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ON LIGHTLIKE SUBMANIFOLDS OF A GRW SPACE-TIME

Tae Ho Kang

Abstract. This paper provides a study of lightlike submanifolds of a generalized Robertson-Walker (GRW) space-time. In particular, we investigate lightlike submanifolds with curvature invariance, parallel second fundamental forms, totally umbilical second fundamental forms, null sectional curvatures and null Ricci curvatures, respectively.

1. Introduction

In general relativity, a space time is a four-dimensional differentiable manifold equipped with a Lorentzian metric. One of the important cosmological models in general relativity is the family of Robertson-Walker space-times:

$$L_1^4(c,f) := (I \times_f F, \bar{g}), \bar{g} = -dt^2 + f^2(t)g_c.$$

Explicitly, $L_1^4(c,f)$ is a warped product with Lorentzian metric $\bar{g}$ of an open interval I and a three-dimensional Riemannian manifold (F, g_c) of constant curvature c with a warping function $f > 0$, which is defined on an open interval I in R_1^1.

Recently, B. Y. Chen and J. Van der Veken ([3]) studied nondegenerate surfaces (i.e., spatial or Lorentzian) of a Robertson-Walker space-time from differential geometric view point. In [9], the author studied lightlike (degenerate, null) hypersurfaces of a generalized Robertson-Walker space-time (GRW), which is also defined as a warped product $L_1^{n+1}(c,f) = I \times_f F$, where F is an n-dimensional Riemannian manifold of constant curvature c.

In [4], B. Y. Chen and S. W. Wei provided a general study of submanifolds in the Riemannian warped product $I \times_f F$, $\bar{g} = dt^2 + f^2(t)g_c$.

In this article we give a study of lightlike submanifolds of a GRW space-time $L_1^{n+1}(c,f)$. In particular, we investigate lightlike submanifolds with curvature invariance and parallel second fundamental forms (Section 4), totally umbilical lightlike submanifolds (Section 5), null sectional curvatures and null Ricci curvatures (Section 6), respectively.

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2. Basics on GRW space-times

In this section, we review some results of the connection and curvature of a GRW space-time, which follow from general results on warped product ([11]).

Consider a GRW space-time

$$L_1^{n+1}(c,f) = (I \times_f F, \bar{g}), \bar{g} = -dt^2 + f^2(t) g_c,$$

where f is a smooth positive function on I, and (F, g_c) is an n-dimensional Riemannian manifold of constant sectional curvature c. The standard choices for F are S^n, E^n and H^n, with curvature 1, 0,−1, respectively.

Let π and σ be the natural projections of $I \times F$ onto I and F, respectively. Let $\mathfrak{L}(I)$ and $\mathfrak{L}(F)$ be the set of horizontal and vertical lifts of vector fields on I and F to $I \times_f F$, respectively. Let $\partial_t \in \mathfrak{L}(I)$ denote the horizontal lift vector field to $I \times_f F$ of the standard vector field $\frac{d}{dt}$ on I.

By a *spacelike slice* of $L_1^{n+1}(c,f) = (I \times_f F, \bar{g})$ we mean a hypersurface of $L_1^{n+1}(c,f)$ given by a fibre $S(t_0) := \pi^{-1}(t_0)$ with metric $f^2(t_0) g_c$.

For each vector X tangent to $L_1^{n+1}(c,f)$, we put

$$X = \phi_X \partial_t + \hat{X}, \tag{2.1}$$

where $\phi_X = -\bar{g}(X, \partial_t)$ and $\hat{X}$ is the vertical component of X.

The following two lemmas are well-known ([11]).

Lemma 2.1. *Let $\bar{\nabla}$ be the Levi Civita connection of $L_1^{n+1}(c,f)$. For vectors fields $X, Y \in \mathfrak{L}(F)$ we have*

(1) $\bar{\nabla}_{\partial_t} \partial_t = 0$,

(2) $\bar{\nabla}_{\partial_t} X = \bar{\nabla}_X \partial_t = (\ln f)' X$,

(3) $\bar{g}(\bar{\nabla}_X Y, \partial_t) = -\bar{g}(X,Y)(\ln f)'$,

(4) $\widehat{\bar{\nabla}_X Y}$ *is the vertical lift of $\nabla^F_X Y$, where ∇^F is the Levi Civita connection of F.*

Lemma 2.2. *Let $\bar{R}$ be the curvature tensor of $L_1^{n+1}(c,f)$. If $X, Y, Z \in \mathfrak{L}(F)$, then*

(1) $\bar{R}(\partial_t, X)\partial_t = \frac{f''}{f} X$,

(2) $\bar{R}(X, \partial_t) Y = -\bar{g}(X,Y) \frac{f''}{f} \partial_t$,

(3) $\bar{R}(X,Y)\partial_t = 0$,

(4) $\bar{R}(X,Y)Z = \frac{(f')^2 + c}{f^2} (\bar{g}(Y,Z)X - \bar{g}(X,Z)Y)$.

It follows from (2.1) and (2) in Lemma 2.1 that

$$\bar{\nabla}_X (f(t)\partial_t) = f'(t) X \tag{2.2}$$

for any vector field X on $L_1^{n+1}(c,f)$.

On the other hand we can agglomerate (1) $\sim$ (4) in Lemma 2.2 together into a single form (2.3) ([9]).

Proposition 2.3. *For any vector fields* X, Y, Z *on* $L_1^{n+1}(c, f)$

$$\bar{R}(X,Y)Z = \lambda\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} + \mu\{\phi_X\phi_Z Y - \phi_Y\phi_Z X + (\phi_X\bar{g}(Y,Z) - \phi_Y\bar{g}(X,Z))\partial_t\}, \tag{2.3}$$

where $\lambda = \frac{f'^2+c}{f^2}$, $\mu = \frac{ff''-(f'^2+c)}{f^2}$.

Remark 2.4. (1) $L_1^{n+1}(c,f)$ is flat if and only if $f(t) = at + b(c = -a^2)$,

(2) $L_1^{n+1}(c,f)$ has constant curvature $k^2 > 0$ if and only if $f(t) = ae^{kt} + be^{-kt}$, $c = 4k^2ab$,

(3) $L_1^{n+1}(c,f)$ has constant curvature $-k^2 < 0$ if and only if $f(t) = a\sin(kt) + bcos(kt)$, $c = -4k^2(a^2+b^2)$.

