Commun. Korean Math. Soc. **29** (2014), No. 2, pp. 359–366
http://dx.doi.org/10.4134/CKMS.2014.29.2.359

ON MINUS TOTAL DOMINATION OF DIRECTED GRAPHS

WenSheng Li, Huaming Xing, and Moo Young Sohn

Abstract. A three-valued function f defined on the vertices of a digraph $D = (V, A)$, $f : V \to \{-1, 0, +1\}$ is a minus total dominating function(MTDF) if $f(N^-(v)) \geq 1$ for each vertex $v \in V$. The minus total domination number of a digraph D equals the minimum weight of an MTDF of D. In this paper, we discuss some properties of the minus total domination number and obtain a few lower bounds of the minus total domination number on a digraph D.

1. Introduction

For terminology and notation on graph theory not given here we follow [1]. Let $G = (V, E)$ be a simple graph (digraph or undirected graph). Let v be a vertex in V. The *open neighborhood* of v is the set $N(v) = \{u \in V \mid uv \in E\}$ and the degree of v is $d_G(v) = |N(v)|$. The maximum degree and minimum degree of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. When no ambiguity can occur, we often simply write $d(v)$, Δ, and δ instead of $d_G(v)$, $\Delta(G)$, and $\delta(G)$, respectively. For any $S \subseteq V$, $G[S]$ is the subgraph induced by S. We denote $\chi(G)$ by the chromatic number of G. Let $D = (V, A)$ be a digraph. For each vertex $v \in V$, let $N^-(v)$ be the in-neighbor set consisting of all vertices of D from which arcs go into v and $N^+(v)$ be the out-neighbor set consisting of all vertices of D into which arcs go from v. If $uv \in A$, we say that v is an out-neighborhood of u and u is an in-neighborhood of v. We write $d^+(v)$ for the outdegree of a vertex v and $d^-(v)$ for its indegree. Then $d^+(v) + d^-(v) = d(v)$. The maximum outdegree, the maximum indegree, the minimum outdegree, and the minimum indegree of D are denoted by Δ^+, Δ^-, δ^+, and δ^-, respectively. If $S \subseteq V$ and $v \in V$, then $E(S, v)$ is the set of arcs from S to v. For a real-valued function $f : V \to \mathbb{R}$, the weight of f is $w(f) = \sum_{v \in V} f(v)$. For $S \subseteq V$, we denote $f(S) = \sum_{v \in S} f(v)$. Then $w(f) = f(V)$.

A *total dominating function*, abbreviated as TDF, of a digraph D was introduced by Huang and Xu [4] as function $f : V \to \{0, 1\}$ such that $f(N^-(v)) \geq 1$

Received August 29, 2013; Revised March 5, 2014.
2010 *Mathematics Subject Classification.* 05C50, 05C69.
Key words and phrases. minus total domination, digraph, tournament, lower bound.

for every $v \in V$. The *total domination number* of D is $\gamma_t(D) = \min\{w(f) \mid f$ is a TDF on $D\}$.

A *signed total dominating function*, abbreviated as STDF, of a digraph D was introduced by S. Sheikholeslami [8] as function $f : V \to \{-1, +1\}$ such that $f(N^-(v)) \geq 1$ for every $v \in V$. The *signed total domination number* of D is $\gamma_{st}(D) = \min\{w(f) \mid f$ is a STDF on $D\}$.

A *minus total dominating function* of an undirected graph $G = (V, E)$ is defined in [2] as a function of the form $f : V \to \{-1, 0, +1\}$ such that $f(N(v)) \geq 1$ for every $v \in V$. The *minus total domination number* of G is $\gamma_t^-(G) = \min\{w(f) \mid f$ is a minus total dominating function on $G\}$. Minus total domination of a graph has been studied in [5, 6, 7, 10, 11] and elsewhere, but the respective analogs on directed graphs have not received any attention.

Now, we introduce the notion of a minus total domination number of digraphs.

