Commun. Korean Math. Soc. **29** (2014), No. 2, pp. 263–267
http://dx.doi.org/10.4134/CKMS.2014.29.2.263

ON THE GROWTH RATE OF SOLUTIONS TO GROSS-NEVEU AND THIRRING EQUATIONS

Hyungjin Huh

Abstract. We study the growth rate of H^1 Sobolev norm of the solutions to Gross-Neveu and Thirring equations. A well-known result is the double exponential rate. We show that the H^1 Sobolev norm grows at most an exponential rate $\exp(c\,t^2)$.

1. Introduction

We study the following Cauchy problem of the nonlinear Dirac equations

$$\begin{aligned} i(\partial_t u + \partial_x u) + mv &= \partial_{\bar u} W(u,\, v), \\ i(\partial_t v - \partial_x v) + mu &= \partial_{\bar v} W(u,\, v), \\ u(x,0) = u_0(x), &\quad v(x,0) = v_0(x), \end{aligned} \tag{1.1}$$

where $u,\, v : \mathbb{R}^{1+1} \to \mathbb{C}$ and m (≥ 0) is a mass. The potential W takes the form

$$W = a_1|u|^2|v|^2 + a_2(\bar u v + \bar v u)^2,$$

where a_1, a_2 are real constants and $\bar u$ is a complex conjugate of u.

The system (1.1) with $W = 4|u|^2|v|^2$ is called Thirring model and an initial value problem of it has been studied by several authors [3, 5, 8, 12]. It is well known in [5] that the problem is globally well posed in Sobolev space $H^1(\mathbb{R})$. Low regularity well-posedness was discussed in [3, 8, 12] showing that there exists a time $T > 0$ and solution $u,\, v \in C([0,T],\, H^s(\mathbb{R}))$ $(s \geq 0)$. Especially global existence from L^2 initial data has recently been proved in [3].

The system (1.1) with $W = \frac{1}{4}(\bar u v + \bar v u)^2$ is called Gross-Neveu model [7] and an initial value problem of it has been studied in [9, 13]. The global existence of the solution in H^1 was proved in [9] where an L^∞ bound of the solution is obtained by applying local form of charge conservation.

The spectral stability of solitary wave solutions to nonlinear Dirac equations has been studied in [1, 2, 4]. They show that the solitary waves are spectrally stable through analysis of the spectrum of linearization at a solitary wave. The orbital stability of solitary wave of Thirring model is proved recently in [11].

Received December 11, 2013.
2010 *Mathematics Subject Classification.* 35L45, 35Q41, 35F25, 81Q05.
Key words and phrases. Gross-Neveu, Thirring, Sobolev norm, L^∞ bound.

Here we are interested in the growth rate of the H^1 norm. Note that the system (1.1) has the charge conservation

$$\int_{\mathbb{R}} |u(x,\, t)|^2 + |v(x,\, t)|^2 dx = \int_{\mathbb{R}} |u_0(x)|^2 + |v_0(x)|^2 dx. \tag{1.2}$$

A well-known result in [5, 9] is that the solution of (1.1) grows at most in a double exponential, i.e.,

$$\|u(t)\|_{H^1(\mathbb{R})} \le c_1 \exp(\exp(c_2 t)), \tag{1.3}$$

where c_1 and c_2 are constants. For the massless ($m = 0$) Thirring model in [8], we can derive an uniform bound $\|u(t)\|_{H^1} \le c\|u(0)\|_{H^1}$ by using explicit solution representation. A global bound on the H^1 norm of the small L^2 solutions to the massive ($m > 0$) Thirring equation is obtained in [11] by deriving a new conserved quantity. The following is our main result.

Theorem 1.1. *Consider the initial value problem* (1.1) *with* $u_0,\, v_0 \in H^1(\mathbb{R})$. *Then we have the following upper bound*

$$\|u(t)\|_{H^1(\mathbb{R})} + \|v(t)\|_{H^1(\mathbb{R})} \le c_1 \exp(c_2 t^2).$$

To prove Theorem 1.1, we will estimate $\|u(t)\|_{L^\infty}$. For the massive Thirring [5, 6, 10] and Gross-Neveu [9] equations, L^∞ norm of solutions was controlled by L^∞ norm of initial data in the following way

$$\|u(t)\|_{L^\infty} \le c_1 e^{c_2 t}.$$

Here we improve the above bound by $\|u(t)\|_{L^\infty} \le c_1 + c_2 t^{\frac{1}{2}}$. Then Theorem 1.1 is proved in Section 2.

2. Proof of Theorem 1.1

To begin with, let us recall basic known facts. Global existence of the solution to (1.1) in Sobolev space $H^1(\mathbb{R})$ was proved in [5, 9].