(4) $L_1^{n+1}(c,f)$ is of constant curvature if and only if $\mu = 0$.

3. Basics on lightlike submanifolds

Let $(\bar{M}, \bar{g})$ be an $(m+n)$-dimensional semi-Riemannian manifold of constant index $\nu, 1 \leq \nu < m+n$ and (M, g) be a submanifold of $(\bar{M}, \bar{g})$ of codimension n.

Consider the so-called *radical distribution* $Rad(TM) := TM \cap TM^{\perp}$. We say that (M, g) is a *lightlike submanifold* of $(\bar{M}, \bar{g})$ if $Rad(TM)$ defines a nonzero differentiable distribution on M of $\mathrm{rank}(Rad(TM)) =: r > 0$.

Let $S(TM)$ be a complementary distribution of $Rad(TM)$ in TM. Then $S(TM)$ is orthogonal to $Rad(TM)$ and nondegenerate with respect to $\bar{g}$ and TM has the orthogonal direct sum

$$TM = Rad(TM) \perp S(TM).$$

Let $S(TM^{\perp})$ be a complementary distribution of $Rad(TM)$ in $TM^{\perp}$. Then $TM^{\perp}$ has the following orthogonal direct decomposition

$$TM^{\perp} = Rad(TM) \perp S(TM^{\perp}).$$

Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles (resp. the *transversal vector bundle* and the *lightlike transversal vector bundle*) to TM in $T\bar{M}\mid_M$ and $Rad(TM)$ in $S(TM^{\perp})^{\perp}$, respectively, where $S(TM^{\perp})^{\perp}$ denotes the orthogonal complementary vector subbundle to $S(TM^{\perp})$ in $S(TM)^{\perp}$, i.e., $S(TM)^{\perp} = S(TM^{\perp}) \perp S(TM^{\perp})^{\perp}$.

Then we have the following decompositions:

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{3.1}$$

$$\begin{aligned} T\bar{M}\mid_M &= S(TM) \perp \{Rad(TM) \oplus ltr(TM)\} \perp S(TM^{\perp}) \\ &= TM \oplus tr(TM). \end{aligned} \tag{3.2}$$

There are four possible cases on lightlike submanifolds. If an m-dimensional lightlike submanifold (M, g) of $(\bar{M}, \bar{g})$ with codimension n is

case 1 : *r-lightlike* if $1 \le r < min\{m, n\}$,
case 2 : *co-isotropic* if $r = n < m$, $S(TM^{\perp}) = \{0\}$,
case 3 : *isotropic* if $r = m < n$, $S(TM) = \{0\}$,
case 4 : *totally lightlike* if $r = m = n$, $S(TM) = \{0\}$ and $S(TM^{\perp}) = \{0\}$.

According to the decomposition (3.1) we put

$$\begin{aligned}\bar{\nabla}_X Y &= \nabla_X Y + h(X, Y) \\ &= \nabla_X Y + h^l(X, Y) + h^s(X, Y), \ \ \forall X, Y \in \Gamma(TM),\end{aligned} \tag{3.3}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \ \ N \in \Gamma(ltr(TM)), \tag{3.4}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \ \ W \in \Gamma(S(TM^{\perp})), \tag{3.5}$$

where $\nabla_X Y$, $A_N X, A_W X \in \Gamma(TM)$, $h(X, Y) \in \Gamma(tr(TM))$, $h^l(X, Y)$, $\nabla^l_X N$, $D^l(X, W) \in \Gamma(ltr(TM))$, and $h^s(X, Y)$, $D^s(X, N)$, $\nabla^s_X W \in \Gamma(S(TM^{\perp}))$.

We note that the lightlike second fundamental form h^l of a lightlike submanifold M does not depend on $S(TM), S(TM^{\perp})$ and $ltr(TM)$.

Making use of (3.3) $\sim$ (3.5) and the fact that $\bar{\nabla}$ is a metric connection, we obtain

$$\bar{g}(h^l(X, Y), \xi) + \bar{g}(Y, h^l(X, \xi)) + g(Y, \nabla_X \xi) = 0, \tag{3.6}$$

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{3.7}$$

$$\bar{g}(A_N X, N') + \bar{g}(A_{N'} X, N) = 0, \tag{3.8}$$

$$\bar{g}(D^s(X, N), W) = \bar{g}(N, A_W X), \tag{3.9}$$

where $\xi \in \Gamma(Rad(TM))$, $W \in \Gamma(S(TM^{\perp}))$ and $N, N' \in \Gamma(ltr(TM))$.

From the decomposition $TM = S(TM) \perp Rad(TM)$, we set

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY), \tag{3.10}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi \tag{3.11}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X PY, A^*_\xi X\}$ and $\{h^*(X, PY), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively.

It follows that ∇^* and ∇^{*t} are linear connections on distributions $S(TM)$ and $Rad(TM)$, respectively. From (3.10) and (3.11) we obtain

$$\bar{g}(h^l(X, PY), \xi) = g(A^*_\xi X, PY), \tag{3.12}$$

$$\bar{g}(h^*(X, PY), N) = g(A_N X, PY), \quad \forall X, Y \in \Gamma(TM). \tag{3.13}$$

In general, the induced connection ∇ on M is not a metric connection, since

$$(\nabla_X g)(Y, Z) = \bar{g}(h^l(X, Y), Z) + \bar{g}(h^l(X, Z), Y). \tag{3.14}$$

Now we give some structure equations of lightlike submanifolds.