Let $D = (V, A)$ be a simple digraph. A *minus total dominating function*, abbreviated as MTDF, of D is a function of the form $f : V \to \{-1, 0, +1\}$ such that $f(N^-(v)) \geq 1$ for every $v \in V$. The *minus total domination number* $\gamma_t^-(D) = \min\{w(f) \mid f$ is an MTDF on $D\}$. We call an MTDF f to be a γ_t^--function of D if $w(f) = \gamma_t^-(D)$. An MTDF f is minimal if no $g < f$ is also an MTDF on D.

To ensure existence of an MTDF, we henceforth restrict our attention to digraphs with $\delta^- \geq 1$. Throughout this paper, if f is an MTDF on D, then we let P_f, Q_f, and M_f denote the sets of those vertices in D which are assigned under f the value $+1, 0$, and -1, respectively. When no ambiguity is possible, P_f, Q_f, and M_f are abbreviated as P, Q, and M, respectively.

In Section 2, we establish some properties of minus total domination on a digraph D. It is shown that if $CT(n)$ is a circulant tournament for odd $n \geq 3$, then $\gamma_t^-(CT(n)) = 3$. In Section 3, we obtain a few lower bounds for minus total domination number on a digraph D. It is shown that if D is an oriented graph of order $n \geq 3$ with $\delta^- \geq 1$ and $\delta^+ \geq 1$ and if G is the underlying graph of D, then $\gamma_t^-(D) \geq \chi(G) + \delta^+ - \Delta(G) - n + 3$. Furthermore, we prove that if D is a digraph with $\delta^+ \geq 1$, then $\gamma_t^-(D) \geq \frac{\delta^+ - \Delta^+ + 2}{\delta^+ + \Delta^+} n$, and this bound is sharp. Our research expands the results of signed total domination of directed graphs as Corollaries 2-4, and enriches the theory of domination on directed graphs.

The motivation for studying this variation of the total domination varied from a modelling perspective. For example, by assigning the values -1, 0 or $+1$ to the vertices of a directed graph which we can model networks of people or organization in which global decisions must be made (e.g., positive, negative or neutral responses or preferences). We assume that each individual has one vote and that each individual has an initial opinion. We assign $+1$ to vertices (individuals) which have a positive opinion, 0 to vertices which have no opinion, and -1 to vertices which have a negative opinion. We also assume, however, that an individual's vote is affected by the opinions of in-neighboring individuals. In particular, the opinions of neighboring individuals (thus individuals of high outdegree have greater 'influence') have equal weight. A voter votes 'aye' if there are more vertices in its in-neighborhood set with positive opinion than with negative opinion, otherwise the vote is 'nay'. We seek an assignment of opinions that guarantee a unanimous decision: that is, for which every vertex votes 'aye'. We call such an assignment of opinions a uniformly positive assignment. Among all uniformly positive assignments of opinions, we are interested primarily in the minimum number of vertices (individuals) who have a positive initial opinion. The minus total domination number can help us to find the minimum number of individuals which have positive opinions.

2. Properties on minus total domination of a digraph

Lemma 1. *An MTDF f on D is minimal if and only if for every vertex $v \in V$ with $f(v) \geq 0$, there exists a vertex $u \in N^+(v)$ with $f(N^-(u)) = 1$.*

Proof. Let f be a minimal MTDF and assume that there is a vertex v_0 with $f(v_0) \geq 0$ and $f(N^-(u)) \geq 2$ for every $u \in N^+(v_0)$. Consider the function $g : V \to \{-1, 0, +1\}$ such that $g(v_0) = f(v_0) - 1$ and $g(v) = f(v)$ for every $v \in V - \{v_0\}$. Then for each $u \in N^+(v_0)$, $g(N^-(u)) = f(N^-(u)) - 1 \geq 1$ and $g(N^-(v)) = f(N^-(v))$ for every $v \in V - N^+(v_0)$. Thus g is an MTDF on D. Since $g < f$, the minimality of f is contradicted.