Theorem 2.1. *For initial data* $u_0,\, v_0 \in H^1(\mathbb{R})$, *there exists a global solution* $(u,\, v)$ *of* (1.1) *satisfying*

$$u,\, v \in C([0, \infty),\, H^1(\mathbb{R})),$$

where $u,\, v$ *depend continuously on the initial data.*

For a simple presentation of proof of Theorem 1.1, we only consider the massive Gross-Neveu equation. For the case of the massive Thirring model, the proof is similar and easier.

For the potential $W = \frac{1}{4}(\bar{u}v + \bar{v}u)^2$, the equation (1.1) takes the form

$$\begin{aligned} i(\partial_t u + \partial_x u) + mv &= \mathrm{Re}(\bar{u}v)v, \\ i(\partial_t v - \partial_x v) + mu &= \mathrm{Re}(\bar{u}v)u. \end{aligned} \tag{2.1}$$

To estimate $\|\partial_x u(t)\|_{L^2}$ and $\|\partial_x v(t)\|_{L^2}$, we take derivative ∂_x on (2.1) and obtain

$$\partial_t|u_x|^2 + \partial_x|u_x|^2 + 2m\mathrm{Im}(v_x\bar{u}_x) = 2\mathrm{Re}(\bar{u}v)\mathrm{Im}(v_x\bar{u}_x) + 2\partial_x(\mathrm{Re}(\bar{u}v))\mathrm{Im}(v\bar{u}_x),$$
$$\partial_t|v_x|^2 - \partial_x|v_x|^2 + 2m\mathrm{Im}(u_x\bar{v}_x) = 2\mathrm{Re}(\bar{u}v)\mathrm{Im}(u_x\bar{v}_x) + 2\partial_x(\mathrm{Re}(\bar{u}v))\mathrm{Im}(u\bar{v}_x),$$

which leads to

$$\begin{aligned}&\partial_t(|u_x|^2+|v_x|^2) + \partial_x(|u_x|^2-|v_x|^2)\\ &= 2\partial_x(\mathrm{Re}(\bar{u}v))\,\mathrm{Im}(v\bar{u}_x) + 2\partial_x(\mathrm{Re}(\bar{u}v))\,\mathrm{Im}(u\bar{v}_x)\\ &\le 2\left(|v|^2|u_x|^2 + 2|u||v||u_x||v_x| + |u|^2|v_x|^2\right).\end{aligned}$$

Integrating on $\mathbb{R}$, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}(\,|u_x|^2+|v_x|^2\,)(x,t)\,dx &\le 4\int_{\mathbb{R}}\left(|v|^2|u_x|^2+|u|^2|v_x|^2\right)(x,t)\,dx\\ &\le 4(\|u\|^2_{L^\infty}+\|v\|^2_{L^\infty})\int_{\mathbb{R}}|u_x|^2+|v_x|^2\,dx.\end{aligned}$$

Then Gronwall's inequality gives a bound

$$\begin{aligned}&\|\partial_x u(t)\|^2_{L^2} + \|\partial_x v(t)\|^2_{L^2}\\ &\le \exp\left(4\int_0^t \|u(s)\|^2_{L^\infty} + \|v(s)\|^2_{L^\infty}ds\right)\left(\|\partial_x u_0\|^2_{L^2} + \|\partial_x v_0\|^2_{L^2}\right).\end{aligned} \tag{2.2}$$

To complete the proof of Theorem 1.1, we will estimate L^∞ bound of the solution in (2.2) by applying an idea in [9]. Multiplying (2.1) by $\bar{u}$ and $\bar{v}$ respectively, we have

$$\partial_t|u|^2 + \partial_x|u|^2 + 2m\mathrm{Im}(\bar{u}v) = 2\mathrm{Re}(\bar{u}v)\mathrm{Im}(\bar{u}v), \tag{2.3}$$
$$\partial_t|v|^2 - \partial_x|v|^2 + 2m\mathrm{Im}(\bar{v}u) = 2\mathrm{Re}(\bar{u}v)\mathrm{Im}(u\bar{v}), \tag{2.4}$$

which implies

$$\partial_t(|u|^2+|v|^2) + \partial_x(|u|^2-|v|^2) = 0. \tag{2.5}$$

Integrating (2.5) on the domain

$$D(x_0,t_0) = \{(x,t)\,|\,0<t<t_0,\ x_0-t_0+t<x<x_0+t_0-t\},$$

we have by applying Green's Theorem

$$\begin{aligned}&2\int_0^{t_0}|u|^2(x_0+t_0-s,s)\,ds + 2\int_0^{t_0}|v|^2(x_0-t_0+s,s)\,ds\\ &= \int_{x_0-t_0}^{x_0+t_0}(\,|u_0(s)|^2+|v_0(s)|^2\,)\,ds \le M,\end{aligned} \tag{2.6}$$

where we denote $M = \int_{\mathbb{R}}(|u_0(y)|^2 + |v_0(y)|^2)\,dy$. Integrating (2.3) along characteristic, we have

$$\frac{d}{dt}|u(x+t,t)|^2 \le 2|v(x+t,t)|^2|u(x+t,t)|^2 + 2m|u(x+t,t)||v(x+t,t)|$$

which implies

$$\frac{d}{dt}|u(x+t,t)| \leq |v(x+t,t)|^2|u(x+t,t)| + m|v(x+t,t)|.$$

Then we have

$$\frac{d}{dt}\left(e^{-\int_0^t |v(x+s,s)|^2\,ds}\,|u(x+t,t)|\right) \leq me^{-\int_0^t |v(x+s,s)|^2\,ds}|v(x+t,t)|.$$