Let $(M, g, S(TM), S(TM^{\perp}))$ be an m-dimensional r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Denote by $\bar{R}, R, R^l$ and R^{*t} the curvature tensors of $\bar{\nabla}, \nabla, \nabla^l$ and ∇^{*t}, respectively. The following structure equations hold:

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z &+ A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X \\ &+ A_{h^s(X,Z)}Y - A_{h^s(Y,Z)}X \\ &- (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) \\ &+ D^l(X, h^s(Y,Z)) - D^l(Y, h^s(X,Z)) - (\nabla_X h^s)(Y,Z) \\ &- (\nabla_Y h^s)(X,Z) + D^s(X, h^l(Y,Z)) - D^s(Y, h^l(X,Z)),\end{aligned} \tag{3.15}$$

$$\begin{aligned}\bar{R}(X,Y)N = R^l(X,Y)N &+ h^l(Y, A_N X) - h^l(X, A_N Y) \\ &+ D^l(X, D^s(Y,N)) - D^l(Y, D^s(X,N)) \\ &+ (\nabla_Y A)(N,X) - (\nabla_X A)(N,Y) \\ &+ A_{D^s(X,N)}Y - A_{D^s(Y,N)}X + (\nabla_X D^s)(Y,N) \\ &- (\nabla_Y D^s)(X,N) + h^s(Y, A_N X) - h^s(X, A_N Y),\end{aligned} \tag{3.16}$$

$$\begin{aligned}\bar{g}(\bar{R}(X,Y)N,\xi) = \bar{g}(R^l(X,Y)N,\xi) &\\ + \bar{g}(h^l(Y,A_N X),\xi) &- \bar{g}(h^l(X,A_N Y,\xi) \\ + \bar{g}(D^s(X,N),h^s(Y,\xi)) &- \bar{g}(D^s(Y,N),h^s(X,\xi)),\end{aligned} \tag{3.17}$$

$$\begin{aligned}\bar{g}(\bar{R}(X,Y)W',W) = \bar{g}(R^s(X,Y)W',W) &\\ + g(A_{W'}X, A_W Y) &- g(A_W X, A_{W'}Y) \\ + \bar{g}(D^l(X,W), A_{W'}Y) &- \bar{g}(D^l(Y,W), A_{W'}X) \\ + \bar{g}(D^l(Y,W'), A_W X) &- \bar{g}(D^l(X,W'), A_W Y)\end{aligned} \tag{3.18}$$

for any vector fields $X, Y, Z \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$, $N \in \Gamma(ltr(TM))$ and $W, W' \in \Gamma(S(TM^\perp))$.

Let (M, g) be an r-lightlike submanifold of $L_1^{n+1}(c, f)$. Then r must be one, because the index of $L_1^{n+1}(c, f)$ is equal to 1.

Here and in the sequel, we mean a lightlike submanifold M of $L_1^{n+1}(c, f)$ by a 1-lightlike one unless otherwise stated.

From the decomposition (3.2) we get

$$indL_1^{n+1}(c,f) = indS(TM) + ind\{Rad(TM) \oplus ltr(TM)\} + indS(TM^\perp),$$

where $ind(\bullet)$ denotes the index of the metric tensor $\bar{g}$ on $\bullet$. Since $\{Rad(TM) \oplus ltr(TM)\}$ is nondegenerate and of constant index one, both $S(TM)$ and $S(TM^\perp)$ are Riemannian vector bundles over M.

4. Curvature invariance and parallel second fundamental forms

To begin with we prepare the following lemma ([9]).

Lemma 4.1. *Let M be a lightlike submanifold of $L_1^{n+1}(c,f)$. Then*
(1) *∂_t can not be tangent to M, i.e., $\partial_t^{tr} \neq 0$,*
(2) *∂_t can not be orthogonal to M,*
(3) *$\phi_U \neq 0$ for any nonzero null vector U on $L_1^{n+1}(c,f)$,*
where ∂_t^{tr} denotes the transversal projection of ∂_t with respect to the decomposition (3.2).

Let (M,g) be a submanifold of a semi-Riemannian manifold $(\bar{M},\bar{g})$. If for any vector fields X and Y on M $\bar{R}(X,Y)T_pM \subset T_pM$ for each $p \in M$, then the submanifold M is said to be *curvature invariant* ([12]), where T_pM denotes the tangent space of M at the point $p \in M$.

Proposition 4.2. *Let $(M,g,S(TM),S(TM^\perp))$ be a lightlike submanifold of $L_1^{n+1}(c,f)$. Then M is curvature invariant if and only if $L_1^{n+1}(c,f)$ is of constant curvature.*

Proof. If M is curvature invariant, then

$$\bar{g}(\bar{R}(X,Y)Z,\xi) = 0, \ \forall Z \in \Gamma(TM), \ \xi \in \Gamma(Rad(TM)). \tag{4.1}$$

From which, using (2.1) and (2.3) we obtain

$$\mu\phi_\xi\{\phi_X\bar{g}(Y,Z) - \phi_Y\bar{g}(X,Z)\} = 0.$$

Putting $X=\xi$ gives $\mu\phi_\xi^2\bar{g}(Y,Z)=0$. Again, putting $Y=Z=PY(\neq 0)$ gives $\mu=0$, since $S(TM)$ is Riemannian and $\phi_\xi \neq 0$ (Lemma 4.1(3)). The converse follows from (2.3) and (4) in Remark 2.4. □

A lightlike submanifold (M,g) of a semi-Riemannian manifold $(\bar{M},\bar{g})$ is *irrotational* ([10]) if

$$\bar{\nabla}_X\xi \in \Gamma(TM), \quad \forall X \in \Gamma(TM), \ \xi \in \Gamma(Rad(TM)),$$

which is equivalent to

$$h^s(X,\xi)=0, h^l(X,\xi)=0, \quad \forall X \in \Gamma(TM), \ \xi \in \Gamma(Rad(TM)) \tag{4.2}$$

with the aid of (3.1) and (3.3).

Proposition 4.3. *Let $(M,g,S(TM),S(TM^\perp))$ be a lightlike submanifold of $L_1^{n+1}(c,f)$. If M is irrotational and the lightlike second fundamental form h^l is parallel, then $L_1^{n+1}(c,f)$ has constant curvature.*

Proof. Since the lightlike second fundamental form h^l is parallel, i.e.,

$$(\nabla_X h^l)(Y,Z) = 0, \ \forall X,Y,Z \in \Gamma(TM),$$

it follows from (3.15) that

$$\bar{g}(\bar{R}(X,Y)Z,\xi) = \bar{g}(D^l(X,h^s(Y,Z)),\xi) - \bar{g}(D^l(Y,h^s(X,Z)),\xi). \tag{4.3}$$

From which and (3.7) we get

$$\bar{g}(\bar{R}(X,Y)Z,\xi) = \bar{g}(h^s(X,Z),h^s(Y,\xi)) - \bar{g}(h^s(Y,Z),h^s(X,\xi)) = 0,$$

which implies that M is curvature invariant. Hence we conclude from Proposition 4.2 that $L_1^{n+1}(c,f)$ has constant curvature. □

Corollary 4.4. *Let $(M,g,S(TM),S(TM^\perp))$ be a co-isotropic submanifold of $L_1^{n+1}(c,f)$, i.e., $S(TM^\perp)=\{0\}$. If the lightlike second fundamental form h^l is parallel, then $L_1^{n+1}(c,f)$ has constant curvature.*

Proof. It follows from (4.3) with $h^s=0$. □

5. Totally umbilical lightlike submanifolds

Let $(M,g,S(TM),S(TM^\perp))$ be an m-dimensional lightlike submanifold of a GRW space-time $L_1^{n+1}(c,f)$.