Conversely, let f be an MTDF on D such that for every $v \in V$ with $f(v) \geq 0$, there exists a vertex $u \in N^+(v)$ with $f(N^-(u)) = 1$. Assume that f is not minimal, i.e., there is an MTDF g on D such that $g < f$. Then $g(v) \leq f(v)$ for all $v \in V$, and there exists at least one vertex $v_0 \in V$ with $g(v_0) < f(v_0)$. Therefore, $f(v_0) \geq 0$ and by assumption, there exists a vertex $u_0 \in N^+(v_0)$ with $f(N^-(u_0)) = 1$. Then $g(N^-(u_0)) \leq f(N^-(u_0)) - 1 = 0$, a contradiction. □

Lemma 2. *Let D be a digraph of order n. If for every vertex $v \in V$, there exists a vertex $u \in N^+(v)$ with $d^-(u) = 1$, then $\gamma_t^-(D) = n$.*

Proof. Let f be a γ_t^--function on D. By assumption, for every vertex $v \in V$, we have $f(v) = f(N^-(u)) \geq 1$, which implies that $f(v) = 1$. Thus $\gamma_t^-(D) = w(f) = n$. □

Corollary 1. *If C_n is the directed circle on n vertices, then $\gamma_t^-(C_n) = n$.*

Since every TDF (or STDF) on a digraph is also an MTDF, the total domination number, signed total domination number and minus total domination number of a digraph are related as follows.

Lemma 3. *Let D be a digraph. Then $\gamma_t^-(D) \leq \min\{\gamma_t(D), \gamma_{st}(D)\}$.*

Theorem 1. *Let D be a digraph. If $\Delta^- \leq 2$ and $\delta^+ \geq 1$, then $\gamma_t^-(D) = \gamma_t(D)$.*

Proof. Let f be a γ_t^--function on D. Then $M = \emptyset$. Suppose to the contrary that $M \neq \emptyset$. Let $u \in M$. Since $\delta^+ \geq 1$, we have $N^+(u) \neq \emptyset$. Let $w \in N^+(u)$. Since $N^-(w) \leq \Delta^- \leq 2$, $f(N^-(w)) \leq f(u) + 1 = 0$, a contradiction. Consider the mapping $g : V \to \{0, 1\}$ such that $g(v) = f(v)$ for every $v \in V$. Then $T = \{v \in V \mid g(v) = 1\}$ is a total dominating set of D. Therefore, $\gamma_t(D) \leq |T| = w(g) = w(f) = \gamma_t^-(D)$. By Lemma 3, $\gamma_t^-(D) \leq \gamma_t(D)$, which implies that $\gamma_t^-(D) = \gamma_t(D)$. □

Let $n (n \geq 3)$ be an odd integer. We have $n = 2k + 1$, where k is a positive integer. We define the circulant tournament $CT(n)$ with n vertices. The vertex set of $CT(n)$ is $V(CT(n)) = \{v_0, v_1, \ldots, v_{n-1}\}$. For each $i\,(0 \leq i \leq n-1)$, the arcs go from v_i to the vertices $v_{i+1}, v_{i+2}, \ldots, v_{i+k}$, the sums being taken modulo n. Then we have the following theorem.

Theorem 2. *Let $CT(n)$ for odd $n \geq 3$ be a circulant tournament. Then $\gamma_t^-(CT(n)) = 3$.*

Proof. Let f be a γ_t^--function on $CT(n)$. For each $v_i \in V(CT(n))$, $N^-(v_i) = \{v_{i-k}, \ldots, v_{i-1} \mid \text{subscripts modulo } n\}$. Therefore,

$$\begin{array}{rcl} f(N^-(v_0)) & = & f(v_{n-k}) + \cdots + f(v_{n-1}) \geq 1 \\ f(N^-(v_1)) & = & f(v_{n-k+1}) + \cdots + f(v_0) \geq 1 \\ & \vdots & \\ f(N^-(v_{n-1})) & = & f(v_{n-k-1}) + \cdots + f(v_{n-2}) \geq 1. \end{array}$$

