Commun. Korean Math. Soc. **29** (2014), No. 2, pp. 345–349
http://dx.doi.org/10.4134/CKMS.2014.29.2.345

POSITIVELY MEASURE EXPANSIVE AND EXPANDING

Jiweon Ahn, Keonhee Lee, and Manseob Lee

Abstract. We show that C^1-generically, a differentiable map is positively measure expansive if and only if it is expanding.

1. Introduction

Let M be a compact connected C^∞ Riemannian manifold without boundary and $C^1(M)$ the space of differentiable maps of M endowed with the C^1-topology. Denote by d the distance on M induced from the Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Given $x \in M$ and $\delta > 0$, define the *dynamical δ-ball*, $\Gamma_\delta(x) = \{y \in M : d(f^i(x), f^i(y)) \leq \delta \text{ for all } i \geq 0\}$. Let μ be a Borel probability measure which is not necessary f-invariant. Let $f \in C^1(M)$. We say that f is *positively measure expansive* (or, *positively μ-expansive*) if there is $\delta > 0$ (called *expansive constant*) such that for all $x \in M$, $\mu(\Gamma_\delta(x)) = 0$. It is known that if f is positively expansive, then f is open and locally one-to-one, that is, f is a local homeomorphism since M is a manifold without boundary. Since M is connected, it can be checked that the set of periodic points, $P(f)$, of f is dense (see [4]).

We say that f is *expanding* if there are constants $C > 0$ and $\lambda > 1$ such that for any $v \in T_xM (x \in M)$, $\|D_xf^n(v)\| \geq C\lambda^n\|v\|$ for any $n \geq 0$. It is known that every expanding map is positively measure expansive, but the converse is not true. Since every expanding map f is structurally stable, there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that any $g \in \mathcal{U}(f)$ is positively measure expansive. Sakai [4] and Arbieto [1] proved that C^1-generically, a positively expansive map is expanding. In this paper, we study the space of positively measure expansive differentiable maps of M.

A subset $\mathcal{R} \subset C^1(M)$ is called *residual* if it contains a countable intersection of open and dense subsets of $C^1(M)$. A property is called (C^1)*generic* if it holds in a residual subset of $C^1(M)$. Recall that a positively measure expansive differentiable map is not necessarily expanding. However, every positively

Received July 8, 2013; Revised October 30, 2013.
2010 *Mathematics Subject Classification.* 37D20.
Key words and phrases. expansive, measure expansive, generic, expanding.

measure expansive differentiable map is expanding in the C^1-generic case. The following is the main result in this paper.

Theorem 1.1. *For C^1-generic $f \in C^1(M)$, f is positively measure expansive if and only if f is expanding.*

2. Proof of Theorem 1.1

Let M be as before and let $f \in C^1(M)$. Hereafter, we denote by $\mathcal{U}(f)$ a C^1-neighborhood of $f \in C^1(M)$.

Lemma 2.1. *Let $f \in C^1(M)$ and $\mathcal{U}(f)$ be given. Then there are $\delta_0 > 0$ and $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ such that for any $g \in \mathcal{U}_0(f)$, a finite set $\{x_1, x_2, \ldots, x_l\}$, a neighborhood U of $\{x_1, x_2, \ldots, x_l\}$ and linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i - D_{x_i}g\| \le \delta_0$ for all $1 \le i \le l$, there are $\epsilon_0 > 0$ and $\overline{g} \in \mathcal{U}(f)$ such that*

(a) *$\overline{g}(x) = g(x)$ if $x \in M \backslash U$, and*
(b) *$\overline{g}(x) = \exp_{g(x_i)} \circ L_i \circ \exp_{x_i}^{-1}(x)$ if $x \in B_{\epsilon_0}(x_i)$ for all $1 \le i \le l$.*

The assertion (b) implies that $\overline{g}(x) = g(x)$ for $x \in \{x_1, x_2, \ldots, x_l\}$, and that $D_{x_i}\overline{g} = L_i$ for all $1 \le i \le l$.