Integrating both sides and considering (2.6), we obtain

$$\begin{aligned} |u(x+t,t)| &\leq e^{\int_0^t |v(x+s,s)|^2\,ds}\left(|u(x,0)| + \int_0^t m|v(x+s,s)|ds\right) \\ &\leq e^{M/2}\left(|u(x,0)| + m\left(\int_0^t |v(x+s,s)|^2 ds\right)^{\frac{1}{2}} t^{\frac{1}{2}}\right) \qquad (2.7) \\ &\leq e^{M/2}\left(|u(x,0)| + m\,(M/2)^{\frac{1}{2}}\,t^{\frac{1}{2}}\right). \end{aligned}$$

The similar argument applied to (2.4) leads us to

$$|v(x-t,t)| \leq e^{M/2}\left(|v(x,0)| + m\,(M/2)^{\frac{1}{2}}\,t^{\frac{1}{2}}\right). \tag{2.8}$$

Then we have, from (2.7) and (2.8),

$$\|u(t)\|_{L^\infty}^2 \leq e^M(2\|u_0\|_{L^\infty}^2 + m^2Mt), \ \ \|v(t)\|_{L^\infty}^2 \leq e^M(2\|v_0\|_{L^\infty}^2 + m^2Mt). \tag{2.9}$$

Plugging (2.9) into (2.2), we have

$$\begin{aligned} &\|\partial_x u(t)\|_{L^2}^2 + \|\partial_x v(t)\|_{L^2}^2 \\ \leq\ &\exp\left(8e^M\int_0^t m^2Ms + \|u_0\|_{L^\infty}^2 + \|v_0\|_{L^\infty}^2\,ds\right)\left(\|\partial_x u_0\|_{L^2}^2 + \|\partial_x v_0\|_{L^2}^2\right), \end{aligned}$$

which proves Theorem 1.1.

Acknowledgement. This work was supported by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2010-330-B00128).

References

[1] G. Berkolaiko and A. Comech, *On spectral stability of solitary waves of nonlinear Dirac equation in 1D*, Math. Model. Nat. Phenom. **7** (2012), no. 2, 13–31.

[2] N. Boussaid and A. Comech, *On spectral stability of the nonlinear Dirac equation*, arXiv:1211.3336.

[3] T. Candy, *Global existence for an L^2 critical nonlinear Dirac equation in one dimension*, Adv. Differential Equations **16** (2011), no. 7-8, 643–666.

[4] A. Comech, M. Guan, and S. Gustafson, *On linear instability of solitary waves for the nonlinear Dirac equation*, arXiv:1209.1146.

[5] V. Delgado, *Global solutions of the Cauchy problem for the* (*classical*) *coupled Maxwell-Dirac and other nonlinear Dirac equations in one space dimension*, Proc. Amer. Math. Soc. **69** (1978), no. 2, 289–296.

[6] R. H. Goodman, M. I. Weinstein, and P. J. Holmes, *Nonlinear propagation of light in one-dimensional periodic structures*, J. Nonlinear Sci. **11** (2001), no. 2, 123–168.

[7] D. J. Gross and A. Neveu, *Dynamical symmetry breaking in asymptotically free field theories*, Phys. Rev. D **10** (1974), 3235–3253.
[8] H. Huh, *Global strong solution to the Thirring model in critical space*, J. Math. Anal. Appl. **381** (2011), no. 2, 513–520.
[9] ———, *Global solutions to Gross-Neveu equations*, Lett. Math. Phys. **103** (2013), no. 8, 927–931.
[10] D. E. Pelinovsky, *Survey on global existence in the nonlinear Dirac equations in one spatial dimension*, Harmonic analysis and nonlinear partial differential equations, 3750, RIMS Kôkyûroku Bessatsu, B26, Kyoto, 2011.
[11] D. E. Pelinovsky and Y. Shimabukuro, *Orbital stability of Dirac solitons*, Lett. Math. Phys. **104** (2014), no. 1, 21–41.
[12] S. Selberg and A. Tesfahun, *Low regularity well-posedness for some nonlinear Dirac equations in one space dimension*, Differential Integral Equations **23** (2010), no. 3-4, 265–278.
[13] Y. Zhang, *Global solution to a cubic nonlinear Dirac equation in* $1 + 1$ *dimensions*, arXiv:1304.1989.

Department of Mathematics
Chung-Ang University
Seoul 156-756, Korea
E-mail address: huh@cau.ac.kr