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FURTHER EXPANSION AND SUMMATION FORMULAS INVOLVING THE HYPERHARMONIC FUNCTION

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Abstract. The aim of the paper is to present several new relationships involving the hyperharmonic function introduced by Mezö in (I. Mezö, Analytic extension of hyperharmonic numbers, Online J. Anal. Comb. 4, 2009) which is an analytic extension of the hyperharmonic numbers. These relations are obtained by using some fractional calculus theorems as Leibniz rules and Taylor like series expansions.

1. Introduction

In 1996, Conway and Guy [2] defined the notion of hyperharmonic numbers. The n-th hyperharmonic number of order r is defined recursively for r a positive integer greater that 1 by

$$H_n^{(r)} = \sum_{k=1}^{n} H_{(r-1)}, \tag{1.1}$$

where $H_n^{(1)} := H_n$ denotes the n-th harmonic number given by the n-th partial sum of the harmonic series:

$$H_n = \sum_{k=1}^{n} \frac{1}{k}. \tag{1.2}$$

These numbers can be expressed by the binomial coefficients as well as by the ordinary harmonic numbers as follows

$$H_n^{(r)} = \binom{n+r-1}{r-1} \left(H_{n+r-1} - H_{r-1}\right). \tag{1.3}$$

Very interesting combinatorial properties of these numbers have been investigated by many authors [1, 10, 11].

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Recently, Mezö [10] obtained an analytic extension of the hyperharmonic numbers. Using the fact that, for integers n,

$$H_{n-1} = \Psi(n) - \gamma, \tag{1.4}$$

where $\gamma = 0,577215\cdots$ is the Euler-Mascheroni constant and $\Psi(z)$ denotes the digamma function defined by the logarithmic derivative of the celebrated gamma function

$$\Psi(z) = \frac{d}{dz}\log(\Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{1.5}$$

we can rewrite (1.3) as

$$H_n^{(r)} = \frac{(n)_r}{n\Gamma(r)}\left(\Psi(n+r) - \Psi(r)\right), \tag{1.6}$$

where $(x)_n$ is the Pochhammer symbol

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\cdots(x+n-1). \tag{1.7}$$

Finally, since the gamma and digamma functions are analytic except for poles at $z \in \mathbb{Z}^- = \{0, -1, -2, \ldots\}$, Mezö extended the hyperharmonic numbers as follows:

Definition 1.1. The hyperharmonic function $H_\alpha^{(\beta)}$ is defined by

$$H_\alpha^{(\beta)} := \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)}\left(\Psi(\alpha+\beta) - \Psi(\beta)\right) \tag{1.8}$$

with $\alpha + \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

In this paper, we make use of fractional derivatives techniques to obtain further interesting new relationships involving the hyperharmonic function. In Section 2, we introduce the Pochhammer based representation for fractional derivatives and we express the hyperharmonic in terms of fractional derivative operator. Next, we recall some important theorems as the generalized Leibniz rules as well as their integral analogue and the Taylor like power series expansions. Finally, in Section 4, we provide ten new results involving the hyperharmonic functions.

2. Pochhammer contour integral representation for fractional derivative

The use of contour of integration in the complex plane provides a very powerful tool in both classical and fractional calculus. The most familiar representation for fractional derivative of order α of $z^p f(z)$ is the Riemann-Liouville integral [3, 9, 20] that is

$$\mathcal{D}_z^\alpha z^p f(z) = \frac{1}{\Gamma(-\alpha)}\int_0^z f(\xi)\xi^p(\xi - z)^{-\alpha-1}d\xi, \tag{2.1}$$

which is valid for $\text{Re}(\alpha) < 0$, $\text{Re}(p) > 1$ and where the integration is done along a straight line from 0 to z in the ξ-plane. By integrating by part m times, we obtain

$$\mathcal{D}_z^\alpha z^p f(z) = \frac{d^m}{dz^m}\mathcal{D}_z^{\alpha-m} z^p f(z). \tag{2.2}$$

This allows to modify the restriction $\text{Re}(\alpha) < 0$ to $\text{Re}(\alpha) < m$ [20]. Another used representation for the fractional derivative is the one based on the Cauchy integral formula widely used by Osler [13, 14, 15, 16]. These two representations have been used in many interesting research papers. It appears that the less restrictive representation of fractional derivative according to parameters is the Pochhammer's contour definition introduced in [22, 23].

Definition 2.1. Let $f(z)$ be analytic in a simply connected region $\mathcal{R}$. Let $g(z)$ be regular and univalent on $\mathcal{R}$ and let $g^{-1}(0)$ be an interior point of $\mathcal{R}$ then if α is not a negative integer, p is not an integer, and z is in $\mathcal{R} - \{g^{-1}(0)\}$, we define the fractional derivative of order α of $g(z)^p f(z)$ with respect to $g(z)$ by

$$\begin{aligned} &D_{g(z)}^\alpha g(z)^p f(z) \\ &\quad = \frac{e^{-i\pi p}\Gamma(1+\alpha)}{4\pi \sin(\pi p)} \int_{C(z+,g^{-1}(0)+,z-,g^{-1}(0)-;F(a),F(a))} \frac{f(\xi)g(\xi)^p g'(\xi)}{(g(\xi)-g(z))^{\alpha+1}} \mathrm{d}\xi. \end{aligned} \tag{2.3}$$

For non-integer α and p, the functions $g(\xi)^p$ and $(g(\xi) - g(z))^{-\alpha-1}$ in the integrand have two branch lines which begin respectively at $\xi = z$ and $\xi = g^{-1}(0)$, and both pass through the point $\xi = a$ without crossing the Pochhammer contour $P(a) = \{C_1 \cup C_2 \cup C_3 \cup C_4\}$ at any other point as shown in Figure 1. $F(a)$ denotes the principal value of the integrand in (2.3) at the beginning and ending point of the Pochhammer contour $P(a)$ which is closed on Riemann surface of the multiple-valued function $F(\xi)$.

