Commun. Korean Math. Soc. **29** (2014), No. 2, pp. 257–262
http://dx.doi.org/10.4134/CKMS.2014.29.2.257

GENERALIZED RIESZ POINTS FOR PERTURBATIONS OF TOEPLITZ OPERATORS

An Hyun Kim

Abstract. In this note we consider "generalized Riesz points" for compact and quasinilpotent perturbations of Toeplitz operators acting on the Hardy space of the unit circle.

1. Introduction

If M is a subset of $\mathbb{C}$, write $\operatorname{iso} M$, $\operatorname{acc} M$, and ∂M for the isolated points, the accumulation points, and the boundary of M, respectively. Let $\mathcal{X}$ be an infinite dimensional complex Banach space and write $\mathcal{B}(\mathcal{X})$ for the set of all bounded linear operators acting on $\mathcal{X}$. We recall ([1], [5], [6]) that an operator $T \in \mathcal{B}(\mathcal{X})$ is *Fredholm* if $T(\mathcal{X})$ is closed and both $T^{-1}(0)$ and $\mathcal{X}/T(\mathcal{X})$ are finite dimensional. If $T \in \mathcal{B}(\mathcal{X})$ is Fredholm we can define the *index* of T by $\operatorname{index}(T) = \dim T^{-1}(0) - \dim \mathcal{X}/T(\mathcal{X})$. An operator $T \in \mathcal{B}(\mathcal{X})$ is called *Weyl* if it is Fredholm of index zero. The essential spectrum $\sigma_e(T)$ and the Weyl spectrum $\omega(T)$ of $T \in \mathcal{B}(\mathcal{X})$ are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\} \tag{1}$$

and

$$\omega(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}. \tag{2}$$

If $T \in \mathcal{B}(\mathcal{X})$ we write

$$\pi^{\text{left}}(T) = \{\lambda \in \mathbb{C} : (T - \lambda I)^{-1}(0) \neq \{0\}\} \tag{3}$$

for the set of all eigenvalues of T,

$$\pi_0^{\text{left}}(T) = \{\lambda \in \text{iso } \sigma(T) : 0 < \dim(T - \lambda I)^{-1}(0) < \infty\} \tag{4}$$

for the set of all isolated eigenvalues of finite multiplicity and

$$\pi_{00}(T) = \operatorname{iso} \sigma(T) \setminus \sigma_e(T) \tag{5}$$

Received April 2, 2013.
2010 *Mathematics Subject Classification.* Primary 47A10, 47A55, 47B35.
Key words and phrases. generalized Riesz points, Toeplitz operators, perturbations, Weyl's theorem, Browder's theorem.
The author was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST)(No.2012-0007625).

for the set of *Riesz points* of T. From the continuity of the index we have

$$\pi_{00}(T) = \operatorname{iso}\sigma(T) \setminus \omega(T). \tag{6}$$

In [8], the following notion was introduced.

Definition 1.1. The *generalized Riesz points* of $T \in \mathcal{B}(\mathcal{X})$ are the complement of the Weyl spectrum in the spectrum of T:

$$\pi_0(T) := \sigma(T) \setminus \omega(T). \tag{7}$$

We note that a necessary and sufficient condition for $0 \in \pi_0(T)$ is

$$0 < \dim\, T^{-1}(0) = \dim\, \mathcal{X}/T(\mathcal{X}) < \infty. \tag{8}$$

We recall ([3], [6], [7]) that "Weyl's theorem holds for T" if and only if

$$\pi_0(T) = \pi_0^{\mathrm{left}}(T), \tag{9}$$

and ([8, Definition 1]) "Browder's theorem holds for T" if and only if

$$\pi_0(T) = \pi_{00}(T). \tag{10}$$

In [8], the following problem was raised:

Problem 1.2. For which operators $T \in \mathcal{B}(\mathcal{X})$ is there implication, for compact or quasinilpotent $K \in \mathcal{B}(\mathcal{X})$,

$$\pi_0(T) = \emptyset \Longrightarrow \pi_0(T + K) = \emptyset, \tag{11}$$

or implication

$$\text{int}\ \pi_0(T) = \emptyset \Longrightarrow \text{int}\ \pi_0(T + K) = \emptyset? \tag{12}$$