Consider the following local quasi-orthonormal field of frames of $L_1^{n+1}(c,f)$ along M ([5]):

$$\{\xi, N, X_1, \dots X_{m-1}, W_1, \dots, W_{n-m}\}, \tag{5.1}$$

where ξ and N are lightlike bases of $\Gamma(Rad(TM))$ and $\Gamma(ltr(TM))$, respectively satisfying

$$\bar{g}(N,\xi)=1, \quad \bar{g}(N,N)=0, \tag{5.2}$$

$\{X_1,\dots,X_{m-1}\}$ and $\{W_1,\dots,W_{n-m}\}$ are orthonormal bases $\Gamma(S(TM))$ and $\Gamma(S(TM^\perp))$, respectively. Throughout this paper, we adopt the following range of indices:

$$a\in\{1,\dots,m-1\}; \quad \alpha\in\{1,\dots,n-m\}.$$

The local expressions corresponding to (3.3) $\sim$ (3.5) and (3.11) are respectively given by (5.3) $\sim$ (5.5) and (5.6):

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N + \sum_{\alpha=1}^{n-m} h_\alpha^s(X,Y)W_\alpha, \tag{5.3}$$

where $B(X,Y)=\bar{g}(h^l(X,Y),\xi)$, $h_\alpha^s(X,Y)=\bar{g}(h^s(X,Y),W_\alpha)$.

$$\bar{\nabla}_X N = -A_N X + \rho(X)N + \sum_\alpha \tau_\alpha(X)W_\alpha, \tag{5.4}$$

$$\bar{\nabla}_X W_\alpha = -A_{W_\alpha}X + \nu_\alpha(X)N + \sum_\beta \theta_{\alpha\beta}(X)W_\beta, \tag{5.5}$$

where

$$\rho(X)=\bar{g}(\nabla_X^l N,\xi), \quad \tau_\alpha(X)=\bar{g}(D^s(X,N),W_\alpha),$$
$$\nu_\alpha(X)=\bar{g}(D^l(X,W_\alpha),\xi), \quad \theta_{\alpha\beta}(X)=\bar{g}(\nabla_X^s W_\alpha, W_\beta),$$

and

$$\nabla_X\xi = -A_\xi^* X - \rho(X)\xi. \tag{5.6}$$

Now we define locally the 1-form

$$\eta(X)=\bar{g}(X,N), \forall X\in\Gamma(TM).$$

Then η defines locally the screen distribution $S(TM)$ because $X \in \Gamma(S(TM))$ if and only if $\eta(X) = 0$. Furthermore we have:

Proposition 5.1. *Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold of $L_1^{n+1}(c, f)$. If $\mathcal{L}_X \eta = 0$ for any vector field X tangent to M, then $S(TM)$ is integrable, where $\mathcal{L}_X$ denotes the Lie derivative in the direction X.*

Proof. It follows from (3.3) and (3.4) that

$$\begin{aligned} 0 &= (\mathcal{L}_X \eta)(Y) \\ &= X\bar{g}(Y, N) - \eta([X, Y]) \\ &= \bar{g}(\nabla_X Y, N) - \bar{g}(Y, A_N X) + \rho(X)\eta(Y) - \eta([X, Y]), \forall X, Y \in \Gamma(TM). \end{aligned}$$

Putting $Y = PY$ in this equation, and using (3.10) and (3.13) yield $\eta([X, PY])$ $= 0$, which means that $S(TM)$ is integrable. □

Let (M, g) be a 1-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then

$$h^l(X, \xi) = 0, \quad \forall X \in \Gamma(TM),\ \xi \in \Gamma(Rad(TM)),$$

which follows from putting $Y = \xi$ in (3.6).

Proposition 5.2. *Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold of $L_1^{n+1}(c, f)$. If the time like vector field ∂_t belongs to the hyperbolic plane bundle $Rad(TM) \oplus ltr(TM)$, then*

$$\phi_{PX} = \phi_W = 0, \quad \phi_N h^s(X, \xi) + \phi_\xi D^s(X, N) = 0, \quad 2\phi_\xi \phi_N = -1 \tag{5.7}$$

where $X \in \Gamma(TM)$, $W \in \Gamma(S(TM^\perp))$, $\xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$.

Proof. Using a local quasi-orthonormal field of frames satisfying (5.1) and (5.2) we have

$$\partial_t = -\sum_a \phi_{X_a} X_a - \phi_N \xi - \sum_\alpha \phi_{W_\alpha} W_\alpha - \phi_\xi N.$$

The assumption shows that

$$\phi_{PX} = 0, \phi_W = 0, \quad \forall X \in \Gamma(TM),\ W \in \Gamma(S(TM^\perp)). \tag{5.8}$$

Moreover substituting $\partial_t = -\phi_N \xi - \phi_\xi N$ into (2.2), we get for any vector field X tangent to M

$$\begin{aligned} f'(t)X = (Xf)(-\phi_N \xi - \phi_\xi N) - f(t)\{(X\phi_N)\xi + \phi_N(\nabla_X \xi + h^s(X, \xi)) \\ + (X\phi_\xi)N + \phi_\xi(-A_N X + \nabla^l_X N + D^s(X, N))\}. \end{aligned}$$

Taking $S(TM^\perp)$-part in both sides, we get the second one. The last one follows from computing $-1 = \bar{g}(\partial_t, \partial_t)$. □

A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be *totally umbilical* ([6]) if there is a smooth transversal vector field $\mathcal{H} \in \Gamma(tr(TM))$ on M such that for all $X, Y \in \Gamma(TM))$

$$h(X, Y) = \mathcal{H} g(X, Y). \tag{5.9}$$

The definition does not depend on the choice of both the screen distribution $S(TM)$ and the screen transversal vector bundle $S(TM^{\perp})$.