Remark 2.2. In Definition 2.1, the function $f(z)$ must be analytic at $\xi = g^{-1}(0)$. However it is interesting to note here that we could also allow $f(z)$ to have an essential singularity at $\xi = g^{-1}(0)$, and the equation (2.3) would still be valid.

Remark 2.3. The Pochhammer contour never crosses the singularities at $\xi = g^{-1}(0)$ and $\xi = z$ in (2.3), then we know that the integral is analytic for all p and for all α and for z in $\mathcal{R} - \{g^{-1}(0)\}$. Indeed, the only possible singularities of $D_{g(z)}^\alpha g(z)^p f(z)$ are $\alpha = -1, -2, \ldots$, and $p = 0, \pm 1, \pm 2, \ldots$ which can directly be identified from the coefficient of the integral (2.3). However, integrating by parts N times the integral in (2.3) by two different ways, we can show that $\alpha = -1, -2, \ldots$, and $p = 0, 1, 2, \ldots$ are removable singularities (see [23]).

It is well known that [12, p. 83, Equation (2.4)]

$$D_z^\alpha z^p = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)} z^{p-\alpha} \qquad (\text{Re}(p) > -1), \tag{2.4}$$

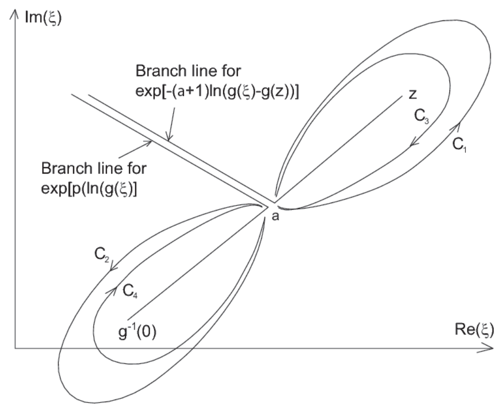

FIGURE 1. Pochhammer's contour

but adopting the Pochhammer based representation for the fractional derivative this last restriction becomes p not a negative integer.

Another well known formula for fractional derivatives [5, vol. 2, pp. 185–200] is given by

$$D_z^\alpha z^p \log z = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)} z^{p-\alpha} \left(\log z + \Psi(1+p) - \Psi(1+p-\alpha)\right) \tag{2.5}$$

which holds for $p \in \mathbb{C} \setminus \mathbb{Z}^-$ and $\alpha, \beta \in \mathbb{C}$.

Setting $p = \alpha+\beta-1$, dividing by $\Gamma(\alpha+1)$ and putting $z = 1$ after operation in the previous formula gives

$$\frac{D_z^\alpha z^{\alpha+\beta-1} \log z\big|_{z=1}}{\Gamma(\alpha+1)} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)} \left(\Psi(\alpha+\beta) - \Psi(\beta)\right), \tag{2.6}$$

where $\alpha+\beta \in \mathbb{C} \setminus \mathbb{Z}^-$ and $\alpha, \beta \in \mathbb{C}$.

We thus obtain the following very useful fractional derivative representation for the generalized hyperharmonic function $H_\alpha^{(\beta)}$

$$H_\alpha^{(\beta)} = \frac{D_z^\alpha z^{\alpha+\beta-1} \log z\big|_{z=1}}{\Gamma(\alpha+1)} \tag{2.7}$$

which holds for $\alpha+\beta \in \mathbb{C} \setminus \mathbb{Z}^-$ and $\alpha, \beta \in \mathbb{C}$.

3. Generalized Leibniz rules, Taylor-like expansions and a general expansion involving fractional derivatives

In this section, we recall many fundamental theorems related to fractional calculus that will play central roles in this work. These theorems consist in generalized Leibniz rules and their integral analogue and Taylor-like expansions in terms of different types of functions. First of all, let us begin by stating a theorem obtained in 1971 by Osler [17].

Theorem 3.1. *Let $u(z)$ and $v(z)$ be analytic in the simply connected region $\mathcal{R}$. Let 0 be an interior point of $\mathcal{R}$. Then, for $z \in \mathcal{R} - \{0\}$, $0 < a \leq 1$, $\alpha \in \mathbb{C}$, α not a negative integer and $\mu \in \mathbb{C}$, the following generalized Leibniz rule*

$$D_z^\alpha u(z)v(z) = a \sum_{n=-\infty}^{\infty} \binom{\alpha}{\mu + an} D_z^{\alpha-\mu-an} u(z) D_z^{\mu+an} v(z) \tag{3.1}$$

holds true.

Osler also proved that it is permitted to take the limit as $a \to 0$ in (3.1). By this way, he found the integral analogue of the generalized Leibniz rule.