In [8], it was shown that the implication (11) fails for compact and for quasinilpotent operators T, while the implication (12) fails for quasinilpotents but holds for compact operators and that both (11) and (12) can fail for self adjoint and for unitary operators. We recall ([2], [4]) that a "Toeplitz operator" T_φ, induced by a function (so-called the *symbol*) $\varphi \in L^\infty \equiv L^\infty(\mathbb{T})$ ($\mathbb{T}$ denotes the unit circle), is the operator on the Hardy space $H^2 \equiv H^2(\mathbb{T})$ given by setting

$$T_\varphi(f) = \mathbf{P}(\varphi f) \quad \text{for each } f \in H^2, \tag{13}$$

where $\mathbf{P}$ is the orthogonal projection from $L^2 \equiv L^2(\mathbb{T})$ onto H^2. It is familiar ([4, Corollary 7.46]) that the spectrum of a Toeplitz operator is always connected, and that the spectrum and the Weyl spectrum coincide, i.e., $\sigma(T_\varphi) = \omega(T_\varphi)$ (cf. [4, Corollary 7.25]; [3, Theorem 4.1]). Thus $\pi_0(T_\varphi) = \emptyset$ for every Toeplitz operator T_φ. Therefore Toeplitz operators satisfy the conditions of (11) and (12). Thus it is natural to ask Problem 1.2 for Toeplitz operators. In this note we consider generalized Riesz points for compact and quasinilpotent perturbations of Toeplitz operators acting on the Hardy space of the unit circle.

2. The main result

We first recall the *connected hull* ηK of compact $K \subseteq \mathbf{C}$, the complement of the unique unbounded connected component of the complement $\mathbf{C} \setminus K$, and also write

$$\eta' K = \eta K \setminus K = \bigcup \text{Hole}(K)$$

for the union of all bounded components of that complement.

We are ready for:

Theorem 2.1. *If T_φ is a Toeplitz operator on H^2 and $K \in \mathcal{B}(H^2)$ is a compact operator, then*

$$acc\ \pi_0(T_\varphi + K) \subseteq \eta'(T_\varphi + K).$$

Proof. First of all observe

$$\sigma_e(T_\varphi + K) = \sigma_e(T_\varphi) \quad \text{and} \quad \omega(T_\varphi + K) = \omega(T_\varphi) = \sigma(T_\varphi). \tag{14}$$

We now claim

$$\sigma(T_\varphi + K) \setminus \eta\sigma(T_\varphi) \subseteq \text{iso}\,\sigma(T_\varphi + K)\,. \tag{15}$$

Indeed if $\lambda \in \sigma(T_\varphi + K)$ but $\lambda \notin \eta\sigma(T_\varphi)$, then $T_\varphi - \lambda I$ is invertible, so that $T_\varphi + K - \lambda I$ is Weyl but not invertible. If $T_\varphi + K - \mu I$ were Weyl but not invertible for each μ in the disk $|\mu - \lambda| < \epsilon$ for some $\epsilon > 0$, then $\partial\sigma(T_\varphi + K)$ could contain a curve which does not intersect $\sigma(T_\varphi)$. But then such a curve should lie in $\sigma_e(T_\varphi + K)$ because by the punctured neighborhood theorem we have that for every operator S on a Hilbert space,

$$\partial\sigma(S) \setminus \sigma_e(S) \subseteq \text{iso}\,\sigma(S).$$

Thus $\sigma_e(T_\varphi + K) \neq \sigma_e(T_\varphi)$, a contradiction. Therefore we must have that $\lambda \in \text{iso}\,\sigma(T_\varphi + K)$. This proves (15). Now in view of (14) and (15), the passage from $\sigma(T_\varphi)$ to $\sigma(T_\varphi + K)$ is either filling in some holes of $\sigma(T_\varphi)$ or putting some isolated points outside $\eta\sigma(T_\varphi)$. This implies

$$\begin{aligned} \pi_0(T_\varphi + K) &= \sigma(T_\varphi + K) \setminus \omega(T_\varphi + K) \\ &= \sigma(T_\varphi + K) \setminus \sigma(T_\varphi) \\ &\subseteq \Big(\eta\sigma(T_\varphi) \setminus \sigma(T_\varphi)\Big) \cup \text{iso}\,(T_\varphi + K), \end{aligned}$$

which implies acc $\pi_0(T_\varphi + K) \subseteq \eta'(T_\varphi + K)$. $\square$