For $p \in P(f)$, denote by $\pi(p) > 0$ the period, that is, $f^{\pi(p)}(p) = p$. We say that p is *hyperbolic* if $D_p f^{\pi(p)} : T_pM \to T_pM$ has no eigenvalues of modulus 1. Thus T_pM splits into the direct sum $E_p^s \oplus E_p^u$ of subspaces such that $D_p f^{\pi(p)}(E_p^s) = E_p^s$ and $D_p f^{\pi(p)}(E_p^u) = E_p^u$, and there are constants $C > 0$, and $0 < \lambda < 1$ such that for any $n > 0$,

- $\| D_p f^n(v) \| \le C\lambda^n \| v \|$ for any $v \in E_p^s$, and
- $\| D_p f^{-n}(v) \| \le C\lambda^n \| v \|$ for any $v \in E_p^u$.

Let $p \in P(f)$ be hyperbolic. We say that p is a *sink* if $T_pM = E_p^s$, a *source* if $T_pM = E_p^u$, and a *saddle* if $E_p^s \ne \{0\}$ and $E_p^u \ne \{0\}$. Note that if f is positively measure expansive with an expansive constant δ, then there are no sinks and saddles. For, if there is an eigenvalue λ with $|\lambda| < 1$, then there is a local stable manifold $W_\nu^s(p)$ of p for some $\nu > 0$, where $W_\nu^s(p) = \{y \in M : d(f^i(p), f^i(y)) \le \nu, i \ge 0\} = \Gamma_\nu(p)$. We may assume that $\nu < \delta$. Then $W_\nu^s(p) = \Gamma_\nu(p) \subset \Gamma_\delta(p)$ implies that $0 < \mu(W_\nu^s(p)) < \mu(\Gamma_\delta(p))$. This is contradiction, since f is positively measure expansive.

For $0 < \delta < 1$, we say that a hyperbolic periodic point p has a *δ-weak expanding eigenvalue* if $D_p f^{\pi(p)}$ has an eigenvalue λ such that $|\lambda| < (1+\delta)^{\pi(p)}$. Hence because f *has a δ-weak expanding eigenvalue* we mean f *has no hyperbolic periodic point with a δ-weak expanding eigenvalue.* Moreover, we say that the periodic point has real spectrum if all of its eigenvalues are real and simple spectrum if all of its eigenvalues have multiplicity one. Note that by Kupka-Smale's theorem for differentiable maps, for C^1-generic $f \in C^1(M)$, every $p \in P(f)$ is hyperbolic, and thus, such p is source if f is positively measure expansive.

Lemma 2.2. *There is a residual set $\mathcal{G}_1 \subset C^1(M)$ such that for any $f \in \mathcal{G}_1$, and for any $\delta > 0$, if for any C^1-neighborhood $\mathcal{U}(f)$ of f, there is $g \in \mathcal{U}(f)$ having $q_g \in P(g)$ with a δ-weak expanding eigenvalue, then f has $q \in P(f)$ with a δ-weak expanding eigenvalue.*

Proof. Let $q \in P(f)$ be hyperbolic. Then for any $g \in C^1(M)$ C^1-nearby f, there is a unique $q_g \in P(g)(\pi(q) = \pi(q_g))$ nearby q such that $D_{q_g} g^{\pi(q_g)} \to D_q f^{\pi(q)}$ as $g \to f$ with respect to the C^1-topology. For $\delta > 0$ and $n \in \mathbb{N}$, let $\mathcal{H}_n$ be the set of $f \in C^1(M)$ such that there is $q \in P(f)$ with a δ-weak expanding eigenvalue. From the stability of the hyperbolic periodic point stated above, $\mathcal{H}_n$ is open in $C^1(M)$. Let $\mathcal{N}_n = C^1(M) \setminus \overline{\mathcal{H}_n}$. Since $\mathcal{H}_n \cup \mathcal{N}_n$ is open and dense in $C^1(M)$ by their definitions,

$$\mathcal{G}_1 = \bigcap_{n \in \mathbb{N}} (\mathcal{H}_n \cup \mathcal{N}_n)$$

is residual. Here $\mathbb{N}$ denotes the set of natural numbers.