Theorem 3.2. *Assume the hypothesis of Theorem 3.1. Then the following integral analogue of the generalized Leibniz rule holds true*

$$D_z^\alpha u(z)v(z) = \int_{-\infty}^{\infty} \binom{\alpha}{\mu + \omega} D_z^{\alpha-\mu-\omega} u(z) D_z^{\mu+\omega} v(z) d\omega. \tag{3.2}$$

Recently, Tremblay et al. obtained a new generalized Leibniz rule for fractional derivatives by making use of the properties of the Pochhammer based representation for fractional derivatives [25, 26]. Explicitly, they proved the following theorem:

Theorem 3.3. *Let $\mathcal{R}$ be a simply connected region containing the origin. Let $u(z)$ and $v(z)$ satisfy the conditions of Definition 2.1 for the existence of the fractional derivative. Let $\mathcal{U} \subset \mathcal{R}$ being the region of analyticity of the function $u(z)$ and $\mathcal{V} \subset \mathcal{R}$ being the one for the function $v(z)$. Then for $z \neq 0$, $z \in \mathcal{U} \cap \mathcal{V}$, $\operatorname{Re}(1-\beta) > 0$ the following product rule holds*

$$\begin{aligned} &D_z^\alpha z^{\alpha+\beta-1} u(z)v(z) \\ &\quad = \frac{z \sin(\beta\pi)\Gamma(1+\alpha)\sin(\mu\pi)\sin((\alpha+\beta-\mu)\pi)}{\sin((\alpha+\beta)\pi)\sin((\beta-\mu-\nu)\pi)\sin((\mu+\nu)\pi)} \\ &\quad\quad \cdot \sum_{n=-\infty}^{\infty} \frac{D_z^{\alpha+\nu+1-n} z^{\alpha+\beta-\mu-1-n} u(z) D_z^{-1-\nu+n} z^{\mu-1+n} v(z)}{\Gamma(2+\alpha+\nu-n)\Gamma(-\nu+n)}. \end{aligned} \tag{3.3}$$

They also proved the following integral analogue.

Theorem 3.4. *Assume the hypothesis of Theorem 3.3. Then the following integral analogue of (3.3) holds*

$$\begin{aligned} &D_z^\alpha z^{\alpha+\beta-1} u(z)v(z) \\ &\quad = \frac{z \sin(\beta\pi)\Gamma(1+\alpha)}{\sin((\alpha+\beta)\pi)\sin((\beta-\mu-\nu)\pi)\sin((\mu+\nu)\pi)} \\ &\quad\quad \cdot \int_{-\infty}^{\infty} \frac{\sin((\mu+\omega)\pi)\sin((\alpha+\beta-\mu-\omega)\pi)}{\Gamma(2+\alpha+\nu-\omega)\Gamma(-\nu+\omega)} \\ &\quad\quad \cdot D_z^{\alpha+\nu+1-\omega} z^{\alpha+\beta-\mu-1-\omega} u(z) D_z^{-\nu-1+\omega} z^{\mu-1+\omega} v(z) d\omega. \end{aligned} \tag{3.4}$$

Now let us shift our focus on some theorems involving Taylor like power series expansions. We begin by giving two results obtained by Osler [17, 18].

Theorem 3.5. *Let $f(z)$ be an analytic function in a simply connected region $\mathcal{R}$. Let α, γ be two arbitrary complex numbers and $\theta(z) = (z - z_0)q(z)$ with $q(z)$ a regular and univalent function without zero in $\mathcal{R}$. Let a be a positive real number and $K = 0, 1, \ldots, [c]$, $[c]$ being the largest integer not greater than c. Let b, z_0 be two points in $\mathcal{R}$ such that $b \neq z_0$ and let $\omega = \exp(2\pi i/a)$. Then the following relationship*

$$\sum_{k\in K} c^{-1}\omega^{-\gamma k} f(\theta^{-1}(\theta(z)\omega^k)) = \sum_{n=-\infty}^{\infty} \frac{D_{z-b}^{cn+\gamma}[f(z)\theta^{'}(z)[(z-z_0)/\theta(z)]^{cn+\gamma+1}]|_{z=z_0}\theta(z)^{cn+\gamma}}{\Gamma(cn+\gamma+1)} \tag{3.5}$$

holds true for $|z - z_0| = |z_0|$.

In particular, if $0 < c \leq 1$ and $\theta(z) = (z - z_0)$, then $K = 0$ and the formula (3.5) reduces to

$$f(z) = c\sum_{n=-\infty}^{\infty} \frac{D_{z-b}^{cn+\gamma} f(z)\big|_{z_0}(z-z_0)^{cn+\gamma}}{\Gamma(cn+\gamma+1)}. \tag{3.6}$$

This last formula is usually called the Taylor-Riemann formula and has been studied in several papers [6, 7, 15, 19, 27].

Theorem 3.6. *Assuming the hypotheses of Theorem 3.5. Then the following integral analogue of Taylor series for fractional derivatives*

$$f(z) = \int_{-\infty}^{\infty} \frac{D_{z-b}^{\omega+\gamma} f(z)\big|_{z_0}(z-z_0)^{\omega+\gamma}}{\Gamma(\omega+\gamma+1)} d\omega. \tag{3.7}$$

holds true.