Making use of (3.3), it is easy to see that M is totally umbilical if and only if, on any coordinate neighborhood $\mathcal{U}$ in M there is smooth vector fields $\mathcal{H}^l \in \Gamma(ltr(TM))$, $\mathcal{H}^s \in \Gamma(S(TM^{\perp}))$, and smooth functions $\mathcal{H}_0^l \in F(ltr(TM))$, $\mathcal{H}_\alpha^s \in F(S(TM^{\perp}))$ such that

$$h^l(X,Y) = \mathcal{H}^l \bar{g}(X,Y), h^s(X,Y) = \mathcal{H}^s \bar{g}(X,Y), \tag{5.10}$$

$$B(X,Y) = \mathcal{H}_0^l \bar{g}(X,Y), h_\alpha^s(X,Y) = \mathcal{H}_\alpha^s \bar{g}(X,Y), \tag{5.11}$$

where $\mathcal{H}_0^l = \bar{g}(\mathcal{H}^l, \xi)$, $\mathcal{H}_\alpha^s = \bar{g}(\mathcal{H}^s, W_\alpha)$.

Moreover it is clear from (3.7) and (3.12) that on each coordinate neighborhood $\mathcal{U}$ in M the followings hold when M is totally umbilical:

$$D^l(X, W_\alpha) = 0 \text{ (i.e., } \nu_\alpha(X) = 0), A_\xi^* X = \mathcal{H}_0^l PX, P(A_{W_\alpha} X) = \mathcal{H}_\alpha^s PX. \tag{5.12}$$

Theorem 5.3. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a totally umbilical lightlike submanifold of $L_1^{n+1}(c,f)$. Then the functions $\mathcal{H}_0^l$ and $\mathcal{H}_\alpha^s$ satisfy the following partial differential equations*:

(1) $\xi \mathcal{H}_0^l + \rho(\xi)\mathcal{H}_0^l - (\mathcal{H}_0^l)^2 + \mu \phi_\xi^2 = 0,$

(2) $\xi \mathcal{H}_\alpha^s + \tau_\alpha(\xi)\mathcal{H}_0^l - \mathcal{H}_0^l \mathcal{H}_\alpha^s - \sum_\beta \mathcal{H}_\beta^s \theta_{\beta\alpha}(\xi) + \mu \phi_\xi \phi_{W_\alpha} = 0,$

(3) $PX(\mathcal{H}_0^l) + \rho(PX)\mathcal{H}_0^l + \mu \phi_{PX} \phi_\xi = 0,$

(4) $PX(\mathcal{H}_\alpha^s) - \sum_\beta \mathcal{H}_\beta^s \theta_{\beta\alpha}(PX) + \tau_\alpha(PX)\mathcal{H}_0^l + \mu \phi_{PX} \phi_{W_\alpha} = 0,$

(5)

$$\begin{aligned} R(X,Y)Z = &\{\lambda X + \mathcal{H}_0^l A_N X + \sum_\alpha \mathcal{H}_\alpha^s A_{W_\alpha} X\} g(Y,Z) \\ &- \{\lambda Y + \mathcal{H}_0^l A_N Y + \sum_\alpha \mathcal{H}_\alpha^s A_{W_\alpha} Y\} g(X,Z) \\ &+ \mu\{\phi_X \phi_Z Y - \phi_Y \phi_Z X - (\phi_X \bar{g}(Y,Z) - \phi_Y \bar{g}(X,Z))\partial_t^T\}, \end{aligned}$$

where ∂_t^T denotes the tangential projection of ∂_t with respect to the decomposition (3.2).

Proof. Making use of (3.15) and our assumption, we obtain

$$\bar{g}(\bar{R}(X,Y)\xi, PZ) = \bar{g}((\nabla_Y h^l)(X, PZ), \xi) - \bar{g}((\nabla_X h^l)(Y, PZ), \xi) \tag{5.13}$$

for $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$ and $PZ \in \Gamma(S(TM))$. The first term in the right hand side of (5.13) is computed as follows:

$$\begin{aligned} &\bar{g}((\nabla_Y h^l)(X, PZ), \xi) \\ &= \{(Y\mathcal{H}_0^l) g(X, PZ) + \rho(Y)\mathcal{H}_0^l g(X, PZ)\} + (\mathcal{H}_0^l)^2 \eta(X) g(Y, PZ). \end{aligned} \tag{5.14}$$

On the other hand, it is clear from (2.3) that

$$\bar{g}(\bar{R}(X,Y)\xi, PZ) = \mu\{\phi_X \phi_\xi g(Y, PZ) - \phi_Y \phi_\xi g(X, PZ)\}. \tag{5.15}$$

Substituting (5.14) and (5.15) into (5.13), we get

$$
\begin{aligned}
(5.16)\qquad &\{Y\mathcal{H}^l_0 + \rho(Y)\mathcal{H}^l_0 - \eta(Y)(\mathcal{H}^l_0)^2 + \mu\phi_Y\phi_\xi\}g(X, PZ)\\
&-\{X\mathcal{H}^l_0 + \rho(X)\mathcal{H}^l_0 - \eta(X)(\mathcal{H}^l_0)^2 + \mu\phi_X\phi_\xi\}g(Y, PZ) = 0.
\end{aligned}
$$

Substituting $X = \xi$ and $Y = PZ$ into (5.16) gives the equation (1). In the similar way calculating $\bar{g}(\bar{R}(X, Y)W, PZ) = -\bar{g}(\bar{R}(X, Y)PZ, W)$ with (3.15) we have

(5.17)

$$
\begin{aligned}
&\{Y\mathcal{H}^s_\alpha - \sum_\beta \mathcal{H}^s_\beta\theta_{\beta\alpha}(Y) - \mathcal{H}^s_\alpha\mathcal{H}^l_0\eta(Y) + \mathcal{H}^l_0\tau_\alpha(Y) + \mu\phi_Y\phi_{W_\alpha}\}g(X, PZ)\\
&-\{X\mathcal{H}^s_\alpha - \sum_\beta \mathcal{H}^s_\beta\theta_{\beta\alpha}(X) - \mathcal{H}^s_\alpha\mathcal{H}^l_0\eta(X) + \mathcal{H}^l_0\tau_\alpha(X) + \mu\phi_X\phi_{W_\alpha}\}g(Y, PZ) = 0.
\end{aligned}
$$