We show that $\mathcal{G}_1$ is what we want. Fix any $f \in \mathcal{G}_1$ and any $\delta > 0$. Suppose that for any $\mathcal{U}(f)$, there is $g \in \mathcal{U}(f)$ having $q_g \in P(g)$ with a δ-weak expanding eigenvalue. Then $f \in \overline{\mathcal{H}_n}$, and so $f \in \mathcal{H}_n$. □

The following was proved by Sakai (see [4, Remark 1]).

Remark 2.3. *There is a residual set $\mathcal{G}_2 \subset C^1(M)$ such that for any $f \in \mathcal{G}_2$, by Lemma 2.2 for any $\delta > 0$, if any $q \in P(f)$ has no 2δ-weak eigenvalue, then there is $\mathcal{U}(f)$ such that for any $g \in \mathcal{U}(f)$, any $q \in P(g)$ has no δ-weak eigenvalue.*

To prove Lemma 2.4, we give some notations on probability measures of M. Denote by $\mathcal{M}(M)$ the set of Borel probability measures of M endowed with the weak topology, and $\mathcal{M}_f(M) \subset \mathcal{M}(M)$ the set of f-invariant measures. We say that $\mu \in \mathcal{M}(M)$ is *atomic* if there exists a point $x \in M$ such that $\mu(\{x\}) > 0$. It is known that the set of non-atomic measures is a residual set in $\mathcal{M}(M)$. Note that for $f \in C^1(M)$, a submanifold $V \subset M$ is said to be a *normally hyperbolic invariant submanifold* if $f(V) = V$ and if there is a splitting $T_xM = T_xV \oplus N_x^s \oplus N_x^u$ for each $x \in V$ such that

- the splitting depends continuously on x,
- the splitting is invariant under Df, i.e., $Df(N_x^s) = N_{f(x)}^s$ and $Df(N_x^u) = N_{f(x)}^u$,
- for some Riemannian metric, and constants $C > 0, \lambda > 1, r \geq 1$, one has for every triple of unit vectors $v \in T_x(V), n^s \in N_x^s$, and $n^u \in N_x^u$ and any $n > 0$

$$\frac{\|(Df^n)n^u\|}{\|(Df^n)v\|} \geq C\lambda^n \text{ and } \frac{\|(Df^n)n^s\|}{\|(Df^n)v\|} \leq C^{-1}\lambda^{-n}.$$

By Mañé's [2] result, the normally hyperbolic V is persistence.

Lemma 2.4. *There is a residual set $\mathcal{G}_3 \subset C^1(M)$ such that for any $f \in \mathcal{G}_3$, if f is positively measure expansive, then there is $\delta > 0$ such that f has no δ-weak expanding eigenvalue.*

Proof. Let $f \in \mathcal{G}_3 = \mathcal{G}_1 \cap \mathcal{G}_2$ be positively measure expansive. To induce the contradiction, suppose that for any $\delta > 0$ there is a periodic point $p \in P(f)$ with a δ-weak expanding eigenvalue. Since δ is arbitrary small, by Lemma 2.1, we can construct $g_1 \in \mathcal{U}(f)$ possessing a non hyperbolic periodic point q. Assume that $D_q g_1^{\pi(q)}$ has an eigenvalue λ with modulus equal to 1. Let E_q^c be the subspace of T_qM generated by eigenvectors corresponding to the eigenvalues λ with $|\lambda| = 1$. Then we have $T_qM = E_q^c \oplus E_q^u$. For simplicity, we assume that $\dim E_q^c = 1$ and the corresponding eigenvalue $\lambda = 1$. Then by Lemma 2.1, we can choose $\epsilon_1 > 0$ and construct $g_2 \in \mathcal{U}(f)$ satisfying the conditions of the Lemma 2.2. Since $\lambda = 1$, there exists a small arc $\mathcal{K}_q \subset B_{\epsilon_1}(q) \cap \exp(E_q^c(\epsilon_1))$ with its center at q such that $g_2^{\pi(q)}(\mathcal{K}_q) = \mathcal{K}_q$ and $\mathcal{K}_q$ is nomally hyperbolic. Then by Mañe's result [2], $\mathcal{K}_q$ is persistence. Thus, f has an arc $\mathcal{J}_p$ such that $f^{\pi(p)}(\mathcal{J}_p) = \mathcal{J}_p$.