Recently, Tremblay et al. [26] obtained the power series of an analytic function $f(z)$ in terms of the rational expression $(\frac{z-z_1}{z-z_2})$ where z_1 and z_2 are two arbitrary points inside the region of analyticity $\mathcal{R}$ of $f(z)$. In particular, they proved the next theorem.

Theorem 3.7. *Let c be real and positive and let $\omega = e^{2\pi i/c}$. Let $f(z)$ be analytic in the simply connected region $\mathcal{R}$ with z_1 and z_2 being interior point of $\mathcal{R}$. Let the set of curves $\{C(t)|\, 0 < t \leq r\}$, $C(t) \subset \mathcal{R}$, defined by*

$$C(t) = C_1(t) \cup C_2(t) = \left\{z \;\big|\; |\,\lambda_t(z_1, z_2; z)| = |\,\lambda_t(z_1, z_2; (z_1+z_2)/2)|\right\}, \tag{3.8}$$

where

$$\lambda_t(z_1, z_2; z) = [z - (z_1+z_2)/2 + t(z_1-z_2)/2]\,[z - (z_1+z_2)/2 - t(z_1-z_2)/2]\,, \tag{3.9}$$

which are lemniscates of Bernoulli type with center located at $(z_1 + z_2)/2$ and with double-loops; one loop $C_1(t)$ leads around the focus point $(z_1 + z_2)/2 +$

$t(z_1-z_2)/2$ and the other loop $C_2(t)$ encircles the focus point $(z_1+z_2)/2 - t(z_1-z_2)/2$, for each t such that $0 < t \leq r$. Let $((z-z_1)(z-z_2))^{\lambda} = \exp\{\,\lambda\,\ln(\theta((z-z_1)(z-z_2)))\}$ denote the principal branch of that function which is continuous and inside $C(r)$, cut by the respective two branch lines $L_{\pm}$ defined by

$$L_{\pm} = \begin{cases} \{z|\ z=(z_1+z_2)/2 \pm t(z_1-z_2)/2\}, & \text{for } 0\leq t\leq 1;\\ \{z|\ z=(z_1+z_2)/2 \pm it(z_1-z_2)/2\}, & \text{for } t<0. \end{cases} \tag{3.10}$$

such that $\ln((z-z_1)(z-z_2))$ is real where $((z-z_1)(z-z_2)) > 0$. Let $f(z)$ satisfies the conditions of Definition 2.1 for the existence of the fractional derivative of $(z-z_2)^p f(z)$ of order α for $z \in \mathcal{R} - \{L_+ \cup L_-\}$, noticed by $D^{\alpha}_{z-z_2}(z-z_2)^p f(z)$ where α and p are real or complex numbers. Let $K = \{k \mid k \in \mathbb{N}$ and $\arg(\lambda_t(z_1,z_2,(z_1+z_2)/2)) < \arg(\lambda_t(z_1,z_2,(z_1+z_2)/2)) + 2\pi k/a < \arg(\lambda_t(z_1,z_2,(z_1+z_2)/2)) + 2\pi\}$. Then for arbitrary complex numbers μ, ν, γ and for z on $C_1(1)$ defined by $\xi = \frac{z_1+z_2}{2} + \frac{z_1-z_2}{2}\sqrt{1+\mathrm{e}^{i\theta}}$, $-\pi < \theta < \pi$, we have

(3.11)

$$\sum_{k\in K} \frac{c^{-1}\,\omega^{-\gamma k}\, f\left(\phi^{-1}(\phi(z)\omega^k)\right)\left(\phi^{-1}(\phi(z)\omega^k)-z_1\right)^{\nu}\left(\phi^{-1}(\phi(z)\omega^k)-z_2\right)^{\mu}}{(z_1-z_2)}$$
$$= \sum_{n=-\infty}^{\infty} \frac{e^{i\pi c(n+1)}\sin((\mu+cn+\gamma)\pi)\, D^{-\nu+cn+\gamma}_{z-z_2}\,(z-z_2)^{\,\mu+cn+\gamma-1}\,f(z)\big|_{z=z_1}}{\sin((\mu-c+\gamma)\pi)\Gamma(1-\nu+cn+\gamma)}\,\phi(z)^{cn+\gamma},$$

where $\phi(z) = \left(\frac{z-z_1}{z-z_2}\right)$.

The case $0 < c \leq 1$ reduces to

$$\begin{aligned} &\frac{c^{-1}\,f(z)(z-z_1)^{\nu}(z-z_2)^{\mu}}{(z_1-z_2)} \\ &\quad= \sum_{n=-\infty}^{\infty} \frac{\mathrm{e}^{i\pi c(n+1)}\sin((\mu+cn+\gamma)\pi)}{\sin((\mu-c+\gamma)\pi)\Gamma(1-\nu+cn+\gamma)} \\ &\qquad\times\ D^{-\nu+cn+\gamma}_{z-z_2}\,(z-z_2)^{\,\mu+cn+\gamma-1}\,f(z)\big|_{z=z_1}\left(\frac{z-z_1}{z-z_2}\right)^{cn+\gamma}. \end{aligned} \tag{3.12}$$

Finally, in 2007, Tremblay and Fugère [24] obtained the power series of an analytic function $f(z)$ in terms of arbitrary function $(z-z_1)(z-z_2)$ where z_1 and z_2 are two arbitrary points inside the analyticity region $\mathcal{R}$ of $f(z)$. Explicitly, they found the following relationship.