Summing these inequalities, we have $k(f(v_0) + f(v_1) + \cdots + f(v_{n-1})) \geq n$. Then $\gamma_t^-(CT(n)) = w(f) \geq \frac{n}{k} = 2 + \frac{1}{k}$. Since $\gamma_t^-(CT(n))$ must be an integer, $\gamma_t^-(CT(n)) \geq 3$.

On the other hand, consider the function $g : V(CT(n)) \to \{-1, 0, +1\}$ such that $g(v_i) = +1$ for $i \in \{0, k, 2k\}$ and otherwise $g(v_i) = 0$. Then g is an MTDF on $CT(n)$ and $w(g) = 3$. Thus $\gamma_t^-(CT(n)) \leq 3$, which implies that $\gamma_t^-(CT(n)) = 3$. □

3. Lower bounds of minus total domination number of a digraph

Theorem 3. *Let D be a digraph of order $n \geq 2$. Then $\gamma_t^-(D) \geq 4 - n$ and this bound is sharp.*

Proof. Let f be a γ_t^--function on D and let $v \in V$. Then, there exists a vertex $u \in N^-(v)$ with $f(u) = 1$, and there also exists a vertex $w \in N^-(u)$ with $f(w) = 1$. Therefore $|P_f| \geq 2$ and $|M_f| \leq n - 2$. Thus $\gamma_t^-(D) = |P_f| - |M_f| \geq 4 - n$.

If $H = C_2 \vee \overline{K}_{n-2}$ in which C_2 is a directed circle and the edges are oriented from $V(C_2)$ to $V(\overline{K}_{n-2})$. Consider the mapping $g : V(D) \to \{-1, 0, +1\}$ by $g(v) = 1$ if $v \in V(C_2)$, and $g(v) = -1$ if $v \in V(\overline{K}_{n-2})$. Then g is an MTDF of D and $w(g) = 4 - n$. Hence, $\gamma_t^-(D) = |P_g| - |M_g| \leq 4 - n$, which implies that $\gamma_t^-(D) = 4 - n$. This completes the proof. □

Corollary 2 ([8])**.** *For any digraph D of order $n \geq 2$, $\gamma_{st}(D) \geq 4 - n$, and this bound is sharp.*

Theorem 4. *Let D be an oriented graph of order $n \geq 3$. Then $\gamma_t^-(D) \geq 6-n$, and this bound is sharp.*

Proof. Let f be a γ_t^--function on D. Then for each $v \in P_f$, there exists a vertex $u \in N^-(v)$ with $f(u) = 1$. Thus $|P_f| \leq |A(D[P_f])|$. Since D is an oriented graph, we have $|A(D[P_f])| \leq \frac{|P_f|(|P_f|-1)}{2}$. Therefore, $|P_f| \leq \frac{|P_f|(|P_f|-1)}{2}$ and $|P_f| \geq 3$. Then $|M_f| = n - |P_f| - |Q_f| \leq n - 3$. Thus $\gamma_t^-(D) = |P_f| - |M_f| \geq 6 - n$.

If $D = C_3 \vee \overline{K_{n-3}}$ in which C_3 is a directed circle and the edges are oriented from $V(C_3)$ to $V(\overline{K_{n-3}})$. Consider the mapping $g : V(D) \to \{-1, 0, +1\}$ by $g(v) = 1$ if $v \in V(C_3)$, and $g(v) = -1$ if $v \in V(\overline{K_{n-3}})$. Then g is an MTDF of D and $w(g) = 6 - n$. Hence, $\gamma_t^-(D) \leq 6-n$, which implies that $\gamma_t^-(D) = 6-n$. This completes the proof. □

Szekeres and Wilf [9] gave the following upper bound on the chromatic number of a simple graph G.