The essential spectrum of the Toeplitz operator induced by a continuous symbol coincides with the range of the function ([4, Theorem 7.26]):

$$\sigma_e(T_\varphi) = \sigma(\varphi) = \varphi(\mathbb{T}).$$

The spectrum and the Weyl spectrum both coincide ([4, Corollary 7.25]) with the *exponential spectrum* ([6, Definition 9.3.1]) of the symbol:

$$\sigma(T_\varphi) = \omega(T_\varphi) = \varepsilon(\varphi)$$

is the set of $\lambda \in \mathbb{C}$ for which either $\varphi - \lambda$ vanishes somewhere on the circle $\mathbb{T}$, or if not, then $\varphi - \lambda$ winds non-trivially around the origin $0 \in \mathbb{C}$.

We then have:

Corollary 2.2. *If T_φ is a Toeplitz operator with a continuous symbol φ such that $\sigma(T_\varphi)$ has no hole or is an annulus M whose boundary contains an inner boundary (i.e., $\partial M \neq \partial \eta M$) and if $K \in \mathcal{B}(H^2)$ is a compact operator, then*

$$int\, \pi_0(T_\varphi + K) = \emptyset.$$

Proof. We note that by Theorem 2.1, the passage from $\sigma(T_\varphi)$ to $\sigma(T_\varphi + K)$ is either filling some holes of $\sigma(T_\varphi)$ or putting some isolated points outside $\eta\sigma(T_\varphi)$. Thus if $\sigma(T_\varphi)$ has no hole, then there is nothing to prove. If instead $\sigma(T_\varphi)$ satisfies $\partial\,\sigma(T_\varphi) \neq \partial\,\eta\sigma(T_\varphi)$, then $\sigma(T_\varphi + K)$ cannot fill in any hole of $\sigma(T_\varphi)$; if it were not so then we would have that $\sigma_e(T_\varphi + K) \neq \sigma_e(T_\varphi)$. Therefore evidently,

$$\pi_0(T_\varphi + K) = \sigma(T_\varphi + K) \setminus \omega(T_\varphi + K) = \sigma(T_\varphi + K) \setminus \sigma(T_\varphi) \subseteq \mathrm{iso}\,(T_\varphi + K),$$

which gives the result. $\square$

Remark 2.3. We need not expect that (11) is true for any Toeplitz operator $Tvarphi$ with a continuous symbol φ. Indeed, in [8, Theorem 11], it was shown that if $\sigma(T) = \omega(T) \neq \eta\sigma(T)$, then (11) fails. For a concrete example, if we take

$$\varphi(e^{i\theta}) = \begin{cases} e^{2i\theta} & (0 \le \theta \le \pi) \\ e^{-2i\theta} & (\pi \le \theta \le 2\pi), \end{cases}$$

then $\sigma(T_\varphi) = \omega(T_\varphi) = \mathbb{T}$, and hence $\eta\sigma(T_\varphi) = \mathrm{cl}\,\mathbb{D}$ (the closed unit disk), so that $\partial\,\sigma(T_\varphi) = \partial\,\eta\sigma(T_\varphi)$.

We have been unable to decide whether or not int $\pi_0(T + K) = \emptyset$ for every Toeplitz operator T and every quasinilpotent K. We however have:

Theorem 2.4. *If $T \equiv T_\varphi$ is a Toeplitz operator with analytic or co-analytic symbol φ (i.e., $\varphi \in H^\infty$ or $\overline{\varphi} \in H^\infty$) and if $K \in \mathcal{B}(H^2)$ is a quasinilpotent operator, then*

$$\pi_0(T + K) \subseteq \{\beta\} \quad \textit{for some } \beta \in \mathbb{C}. \tag{16}$$