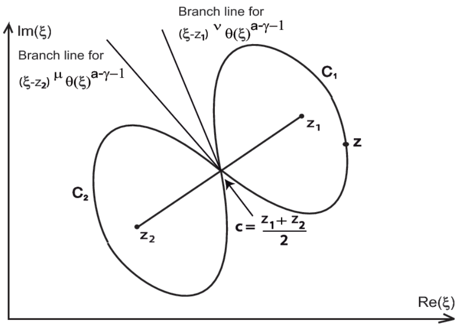

FIGURE 2. Multi-loops contour

Theorem 3.8. *Assuming the assumptions of Theorem 3.7. Then the following expansion*

(3.13)

$$\sum_{k\in K} c^{-1}\omega^{-\gamma k}\Big[f\left(\frac{z_1+z_2+\sqrt{\Delta_k}}{2}\right)\left(\frac{z_2-z_1+\sqrt{\Delta_k}}{2}\right)^{\alpha}\left(\frac{z_1-z_2+\sqrt{\Delta_k}}{2}\right)^{\beta}$$
$$-\,\mathrm{e}^{i\pi(\alpha-\beta)}\frac{\sin((\alpha+c-\gamma)\pi)}{\sin((\beta+c-\gamma)\pi)}f\left(\frac{z_1+z_2-\sqrt{\Delta_k}}{2}\right)\left(\frac{z_2-z_1-\sqrt{\Delta_k}}{2}\right)^{\alpha}\left(\frac{z_1-z_2-\sqrt{\Delta_k}}{2}\right)^{\beta}\Big]$$
$$=\sum_{n=-\infty}^{\infty}\frac{\sin((\beta-cn-\gamma)\pi)}{\sin((\beta-c-\gamma)\pi)}e^{-i\pi c(n+1)}\theta(z)^{cn+\gamma}$$
$$\times\frac{D_{z-z_2}^{-\alpha+cn+\gamma}\left[(z-z_2)^{\beta-cn-\gamma-1}\left(\frac{\theta(z)}{(z-z_2)(z-z_1)}\right)^{-cn-\gamma-1}\theta'(z)f(z)\right]\Big|_{z=z_1}}{\Gamma(1-\alpha+cn+\gamma)},$$

where

$$\Delta_k=(z_1-z_2)^2+4V(\theta(z)\omega^k), \tag{3.14}$$

$$V(z)=\sum_{r=1}^{\infty}D_z^{r-1}\Big(q(z)^{-r}\Big)\Big|_{z=0}z^r/r! \tag{3.15}$$

and

$$\theta(z)=(z-z_1)(z-z_2)q\big((z-z_1)(z-z_2)\big) \tag{3.16}$$

holds true.

As special case, if we set $0<c\leq 1$, $q(z)=1$ $(\theta(z)=(z-z_1)(z-z_2))$ and $z_2=0$ in (3.13), we obtain

$$f(z)=c\,z^{-\beta}(z-z_1)^{-\alpha}\sum_{n=-\infty}^{\infty}\frac{\sin((\beta-cn-\gamma)\pi)}{\sin((\beta+c-\gamma)\pi)}\mathrm{e}^{I\pi c(n+1)}\left[z(z-z_1)\right]^{cn+\gamma}$$

$$\times \frac{D_z^{-\alpha+cn+\gamma}}{\Gamma(1-\alpha+cn+\gamma)} z^{\beta-cn-\gamma-1}(z+w-z_1)f(z)\Bigg|_{\substack{z=z_1\\ w=z}}. \tag{3.17}$$

4. Main results

This section is devoted to the presentation of 10 theorems. By suitably applying Theorems 3.1 to 3.8 as well as some famous results due to Dougall and Ramanujan, we obtain very interesting formulas involving the hyperharmonic function $H_\alpha^{(\beta)}$.

Theorem 4.1. *Let α, β and μ be complex numbers such that $\alpha+\beta$ is not a negative integer and let $0 < a \leq 1$. Then the following relationship holds true for the hyperharmonic function*

$$H_\alpha^{(\beta)} = -\Gamma(\alpha+\beta) \sum_{n=-\infty}^{\infty} \frac{\gamma + \Psi(1-\mu-an)}{\Gamma(1-\alpha-\mu-an)\Gamma(1+\mu+an)\Gamma(\beta+\mu+an)\Gamma(1-\mu-an)}. \tag{4.1}$$

Proof. Let $u(z) = z^{\alpha+\beta-1}$ and $v(z) = \log z$ in Theorem 3.1, we thus obtain

$$D_z^\alpha z^{\alpha+\beta-1} \log z = a \sum_{n=-\infty}^{\infty} \binom{\alpha}{\mu+an} D_z^{\alpha-\mu-an} z^{\alpha+\beta-1} D_z^{\mu+an} \log z. \tag{4.2}$$

Dividing both sides of (4.2) by $\Gamma(\alpha+1)$, using equations (2.4) and (2.5) for the computation of fractional derivatives, setting $z=1$ and with the help of (2.7) yields the desired result. □

Let us recall the celebrated Dougall's summation formula [4, vol. 1, p. 7]

$$\frac{\pi^2\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)\sin(a\pi)\sin(b\pi)} = \sum_{n=-\infty}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(d+n)} \tag{4.3}$$

which is valid for $\mathrm{Re}(c+d-a-b) > 1$.