Theorem 5 ([9])**.** *For any simple graph G, $\chi(G) \leq 1 + \max\{\delta(H) \,|\, H$ is a subgraph of $G\}$.*

Theorem 6. *Let D be an oriented graph of order $n \geq 3$ with $\delta^- \geq 1$ and $\delta^+ \geq 1$ and let G be the underlying graph of D. Then $\gamma_t^-(D) \geq \chi(G)+\delta^+-\Delta(G)-n+3$.*

Proof. Let f be a γ_t^--function on D. Then $|E(P, v)| \geq |E(M, v)| + 1$ for every $v \in V$. Thus for each $v \in V$,

$$\begin{aligned}\Delta(G) \geq d(v) &\geq |E(P, v)| + |E(M, v)| + |E(Q, v)| + \delta^+(v)\\ &\geq |E(P, v)| + |E(M, v)| + \delta^+(v)\\ &\geq 2|E(M, v)| + 1 + \delta^+.\end{aligned}$$

Therefore, $|E(M, v)| \leq \frac{\Delta(G)-\delta^+-1}{2}$ for every $v \in V$. Let $\mu = \frac{\Delta(G)-\delta^+-1}{2}$, then $|E(M, v)| \leq \mu$. Let $H = D[M]$ and $T = G[M]$, which is the induced graph of G by M. Let H' be an induced subgraph of H and let T' be the underlying graph of H'. Then for every $v \in V(H')$, $d^-_{H'}(v) \leq d^-_H(v) = |E(M, x)| \leq \mu$. Therefore

$$\sum_{v\in V(H')} d^+_{H'}(v) = \sum_{v\in V(H')} d^-_{H'}(v) \leq \sum_{v\in V(H')} \mu = |V(H')|\mu.$$

Thus, there exists a vertex $u \in V(H')$ with $d^+_{H'}(u) \leq \mu$ and we have $d_{T'}(u) = d^-_{H'}(u) + d^+_{H'}(u) \leq 2\mu$. By Theorem 5,

$$\begin{aligned}\chi(T) &\leq 1 + \max\{\delta(L) \,|\, L \text{ is a subgraph of } T\}\\ &= 1 + \max\{\delta(T') \,|\, T' \text{ is an induced subgraph of } T\}\\ &\leq 1 + 2\mu.\end{aligned}$$

Since $\gamma_t^-(D)=|P|-|M|=2|P|+|Q|-n$, we have $|P|+|Q|=\gamma_t^-(D)+n-|P|$. Notice that $|P|\geq 3$, which is proved in Theorem 4. Thus

$$\begin{aligned}\chi(G)&\leq \chi(G[M])+\chi(G[P])+\chi(G[Q])\\&\leq \chi(G[M])+|P|+|Q|\\&=\chi(T)+\gamma_t^-(D)+n-|P|\\&\leq 2\mu+\gamma_t^-(D)+n-2\\&=\Delta(G)-\delta^++\gamma_t^-(D)+n-3.\end{aligned}$$

Therefore, $\gamma_t^-(D)\geq \chi(G)+\delta^+-\Delta(G)-n+3$. $\square$

Theorem 7. *Let D be a digraph with $\delta^+\geq 1$. Then $\gamma_t^-(D)\geq \frac{\delta^+-\Delta^++2}{\delta^++\Delta^+}n$, and this bound is sharp.*

Proof. Let f be a γ_t^--function on D.

Case 1. $\delta^+<\Delta^+$.

