in the variable $(x,z)$, and $f_3(x,y,z)$ is linear (concave) in the variable $z$, and linear (concave) in the variable $(x,y)$ so that we can not directly apply the existence theorem (Theorem 4.2) of Berge strong equilibrium for the game $\mathcal{G}_2$. However, we can apply the existence theorem of Nash equilibrium for the game $\mathcal{G}_2$ (e.g., Theorem 6 in [14]) so that there exits a Nash equilibrium $\{(1,1,1)\}$ for the game $\mathcal{G}_2$. In fact, $1 \in T_1(1,1,1) = T_2(1,1,1) = T_3(1,1,1) = [0,1]$, and

$$\begin{aligned} 0 = f_1(1,1,1) \ &\geq\ f_1(x,1,1) = x-1 \qquad \text{for all} \quad x \in X_1, \\ 2 = f_2(1,1,1) \ &\geq\ f_2(1,y,1) = 1+y \qquad \text{for all} \quad y \in X_2, \\ 2 = f_3(1,1,1) \ &\geq\ f_3(1,1,z) = 1+z \qquad \text{for all} \quad z \in X_3. \end{aligned}$$

However, there can not exist a generalized Berge strong equilibrium for this game $\mathcal{G}_2$. Indeed, suppose that there exists a generalized Berge strong equilibrium $(\bar{x},\bar{y},\bar{z}) \in X$ for the game $\mathcal{G}_2$. Then, in particular, for the player 2, we must have

$$\begin{aligned} \bar{x} - \bar{z}^2 = f_1(\bar{x},\bar{y},\bar{z}) \ &\geq\ f_1(x,\bar{y},z) = x - z^2 \quad \text{for all} \ \ (x,z) \in X_1 \times X_3, \\ \bar{y} + \bar{z} = f_3(\bar{x},\bar{y},\bar{z}) \ &\geq\ f_3(x,\bar{y},z) = \bar{y} + z \quad \text{for all} \ \ (x,z) \in X_1 \times X_3; \end{aligned}$$

which implies the following inequalities

$$\bar{x} - \bar{z}^2 \ \geq\ x - z^2, \quad \bar{y} + \bar{z} \ \geq\ \bar{y} + z$$

must hold for all $(x,z) \in [0,1] \times [0,1]$. From the first inequality, we have $\bar{x} = 1$ and $\bar{z} = 0$ so that the second inequality "$\bar{y} + \bar{z} \ \geq\ \bar{y} + z$" can not be true. Therefore, there can not exist a generalized Berge strong equilibrium for the game $\mathcal{G}_2$.

Indeed, Example 5.2 shows that the generalized Berge strong equilibrium for the game $\mathcal{G}$ is a more delicate concept than a Nash equilbrium for the game $\mathcal{G}$. Anyway, a generalized Berge strong equilibrium and a Nash equilbrium for the game $\mathcal{G}$ are very useful concepts to analyze and find optimal strategies for all the players of the game.

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