Also putting $X = \xi$ and $Y = PZ$ in (5.17) yields (2). The equations (3) and (4) can be also obtained from substituting $X = PX$ and $Y = PY$ in (5.16) and (5.17), respectively. The last equation (5) follows from (2.3) and (3.15). Thus we complete the proof. □

From (5.7) we obtain:

Proposition 5.4. *Let $(M, g, S(TM), S(TM^\perp))$ be a totally umbilical lightlike submanifold of $L^{n+1}_1(c, f)$. If ∂_t belongs to the hyperbolic plane bundle $Rad(TM) \oplus ltr(TM)$, then $D^s(X, N) = 0$, or equivalently $\tau_\alpha = 0$.*

In case $\mathcal{H}^l_0 \neq 0$ and $\mathcal{H}^s_\alpha \neq 0$ on any local neighborhood $\mathcal{U}$ of M, we say that M is *proper totally umbilical*.

Theorem 5.5. *Let $(M, g, S(TM), S(TM^\perp))$ be a proper totally umbilical lightlike submanifold of $L^{n+1}_1(c, f)$. The followings are equivalent*:

(1) *$S(TM)$ is integrable.*
(2) *A_N is self-adjoint on $\Gamma(S(TM))$ with respect to g.*
(3) *$d\rho(X, Y) = \frac{\mu}{2}(\phi_Y\eta(X) - \phi_X\eta(Y))\phi_\xi$.*

Proof. The equivalence between (1) and (2) follows from (3.10) and (3.13) (cf. [5], [7]).

By direct calculation we obtain from (5.6)

$$
(5.18)\qquad 2d\rho(X, Y) = X(\rho(Y)) - Y(\rho(X)) - \rho([X, Y]) = \bar{g}(R^l(X, Y)N, \xi).
$$

Substituting (3.16) into (5.18), we have

$$
(5.19)\quad 2d\rho(X, Y) = \bar{g}(\bar{R}(X, Y)N, \xi) + \bar{g}(h^l(X, A_NY), \xi) - \bar{g}(h^l(Y, A_NX), \xi),
$$

where we have used the assumption that M is totally umbilical. Making use of (2.3) and (5.10), the equation (5.19) is reduced to

(5.20)

$$
2d\rho(X, Y) = \mu(\phi_Y\phi_\xi\eta(X) - \phi_X\phi_\xi\eta(Y)) + \mathcal{H}^l_0(g(PX, A_NY) - g(PY, A_NX)).
$$

The equivalence between (2) and (3) follows from (5.20). □

Theorem 5.6. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a totally umbilical lightlike submanifold of $L_1^{n+1}(c, f)$. Then the screen transversal connection ∇^s on M is flat, i.e.,*

$$d\theta_{\alpha\beta} = \frac{1}{2}\sum_{\gamma}\theta_{\alpha\gamma}\wedge\theta_{\gamma\beta}.$$

Proof. From (3.18) we have

$$\begin{aligned}&\bar{g}(R^s(X,Y)W_\alpha, W_\beta)\\&= \bar{g}(\bar{R}(X,Y)W_\alpha, W_\beta) + g(A_{W_\alpha}X, A_{W_\beta}Y) - g(A_{W_\beta}X, A_{W_\alpha}Y).\end{aligned}$$

Substituting (2.3) into this equation and using the third one in (5.12) we obtain

$$\bar{g}(R^s(X,Y)W_\alpha, W_\beta) = 0.$$

This means that $R^s = 0$.

Since $\theta_{\alpha\beta}(X) = \bar{g}(\nabla^s_X W_\alpha, W_\beta)$, we get

$$\begin{aligned}2d\theta_{\alpha\beta}(X,Y) &= X(\theta_{\alpha\beta}(Y)) - Y(\theta_{\alpha\beta}(X)) - \theta_{\alpha\beta}([X,Y])\\&= \bar{g}(R^s(X,Y)W_\alpha, W_\beta) + \bar{g}(\nabla^s_Y W_\alpha, \nabla^s_X W_\beta) - \bar{g}(\nabla^s_X W_\alpha, \nabla^s_Y W_\beta)\\&= \sum_{\gamma}(\theta_{\alpha\gamma}\wedge\theta_{\gamma\beta})(X,Y),\end{aligned}$$

where we have used $\nabla^s_X W_\alpha = \sum_\beta \theta_{\alpha\beta}(X)W_\beta$. Thus we complete the proof. □

Theorem 5.7. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a totally umbilical lightlike submanifold of $L_1^{n+1}(c, f)$. If the lightlike second fundamental form h^l is parallel, then we get*

$$\mathcal{H}^l_0 = 0. \tag{5.21}$$

If the screen second fundamental form h^s is parallel, then the equations hold:

$$\mathcal{H}^l_0\mathcal{H}^s_\alpha = 0, \quad X\mathcal{H}^s_\alpha = \sum_{\beta}\theta_{\alpha\beta}(X)\mathcal{H}^s_\beta. \tag{5.22}$$

Proof. The covariant derivatives of the lightlike second fundamental form h^l and the screen second fundamental form h^s are respectively defined as follows:

$$(\nabla_X h^l)(Y,Z) = \nabla^l_X(h^l(Y,Z)) - h^l(\nabla_X Y, Z) - h^l(Y, \nabla_X Z), \tag{5.23}$$

$$(\nabla_X h^s)(Y,Z) = \nabla^s_X(h^s(Y,Z)) - h^s(\nabla_X Y, Z) - h^s(Y, \nabla_X Z). \tag{5.24}$$

It is clear from our assumptions that (5.23) and (5.24) are reduced to (5.25) and (5.26), respectively.