Let $\mathfrak{M}_{\mathcal{J}_p}$ be the normalized Lebesgue measure on $\mathcal{J}_p$. Define $\mu \in \mathcal{M}_f(M)$ by

$$\mu(c) = \frac{1}{\pi(p)} \sum_{j=0}^{\pi(p)-1} \mathfrak{M}_{\mathcal{J}_p}[f^{-j}(C \cap f^j(\mathcal{J}_p))]$$

for any Borel set C of M. It is clear that μ is a non-atomic measure.

Let $\delta > 0$. By the continuity of f, there exists $\delta_1 > 0$ such that $d(p, y) < \delta_1$ implies that $d(f^i(p), f^i(y)) < \delta$ for $0 \le i \le \pi(p) - 1$. Recall that

$$\Gamma_\delta(x) = \{y \in M : d(f^i(x), f^i(y)) \le \delta \text{ for all } i \ge 0\}.$$

Since $f^{\pi(p)}|_{\mathcal{J}_p} = \mathcal{J}_p$,

$$\{y \in \mathcal{J}_p : d(p, y) < \delta_1\} \subset \Gamma_\delta(p).$$

Thus, we have

$$\mu(\Gamma_\delta(p)) \ge \mu(\{y \in \mathcal{J}_p : d(p, y) < \delta_1\}) > 0.$$

This is the contradiction for the our assumption, f is positively measure expansive. □

The following theorem founded by Arbieto [1, Theorem 1.3] and Sakai [4, Remark 1].

Theorem 2.5. *For C^1-local diffeomorphism f, if the periodic orbits of any local diffeomorphism g C^1-close to f, are expanding, then f is expanding.*

Proof of Theorem 1.1. Let $f \in \mathcal{G}_3$ be positively measure expansive. To derive a contradiction, we may assume that f is not expanding. Then by Theorem 2.5, for any $\delta > 0$, there is $p_g \in P(g)(g$ C^1-close to $f)$ such that p_g has a δ-weak expanding eigenvalue. Since $f \in \mathcal{G}_1$, f has $p \in P(f)$ with δ-weak expanding eigenvalue. This is a contradiction by Lemma 2.4. Thus by Theorem 2.5, f is expanding. □

Acknowledgments. KL is supported by the National Research Foundation (NRF) of Korea funded by the Korean Government (No. 2011-0015193). ML is the corresponding author and supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology, Korea (No. 2011-0007649).

References

[1] A. Arbieto, *Periodic orbits and expansiveness*, Math. Z. **269** (2011), no. 3-4, 801–807.
[2] R. Mane, *Persistent manifolds are normally hyperbolic*, Trans. Amer. Math. Soc. **246** (1978), 261–283.
[3] W. L. Reddy, *Expanding maps on compact metric spaces*, Topology Appl. **13** (1982), no. 3, 327–334.
[4] K. Sakai, *Positively expansive differentiable Maps*, Acta Math. Sin. (Engl. Ser.) **26** (2010), no. 10, 1839–1846.
[5] K. Sakai, N. Sumi, and K. Yamamoto, *Measure-expansive diffeomorphisms*, Preprint.
[6] D. Yang and S. Gan, *Expansive homoclinic classes*, Nonlinearity **22** (2009), no. 4,729–733.

Jiweon Ahn
Department of Mathematics
Chungnam National University
Daejeon 305-764, Korea
E-mail address: `twinangel12@hanmail.net`

Keonhee Lee
Department of Mathematics
Chungnam National University
Daejeon 305-764, Korea
E-mail address: `khlee@cnu.ac.kr`

Manseob Lee
Department of Mathematics
Mokwon University
Daejeon 302-729, Korea
E-mail address: `lmsds@mokwon.ac.kr`