By making use of this summation formula, we obtain the next formula involving the hyperharmonic function $H_\alpha^{(\beta)}$.

Theorem 4.2. *Let α, β and μ be complex numbers such that $\alpha+\beta$ is not a negative integer. Then for $\mathrm{Re}(\beta-\alpha) > 0$, we have*

$$\sum_{n=-\infty}^{\infty} \frac{\Psi(1-\mu-n)}{\Gamma(1-\alpha-\mu-n)\Gamma(1+\mu+n)\Gamma(\beta+\mu+n)\Gamma(1-\mu-n)} = \frac{-H_\alpha^{(\beta)}}{\Gamma(\alpha+\beta)} - \frac{\gamma}{\Gamma(1-\alpha)\Gamma(\beta)}. \tag{4.4}$$

Proof. Rewrite (4.1) as

$$\frac{-H_{\alpha}^{(\beta)}}{\Gamma(\alpha+\beta)} = \sum_{n=-\infty}^{\infty} \frac{\Psi(1-\mu-an)}{\Gamma(1-\alpha-\mu-an)\Gamma(1+\mu+an)\Gamma(\beta+\mu+an)\Gamma(1-\mu-an)} + \gamma \sum_{n=-\infty}^{\infty} \frac{1}{\Gamma(1-\alpha-\mu-an)\Gamma(1+\mu+an)\Gamma(\beta+\mu+an)\Gamma(1-\mu-an)}. \tag{4.5}$$

Setting $a = 1$ and using the following well known property of the gamma function [21, p. 240, Eq. (I.30)]

$$\Gamma(a-n) = \frac{(-1)^n \Gamma(a)\Gamma(1-a)}{\Gamma(1-a+n)} \tag{4.6}$$

the second summation term in the right member of (4.5) becomes

$$\sum_{n=-\infty}^{\infty} \frac{1}{\Gamma(1-\alpha-\mu-an)\Gamma(1+\mu+an)\Gamma(\beta+\mu+an)\Gamma(1-\mu-an)} = \frac{1}{\Gamma(1-\mu)\Gamma(\mu)\Gamma(1-\alpha-\mu)\Gamma(\alpha+\mu)} \sum_{n=-\infty}^{\infty} \frac{\Gamma(\mu+n)\Gamma(\alpha+\mu+n)}{\Gamma(1+\mu+n)\Gamma(\beta+\mu+n)}. \tag{4.7}$$

With the help of summation formula (4.3) the result follows. □

As seen in Section 3, the generalized Leibniz rule (3.1) possesses an integral analogue (3.2) which provides the following interesting result.

Theorem 4.3. *Let α, β and μ be complex numbers such that $\alpha+\beta$ is not a negative integer. Then the following relationship holds true for the hyperharmonic function*

$$H_{\alpha}^{(\beta)} = -\Gamma(\alpha+\beta) \int_{-\infty}^{\infty} \frac{(\gamma + \Psi(1-\mu-\omega))\, d\omega}{\Gamma(1-\alpha-\mu-\omega)\Gamma(1+\mu+\omega)\Gamma(\beta+\mu+\omega)\Gamma(1-\mu-\omega)}. \tag{4.8}$$

Proof. The proof is essentially the same as the one of Theorem 4.1. The difference is that instead of using Theorem 3.1, we use Theorem 3.2. □

Now let us recall a formula due to Ramanujan [5, vol. 2, p. 300]

$$\int_{-\infty}^{\infty} \frac{d\omega}{\Gamma(a+\omega)\Gamma(b+\omega)\Gamma(c-\omega)\Gamma(d-\omega)} = \frac{\Gamma(a+b+c+d-3)}{\Gamma(a+c-1)\Gamma(a+d-1)\Gamma(b+c-1)\Gamma(b+d-1)}. \tag{4.9}$$

which is valid for $\mathrm{Re}(a+b+c+d) > 3$.

With the help of this integral formula, we obtain the following integral formula involving the hyperharmonic function $H_{\alpha}^{(\beta)}$.

Theorem 4.4. *Let α, β and μ be complex numbers such that $\alpha+\beta$ is not a negative integer. Then for* $\mathrm{Re}(\beta-\alpha)>0$*, we have*

$$\int_{-\infty}^{\infty} \frac{\Psi(1-\mu-\omega)\,d\omega}{\Gamma(1-\alpha-\mu-\omega)\Gamma(1+\mu+\omega)\Gamma(\beta+\mu+\omega)\Gamma(1-\mu-\omega)} = \frac{-H_{\alpha}^{(\beta)}}{\Gamma(\alpha+\beta)} - \frac{\gamma}{\Gamma(1-\alpha)\Gamma(\beta)}. \tag{4.10}$$

which holds for $\mathrm{Re}(\beta-\alpha)>0$.

Proof. Rewriting (4.8) in the following form

$$\begin{aligned}\frac{-H_{\alpha}^{(\beta)}}{\Gamma(\alpha+\beta)} &= \int_{-\infty}^{\infty} \frac{\Psi(1-\mu-\omega)\,d\omega}{\Gamma(1-\alpha-\mu-\omega)\Gamma(1+\mu+\omega)\Gamma(\beta+\mu+\omega)\Gamma(1-\mu-\omega)} \\ &\quad + \gamma \int_{-\infty}^{\infty} \frac{d\omega}{\Gamma(1-\alpha-\mu-\omega)\Gamma(1+\mu+\omega)\Gamma(\beta+\mu+\omega)\Gamma(1-\mu-\omega)}\end{aligned} \tag{4.11}$$

and using Ramanujan identity (4.9) on the second term of the right member of (4.11) gives the result. □

From the new generalized Leibniz rule (3.3) as well as for its integral analogue (3.4), we can obtain two interesting formulas involving the hyperharmonic function $H_{\alpha}^{(\beta)}$. These two new formulas are given in the next two theorems.