$$\begin{aligned}&(X\mathcal{H}^l_0)g(Y,Z) + (\mathcal{H}^l_0)^2\{\eta(Z)g(X,Y) + \eta(Y)g(X,Z)\}\\&\quad + \mathcal{H}^l_0\rho(X)g(Y,Z) = 0,\end{aligned} \tag{5.25}$$

$$\begin{aligned}&(X\mathcal{H}^s_\alpha)g(Y,Z) + \mathcal{H}^l_0\mathcal{H}^s_\alpha\{\eta(Z)g(X,Y) + \eta(Y)g(X,Z)\}\\&\quad + \sum_{\beta}\mathcal{H}^s_\beta\theta_{\beta\alpha}(X)g(Y,Z) = 0.\end{aligned} \tag{5.26}$$

Putting $Z=\xi$ in (5.25), we get (5.21). (5.22) follows from putting $Y=PY=Z$ in (5.26). □

Corollary 5.8. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a totally umbilical, co-isotropic submanifold of $L_1^{n+1}(c,f)$. If the lightlike second fundamental form h^l is parallel, then M is totally geodesic.*

Corollary 5.9. *If $(M, g, S(TM), S(TM^{\perp}))$ is a totally umbilical submanifold of $L_1^{n+1}(c,f)$ and the lightlike second fundamental form h^l is parallel, then $L_1^{n+1}(c,f)$ is of constant curvature.*

Proof. It is clear from the equation (1) in Theorem 5.4 and (3) in Lemma 4.1. □

Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of $L_1^{n+1}(c,f)$. Then the screen distribution $S(TM)$ is said to be *totally umbilical* if on any coordinate neighborhood $\mathcal{U}\subset M$ there exists a smooth function Λ_0 such that

$$h_0^*(X,PY)=\Lambda_0 g(X,PY),\quad \forall X,Y\in\Gamma(TM),\tag{5.27}$$

where $h_0^*(X,PY)=\bar{g}(h^*(X,PY),N)$. In case $\Lambda_0=0$ (resp. $\Lambda_0\neq 0$) we say that $S(TM)$ is *totally geodesic* (resp. *proper totally umbilical*) ([6]).

In case $S(TM)$ is totally umbilical, it is clear from (3.8), (3.13) and (5.27) that

$$\bar{g}(A_N X,N)=0, A_N X=\Lambda_0 PX.\tag{5.28}$$

Theorem 5.10. *Let $(M, g, S(TM), S(TM^{\perp}))$ be an $m(>2)$-dimensional totally umbilical lightlike submanifold of $L_1^{n+1}(c,f)$. If the screen distribution $S(TM)$ is totally umbilical and ∂_t belongs to the hyperbolic plane bundle*

$$Rad(TM)\oplus ltr(TM),$$

then Λ_0 satisfies the partial differential equations:

(1) $X\Lambda_0-\rho(X)\Lambda_0-\eta(X)\Lambda_0\mathcal{H}_0^l-\lambda\eta(X)-\frac{1}{2}\mu\eta(X)=0,\quad\forall X\in\Gamma(TM)$,

(2) $\xi\Lambda_0-\rho(\xi)\Lambda_0-\Lambda_0\mathcal{H}_0^l-\lambda-\frac{1}{2}\mu=0$.

Proof. From (3.16), we obtain

$$\bar{g}(\bar{R}(X,Y)N,PZ)=\bar{g}((\nabla_Y A)(N,X)-(\nabla_X A)(N,Y),PZ),\tag{5.29}$$

with the aid of Proposition 5.4. Making use of (3.9), (3.12), (3.16) and (5.28), the equation (5.29) is reduced to

$$\begin{aligned}\bar{g}(\bar{R}(X,Y)N,PZ)=&(Y\Lambda_0)g(PX,PZ)-\rho(Y)\Lambda_0 g(PX,PZ)\\&+\eta(X)\Lambda_0\mathcal{H}_0^l g(Y,PZ)-(X\Lambda_0)g(PY,PZ)\\&-\rho(X)\Lambda_0 g(PY,PZ)+\eta(Y)\Lambda_0\mathcal{H}_0^l g(X,PZ).\end{aligned}$$

Substituting (2.3) into the left hand side in this equation gives

$$\{Y\Lambda_0-\rho(Y)\Lambda_0-\eta(Y)\Lambda_0\mathcal{H}_0^l-\lambda\eta(Y)-\frac{1}{2}\mu\eta(Y)\}PX$$

$$= \{X\Lambda_0 - \rho(X)\Lambda_0 - \eta(X)\Lambda_0\mathcal{H}_0^l - \lambda\eta(X) - \frac{1}{2}\mu\eta(X)\}PY.$$

where we have used (5.7) and Proposition 5.4. Since the rank of $S(TM) > 1$, this equation yields (1). (2) follows from putting $X = \xi$ in (1). □

The *type mumber* $t^*(p)$ of the screen distribution $S(TM)$ is defined by the rank of the shape operator A^*_ξ at the point $p \in M$.

Theorem 5.11. *Let $(M, g, S(TM), S(TM^\perp))$ be a screen totally umbilical 1-lightlike submanifold of a semi-Reimannian manifold $(\bar{M}, \bar{g})$. If the screen second fundamental form h^* is parallel and $t^*(p) \geq 1$ at any point $p \in M$, then $S(TM)$ is totally geodesic.*

Proof. The covariant derivative of the screen second fundamental form h^* is defined as follows:

$$(\nabla_X h^*)(Y, PZ) = \nabla^{*t}_X(h^*(Y, PZ)) - h^*(\nabla_X Y, PZ) - h^*(Y, \nabla^*_X PZ). \tag{5.30}$$

Substituting $h^*(Y, PZ) = \Lambda_0 g(Y, PZ)\xi$ and our assumption into (5.30) gives

$$0 = (X\Lambda_0)g(Y, PZ)\xi + \Lambda_0(\nabla_X g)(Y, PZ)\xi + \Lambda_0 g(Y, PZ)\nabla^{*t}_X\xi.$$

It follows from (3.6) and (3.14) that

$$\{(X\Lambda_0)\xi + \Lambda_0\nabla^{*t}_X\xi\}g(Y, PZ) + \Lambda_0\eta(Y)g(A^*_\xi X, PZ)\xi = 0.$$

Putting $Y = \xi$ in this equation gives $\Lambda_0 g(A^*_\xi X, PZ)\xi = 0$, which means that $\Lambda_0 A^*_\xi X = 0$. The assumption on the type number gives $\Lambda_0 = 0$. □

6. Null sectional curvatures and null Ricci curvatures

Let $(\bar{M}, \bar{g})$ be a semi-Riemannian manifold and $p \in \bar{M}$. Given a nonzero null vector $U \in T_p\bar{M}$ and a null plane H of $T_p\bar{M}$ containing U, the *null sectional curvature* at $p \in \bar{M}$ with respect to U in the plane H is defined by

$$\bar{K}_U(p, H) = \frac{\bar{g}(\bar{R}_p(X, U)U, X)}{\bar{g}(X, X)},$$

where X is any non-null vector in H ([2], [5], [6], [7]). In a similar way we define the null sectional curvature on a lightlike submanifold (M, g) of $(\bar{M}, \bar{g})$ as follows:

$$K_\xi(p, H) = \frac{g(R_p(X, \xi)\xi, X)}{g(X, X)},$$

where H is a null plane of T_pM containing a nonzero null vector ξ and X is any non-null vector in H.