Let $P=P_{\Delta^+}\cup P_{\delta^+}\cup P_\Theta$, where $P_{\Delta^+}=\{v\in P\,|\,d^+(v)=\Delta^+\}$, $P_{\delta^+}=\{v\in P\,|\,d^+(v)=\delta^+\}$ and $P_\Theta=\{v\in P\,|\,\delta^++1\leq d^+(v)\leq \Delta^+-1\}$. Similarly, we define $M=M_{\Delta^+}\cup M_{\delta^+}\cup M_\Theta$ and $Q=Q_{\Delta^+}\cup Q_{\delta^+}\cup Q_\Theta$. Further, for $i\in\{\Delta^+,\delta^+,\Theta\}$, let $V_i=P_i\cup M_i\cup Q_i$. Thus $n=|V_{\Delta^+}|+|V_{\delta^+}|+|V_\Theta|$. Since for every $v\in V$, $f(N^-(v))\geq 1$, we have $\sum_{v\in V}f(N^-(v))\geq n$. Thus

$$\begin{aligned}n&\leq \textstyle\sum_{v\in V}f(N^-(v))\\&=\sum_{v\in V}f(v)d^+(v)\\&=\sum_{v\in P}d^+(v)-\sum_{v\in M}d^+(v)\\&\leq \Delta^+|P_{\Delta^+}|+\delta^+|P_{\delta^+}|+(\Delta^+-1)|P_\Theta|-\Delta^+|M_{\Delta^+}|-\delta^+|M_{\delta^+}|-(\delta^++1)|M_\Theta|.\end{aligned}$$

Since for $i\in\{\Delta^+,\delta^+,\Theta\}$, $|P_i|=|V_i|-|M_i|-|Q_i|$, we have

$$\begin{aligned}&n+\Delta^+|M_{\Delta^+}|+\delta^+|M_{\delta^+}|+(\delta^++1)|M_\Theta|\\&\leq \Delta^+|V_{\Delta^+}|+\delta^+|V_{\delta^+}|+(\Delta^+-1)|V_\Theta|-\Delta^+|M_{\Delta^+}|-\delta^+|M_{\delta^+}|-(\Delta^+-1)|M_\Theta|\\&\quad-\Delta^+|Q_{\Delta^+}|-\delta^+|Q_{\delta^+}|-(\Delta^+-1)|Q_\Theta|.\end{aligned}$$

Therefore

$$\begin{aligned}&\Delta^+|V_{\Delta^+}|+\delta^+|V_{\delta^+}|+(\Delta^+-1)|V_\Theta|\\&\geq n+2\Delta^+|M_{\Delta^+}|+2\delta^+|M_{\delta^+}|+(\Delta^++\delta^+)|M_\Theta|+\Delta^+|Q_{\Delta^+}|\\&\quad+\delta^+|Q_{\delta^+}|+(\Delta^+-1)|Q_\Theta|.\end{aligned}$$

Then

$$\begin{aligned}(\Delta^+-1)n\geq{}&2\Delta^+|M_{\Delta^+}|+2\delta^+|M_{\delta^+}|+(\Delta^++\delta^+)|M_\Theta|\\&+\Delta^+|Q_{\Delta^+}|+\delta^+|Q_{\delta^+}|+(\Delta^+-1)|Q_\Theta|+(\Delta^+-\delta^+)|V_{\delta^+}|+|V_\Theta|\end{aligned}$$

$$
\begin{aligned}
&= 2\Delta^+|M_{\Delta^+}| + (\Delta^+ + \delta^+)|M_{\delta^+}| + (\Delta^+ + \delta^+ + 1)|M_\Theta| \\
&\quad + \Delta^+|Q_{\Delta^+}| + \Delta^+|Q_{\delta^+}| + \Delta^+|Q_\Theta| + (\Delta^+ - \delta^+)|P_{\delta^+}| + |P_\Theta|. \\
&\geq (\Delta^+ + \delta^+)|M_{\Delta^+}| + (\Delta^+ + \delta^+)|M_{\delta^+}| + (\Delta^+ + \delta^+)|M_\Theta| \\
&\quad + \Delta^+|Q_{\Delta^+}| + \Delta^+|Q_{\delta^+}| + \Delta^+|Q_\Theta| + (\Delta^+ - \delta^+)|P_{\delta^+}| + |P_\Theta|. \\
&\geq (\Delta^+ + \delta^+)|M| + \Delta^+|Q| \\
&\geq \frac{\Delta^+ + \delta^+}{2}(2|M| + |Q|).
\end{aligned}
$$