Theorem 4.5. *Let α,β,μ and ν be complex numbers. Then for* $\mathrm{Re}(1-\beta)>0$*, we have the following summation formula*

$$H_{\alpha}^{(\beta)} = \frac{\Gamma(-\mu-\nu)\sin(\beta\pi)\sin((\alpha+\beta+\mu)\pi)}{\Gamma(\alpha+1)\sin((\alpha+\beta)\pi)\ \sin((\beta-\mu-\nu)\pi)} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} H_{\alpha+\nu+1-n}^{(\beta-\mu-\nu-1)}}{\Gamma(1-\mu-n)\Gamma(-\nu+n)}. \tag{4.12}$$

Proof. Let $u(z)=\log z$, $v(z)=1$ in equation (3.3) of Theorem 3.3 and divide both side by $\Gamma(\alpha+1)$. We thus have

$$\begin{aligned}\frac{D_z^{\alpha} z^{\alpha+\beta-1}\log z}{\Gamma(\alpha+1)} &= \frac{z\sin(\beta\pi)\sin(\mu\pi)\sin((\alpha+\beta-\mu)\pi)}{\sin((\alpha+\beta)\pi)\sin((\beta-\mu-\nu)\pi)\sin((\mu+\nu)\pi)} \\ &\quad \cdot \sum_{n=-\infty}^{\infty} \frac{D_z^{\alpha+\nu+1-n} z^{\alpha+\beta-\mu-1-n}\log z\ D_z^{-1-\nu+n} z^{\mu-1+n}}{\Gamma(2+\alpha+\nu-n)\Gamma(-\nu+n)}\end{aligned} \tag{4.13}$$

which holds for $\mathrm{Re}(1-\beta)>0$. By making use of (2.7), appealing to (2.4) for the calculation of the fractional derivative involved in the second term of the right member, and setting $z = 1$ after operation gives

(4.14)

$$H_{\alpha}^{(\beta)} = \frac{\sin(\beta\pi)\sin((\alpha+\beta-\mu)\pi)}{\sin((\alpha+\beta)\pi)\sin((\beta-\mu-\nu)\pi)\sin((\mu+\nu)\pi)} \cdot \sum_{n=-\infty}^{\infty} \frac{D_z^{\alpha+\nu+1-n} z^{\alpha+\beta-\mu-1-n}\log z\big|_{z=1} (-1)^n \sin((\mu+n)\pi)\Gamma(\mu+n)}{\Gamma(1+\nu+\mu)\Gamma(2+\alpha+\nu-n)\Gamma(-\nu+n)}.$$

Finally, observing that

$$D_z^{\alpha+\nu+1-n} z^{\alpha+\beta-\mu-1-n}\log z\Big|_{z=1} = \Gamma(\alpha+2+\nu-n) H_{\alpha+\nu+1-n}^{(\beta-\mu-\nu-1)}, \tag{4.15}$$

we obtain the desired result after simple calculations. □

Theorem 4.6. *Let α, β, μ and ν be complex numbers. Then for* $\mathrm{Re}(1-\beta) > 0$, *we have the following integral formula*

$$H_{\alpha}^{(\beta)} = -\frac{\Gamma(-\mu-\nu)\sin(\beta\pi)}{\Gamma(\alpha+1)\sin((\alpha+\beta)\pi)\sin((\beta-\mu-\nu)\pi)} \times \int_{-\infty}^{\infty} \frac{\sin((\alpha+\beta+\mu-\omega)\pi) H_{\alpha+\nu+1-\omega}^{(\beta-\mu-\nu-1)}}{\Gamma(1-\mu-\omega)\Gamma(-\nu+\omega)} d\omega. \tag{4.16}$$

Proof. The proof is essentially the same as the one of Theorem 4.5. The difference is that instead of using Theorem 3.3, we use Theorem 3.4. □

Let us shift our focus on some applications of Taylor-like expansions in terms of different types of functions stated in Section 3 as Theorems 3.5 to 3.8. These expansions provide interesting summation formulas involving the hyperharmonic function H_{α}^{β}.

Theorem 4.7. *Let α, β and λ be three arbitrary complex numbers such that $\alpha+\beta \in \mathbb{C} \setminus \mathbb{Z}^-$ and let $z \neq 0$. Then the following relation*

$$z^{\alpha+\beta-1}\log z = \sum_{n=-\infty}^{\infty} H_{n+\lambda}^{(\alpha+\beta-\lambda-n)} (z-1)^{n+\lambda} \tag{4.17}$$

holds true for $|z-1| = 1$.