Clearly the null sectional curvature of a null plane H is independent of the choice of non-null vectors in H, but depends quadratically on the null vectors. For a geometric interpretation of the null sectional curvature see [1].

Theorem 6.1. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of $L_1^{n+1}(c, f)$. If M is irrotational, then $L_1^{n+1}(c, f)$ is of constant curvature if and only if at a single point $p \in M$, either $K_\xi(p, H) = 0$ or $\bar{K}_\xi(p, H) = 0$ where $H \subset T_pM$ is a null plane which is spanned by any $\xi \in Rad(T_pM)$ and any non-null vector $X \in T_pM$.*

Proof. Let $\xi \in Rad(T_pM)$ and $X \in T_pM$ be a unit spacelike vector. Then we get from (2.3)

$$\bar{K}_\xi(p, H) = -\mu\phi_\xi^2.$$

Combining this with the Gauss equation (3.15) and (4.2) yields

$$\bar{K}_\xi(p, H) = K_\xi(p, H) = -\mu\phi_\xi^2. \tag{6.1}$$

From (6.1) with $\phi_\xi \neq 0$(Lemma 4.1(3)) we complete the proof. □

The Ricci tensor on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is defined as

$$\bar{R}ic(X, Y) = trace\{Z \to \bar{R}(X, Z)Y\}, \quad \forall X, Y \in \Gamma(T\bar{M}).$$

Making use of a quasi-orthonormal field of frames of $L_1^{n+1}(c, f)$ along M satisfying (5.1) and (5.2), the Ricci tensor $\bar{R}ic$ of $L_1^{n+1}(c, f)$ is given by

$$\begin{aligned} \bar{R}ic(X, Y) = & \sum_a g(\bar{R}(X, X_a)Y, X_a) + \bar{g}(\bar{R}(X, \xi)Y, N) \\ & + \sum_\alpha \bar{g}(\bar{R}(X, W_\alpha)Y, W_\alpha) + \bar{g}(\bar{R}(X, N)Y, \xi). \end{aligned} \tag{6.2}$$

The induced Ricci tensor on a lightlike submaniold M of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is also defined as

$$\breve{R}ic(X, Y) = trace\{Z \to R(X, Z)Y\}, \quad \forall X, Y \in \Gamma TM.$$

Using the Gauss equation (3.15), we get:

Proposition 6.2. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a 1-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the induced Ricci tensor $\breve{R}ic$ is given by*

$$\begin{aligned} \breve{R}ic(X, Y) = & \bar{R}ic(X, Y) - TrA_{h(X,Y)} + g(A_N X, A_\xi^* Y) \\ & + \sum_a \bar{g}(h^s(X_a, Y), h^s(X, X_a)) + \bar{g}(h^s(\xi, Y), D^s(X, N)) \\ & - \sum_\alpha \bar{g}(\bar{R}(X, W_\alpha)Y, W_\alpha) - \bar{g}(\bar{R}(X, N)Y, \xi), \end{aligned} \tag{6.3}$$

where

$$\begin{aligned} TrA_{h(X,Y)} = & \sum_a \{g(A_{h^s(X,Y)}X_a, X_a) + g(A_{h^l(X,Y)}X_a, X_a)\} \\ & + \bar{g}(A_{h^s(X,Y)}\xi, N) + \bar{g}(A_{h^l(X,Y)}\xi, N). \end{aligned}$$

Substituting (2.3) into (6.2) and making use of (3.7) ∼ (3.9) and (3.13), we obtain

Proposition 6.3. *Let* $(M, g, S(TM), S(TM^\perp))$ *be a lightlike submanifold of* $L_1^{n+1}(c, f)$*. Then the induced Ricci tensor* $\breve{R}ic$ *is given by*

(6.4)

$$\begin{aligned}
&\breve{R}ic(X, Y) \\
= &- \sum_a \{\bar{g}(h^s(X_a, X_a), h^s(X, Y)) + \bar{g}(h^*(X_a, X_a), h^l(X, Y))\} \\
&+ \sum_a \bar{g}(h^s(X_a, Y), h^s(X, X_a)) + g(A_N X, A_\xi^* Y) - \bar{g}(D^s(\xi, N), h^s(X, Y)) \\
&+ \mu\{(m-2)\phi_X \phi_Y + \eta(Y)\phi_X \phi_\xi\} \\
&+ \{\lambda(1-m) - \mu(1 + \sum_\alpha \phi_{W_\alpha}^2 + \phi_N \phi_\xi)\}\bar{g}(X, Y).
\end{aligned}$$

Theorem 6.4. *Let* $(M, g, S(TM), S(TM^\perp))$ *be a lightlike submanifold of* $L_1^{n+1}(c, f)$ *with* $m > 1$*. If* M *is irrotational, then*

$$\breve{R}ic(\xi, \xi) = 0, \ \forall \xi \in \Gamma(Rad(TM))$$

if and only if $L_1^{n+1}(c, f)$ *is of constant curvature.*

Proof. From (6.4) we get

$$\breve{R}ic(\xi, \xi) = (m-1)\mu\phi_\xi^2$$

with the aid of (3.12) and (4.2). The proof follows from this equation. □

Remark 6.5. In any two-dimensional Lorentzian manifold Ricci curvature always vanishes in any null direction ([2]).

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Department of Mathematics
University of Ulsan
Ulsan 680-749, Korea
E-mail address: thkang@mail.ulsan.ac.kr