Therefore $2|M|+|Q| \leq \frac{2(\Delta^+-1)}{\Delta^++\delta^+}n$. Thus $\gamma_t^-(D) = n-(2|M|+|Q|) \geq \frac{\delta^+-\Delta^++2}{\Delta^++\delta^+}n$.

Case 2. $\delta^+ = \Delta^+ = k \geq 1$.

Since for every $v \in V$, $f(N^-(v)) \geq 1$, we have $n \leq \sum_{v\in V} f(N^-(v)) = \sum_{v\in V} f(v)d^+(v) = k(|P| - |M|)$. Therefore $\gamma_t^-(D) = |P| - |M| \geq \frac{n}{k}$.

In conclusion, we have $\gamma_t^-(D) \geq \frac{\delta^+-\Delta^++2}{\delta^++\Delta^+}n$.

By Corollary 1, the lower bound is sharp. □

Corollary 3. *Let D be a digraph with $\delta^+ \geq 1$. Then $\gamma_{st}(D) \geq \frac{\delta^+-\Delta^++2}{\delta^++\Delta^+}n$.*

Corollary 4 ([8])**.** *If C_n is the directed circle on n vertices, then $\gamma_{st}(C_n) = n$.*

Acknowledgments. This work was supported by the Natural Science Foundation of Hebei Province (A20124 08002) and the Educational Commission of Hebei Province (ZH2011122). The third author was financially supported by Changwon National University in 2013-2014.

References

[1] J. A. Bondy and V. S. R. Murty, *Graph Theory with Application*, Elsevier, Amsterdam, 1976.

[2] L. Harris and J. H. Hattingh, *The algorithmic complexity of certain functional variations of total domination in graphs*, Australas. J. Combin. **29** (2004), 143–156.

[3] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[4] J. Huang and J. M. Xu, *The total domination and total bondage numbers of extended de Bruijn and Kautz digraphs*, Comput. Math. Appl. **53** (2007), no. 8, 1206–1213.

[5] L. Y. Kang, E. F. Shan, and L. Caccetta, *Total minus domination in k-partite graphs*, Discrete Math. **306** (2006), no. 15, 1771–1775.

[6] C. M. Lee, *Signed and minus total domination on subclasses of bipartite graphs*, Ars Combin. **100** (2011), 129–149.

[7] E. F. Shan and T. C. E. Cheng, *Remarks on the minus (signed) total domination in graphs*, Discrete Math. **308** (2008), no. 15, 3373–3380.

[8] S. M. Sheikholeslami, *Signed total domination numbers of directed graphs*, Util. Math. **85** (2011), 273–279.

[9] G. Szekeres and H. S. Wilf, *An inequality for the chromatic number of a graph*, J. Combinatorial Theory **4** (1968), 1–3.

[10] H. M. Xing and H. L. Liu, *Minus total domination in graphs*, Czechoslovak Math. J. **59** (2009), no. 4, 861–870.

[11] H. Yan, X. Q. Yang, and E. F. Shan, *Upper minus total domination in small-degree regular graphs*, Discrete Math. **307** (2007), no. 21, 2453–2463.

Wensheng Li
Department of Mathematics & Information Sciences
Langfang Normal College
Langfang 065000, P. R. China
E-mail address: wsli@live.cn

Huaming Xing
School of Sciences
Tianjin University of Science & Technology
Tianjin 300222, P. R. China
E-mail address: hmxing@aliyun.com

Moo Young Sohn
Department of Mathematics
Changwon National University
Changwon 641-773, Korea
E-mail address: mysohn@changwon.ac.kr