Proof. Let $f(z) = z^{\alpha+\beta-1}\log z$, replace γ by λ and set $z_0 = 1$, $b = 0$, and $c = 1$ in (3.6). We thus have

$$z^{\alpha+\beta-1}\log z = \sum_{n=-\infty}^{\infty} \frac{D_z^{n+\lambda} z^{\alpha+\beta-1}\log z\big|_{z=1}}{\Gamma(n+\lambda+1)} (z-1)^{1+n+\lambda}. \tag{4.18}$$

Using (2.7), we see that

$$\frac{D_z^{n+\lambda} z^{\alpha+\beta-1}\log z\big|_{z=1}}{\Gamma(n+\lambda+1)} = H_{n+\lambda}^{(\alpha+\beta-\lambda-n)} \tag{4.19}$$

and the result follows easily. □

Theorem 4.8. *Let α, β and λ be three arbitrary complex numbers such that $\alpha+\beta \in \mathbb{C}\setminus\mathbb{Z}^-$ and let $z \neq 0$. Then the following integral analogue of* (4.17)

$$z^{\alpha+\beta-1}\log z = \int_{-\infty}^{\infty} H_{\omega+\lambda}^{(\alpha+\beta-\lambda-\omega)}(z-1)^{\omega+\lambda}d\omega \tag{4.20}$$

holds true for $|z-1| = 1$.

Proof. The proof is the same as that of Theorem 4.7 with the exception of the use of Theorem 3.6 instead of Theorem 3.5. □

Theorem 4.9. *Assuming the hypotheses of Theorem 3.7. Then the following expansion*

$$z^{\mu}(z-1)^{\nu}\log z = \sum_{n=-\infty}^{\infty} H_{-\nu+n+\lambda}^{(\mu+\nu)}\left(\frac{z-1}{z}\right)^{n+\lambda} \tag{4.21}$$

holds true for μ, ν, λ arbitrary complex numbers such that $\mu+\lambda+n \in \mathbb{C}\setminus\mathbb{Z}^-$ and for z on the curve defined by $\xi = \frac{1}{2}+\frac{1}{2}\sqrt{1+\mathrm{e}^{i\theta}}$, $-\pi < \theta < \pi$.

Proof. Letting $f(z) = \log z$, $c = 1$, $z_1 = 1$, $z_2 = 0$ and replacing γ by λ in Theorem 3.7 yields

$$z^{\mu}(z-1)^{\nu}\log z = \sum_{n=-\infty}^{\infty} \frac{D_z^{-\nu+n+\gamma}\ z^{\,\mu+n+\lambda-1}\ \log z\big|_{z=1}}{\Gamma(1-\nu+n+\lambda)}\left(\frac{z-1}{z}\right)^{n+\lambda}. \tag{4.22}$$

With the help of (2.7), we easily see that

$$\frac{D_z^{-\nu+n+\gamma}\ z^{\,\mu+n+\lambda-1}\ \log z\big|_{z=1}}{\Gamma(1-\nu+n+\lambda)} = H_{-\nu+n+\lambda}^{(\mu+\nu)}. \tag{4.23}$$

Combining (4.22) and (4.23) gives the result. □

The last formula related to Taylor like expansion is based on that obtained by Tremblay and Fugère [24].

Theorem 4.10. *Assuming the hypotheses of Theorem 3.8. Then the following expansion*

$$\log z = \sum_{n=-\infty}^{\infty}\left[H_{-\alpha+n}^{(1+\alpha+\beta-2n)} + (z-1)H_{-\alpha+n}^{(\alpha+\beta-2n)}\right] z^{n-\beta}(z-1)^{n-\alpha} \tag{4.24}$$

holds true for α and β two arbitrary complex numbers such that $\alpha+\beta+n \in \mathbb{C}\setminus\mathbb{Z}^-$, $1+\alpha+\beta+n \in \mathbb{C}\setminus\mathbb{Z}^-$ and for z on the curve defined by $\xi = \frac{1}{2}+\frac{1}{2}\sqrt{1+\mathrm{e}^{i\theta}}$, $-\pi < \theta < \pi$.

Proof. Setting $f(z) = \log z$, $c = 1$, $z_1 = 1$, $z_2 = 0$, $q(z) = 1$ and $\gamma = 0$ in Theorem 3.8 gives

$$\log z = \sum_{n=-\infty}^{\infty} z^{n-\beta}(z-1)^{n-\alpha}\frac{D_z^{-\alpha+n} z^{\beta-n-1}(z+w-1)\log z}{\Gamma(1-\alpha+n)}\Bigg|_{\substack{z=1\\ w=z}} \tag{4.25}$$

which can be written in the following form

$$\log z = \sum_{n=-\infty}^{\infty} z^{n-\beta}(z-1)^{n-\alpha}\left[\frac{D_z^{-\alpha+n} z^{\beta-n}\log z}{\Gamma(1-\alpha+n)}\bigg|_{z=1} + (z-1)\frac{D_z^{-\alpha+n} z^{\beta-n-1}\log z}{\Gamma(1-\alpha+n)}\bigg|_{z=1}\right]. \tag{4.26}$$

Using (2.7), we observe that

$$\frac{D_z^{-\alpha+n} z^{\beta-n}\log z}{\Gamma(1-\alpha+n)}\bigg|_{z=1} = H_{-\alpha+n}^{(1+\alpha+\beta-2n)}, \tag{4.27}$$

$$\frac{D_z^{-\alpha+n} z^{\beta-n-1}\log z}{\Gamma(1-\alpha+n)}\bigg|_{z=1} = H_{-\alpha+n}^{(\alpha+\beta-2n)} \tag{4.28}$$

and thus the result follows. □

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