Proof. Suppose $\beta \in \pi_0(T + K)$ (if such a β does not exist, there is nothing to prove). Since $T + K - \beta I$ is Weyl but not invertible, β must be an eigenvalue of $T + K$. Thus for some unit vector $x(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta} \in H^2$,

$$(T - \beta I)x = -Kx. \tag{17}$$

Assume a_k is the first non-zero coefficient of $x(e^{i\theta})$. Then (17) gives

$$\mathbf{P}\left(\sum_{n=k}^{\infty} a_n \varphi(e^{i\theta}) e^{in\theta}\right) - \sum_{n=k}^{\infty} \beta a_n e^{in\theta} = -K\left(\sum_{n=k}^{\infty} a_n e^{in\theta}\right), \tag{18}$$

where $\mathbf{P}$ denotes the orthogonal projection from L^2 to H^2. If $\varphi(e^{i\theta}) = \sum_{n=0}^{\infty} b_n e^{in\theta}$ is analytic we have

$$a_k(b_0-\beta)e^{ik\theta} + \sum_{n=k+1}^{\infty} c_n e^{in\theta} = -K\left(\sum_{n=k}^{\infty} a_n e^{in\theta}\right) \tag{19}$$

for some $c_n \in \mathbb{C}$ $(n = k+1, k+2, \ldots)$. But since K is quasinilpotent we must have that $b_0 = \beta$; indeed, a straightforward calculation shows that

$$||K^n||^{\frac{1}{n}} \geq |a_k|^{\frac{1}{n}} |b_0 - \beta|, \tag{20}$$

which implies $b_0 = \beta$ because the left hand side approaches 0 as $n \to \infty$. Thus we can write

$$\varphi(e^{i\theta}) = \beta + \sum_{n=1}^{\infty} b_n e^{in\theta}. \tag{21}$$

Now we assume that $\gamma \in \pi_0(T+K)$. Then since γ is also an eigenvalue of $T+K$, there is an eigenvector $y(e^{i\theta}) = \sum_{n=0}^{\infty} d_n e^{in\theta} \in H^2$ such that

$$(T - \gamma I)y = -Ky. \tag{22}$$

Assume d_j is the first non-zero coefficient of $y(e^{i\theta})$. Then a similar calculation to (19) shows that

$$d_j(\beta-\gamma)e^{ij\theta} + \sum_{n=j+1}^{\infty} f_n e^{in\theta} = -K\left(\sum_{n=j}^{\infty} d_n e^{in\theta}\right) \tag{23}$$

for some $f_n \in \mathbb{C}$ $(n = j+1, j+2, \ldots)$, which by the same argument as (20) implies that $\gamma = \beta$. Therefore we can conclude that there exists at most one point $\beta \in \pi_0(T+K)$, giving (16). If instead φ is co-analytic (that is, $\bar{\varphi}$ is analytic), then the above argument shows that

$$\pi_0(T_\varphi + K) = \overline{\pi_0((T_\varphi + K)^*)} = \overline{\pi_0(T_{\bar{\varphi}} + K^*)} \subseteq \{\beta\} \quad \text{for some } \beta \in \mathbb{C}. \tag{24}$$ □

We were unable to answer:

Problem 2.5. If T_φ is a Toeplitz operator on H^2 and $K \in \mathcal{B}(H^2)$ is a quasinilpotent operator, does it follow

$$\text{int } \pi_0(T_\varphi + K) = \emptyset \text{ ?}$$

Problem 2.6. If Browder's theorem holds for $T \in \mathcal{B}(\mathcal{X})$ does it also hold for $T+K$ whenever K is Riesz and commutes with T ?

This would be the common generalization of the two cases of Theorem 11 of [7].

Acknowledgements. The author is grateful to Professor Robin Harte for his helpful suggestions.

References

[1] B. Aupetit, *A Primer on Spectral Theory*, Springer, New York, 1991.
[2] A. Brown and P. R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1963-64), 89–102.
[3] L. A. Coburn, *Weyl's theorem for nonnormal operators*, Michigan Math. J. **13** (1966), 285–288.
[4] R. G. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, New York, 1972.
[5] R. E. Harte, *Fredholm, Weyl and Browder theory*, Proc. Roy. Irish Acad. Sect. A **85** (1985), no. 2, 151–176.
[6] ———, *Invertibility and Singularity for Bounded Linear Operators*, Dekker, New York, 1988.
[7] R. E. Harte and W. Y. Lee, *Another note on Weyl's theorem*, Trans. Amer. Math. Soc. **349** (1997), no. 5, 2115–2124.
[8] R. E. Harte, W. Y. Lee, and L. L. Littlejohn, *On generalized Riesz points*, J. Operator Theory **42** (2002), no. 1, 187–196.

Department of Mathematics
Changwon National University
Changwon 641-773, Korea
E-mail address: ahkim@changwon.ac.kr