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NONNEGATIVE INTEGRAL MATRICES HAVING GENERALIZED INVERSES

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Abstract. For an $m \times n$ nonnegative integral matrix A, a *generalized inverse* of A is an $n \times m$ nonnegative integral matrix G satisfying $AGA = A$. In this paper, we characterize nonnegative integral matrices having generalized inverses using the structure of nonnegative integral idempotent matrices. We also define a *space decomposition* of a nonnegative integral matrix, and prove that a nonnegative integral matrix has a generalized inverse if and only if it has a space decomposition. Using this decomposition, we characterize nonnegative integral matrices having reflexive generalized inverses. And we obtain conditions to have various types of generalized inverses.

1. Introduction

Given an $m \times n$ matrix A over a set $\mathbb{S}$, consider an $n \times m$ matrix G over $\mathbb{S}$ in the equations

$$(1)\ AGA = A \qquad (2)\ GAG = G \qquad (3)\ (AG)^t = AG \qquad (4)\ (GA)^t = GA,$$

where A^t denotes the transpose of A. A matrix G satisfying (1) is called a *generalized inverse* (simply, *g-inverse*) of A. In this case, we call A *regular*. If G satisfies (1) and (2), then it is called a *reflexive g-inverse* of A. If G satisfies (1) and (3), then it is called a $\{1, 3\}$-*inverse* of A. Also G is called a $\{1, 4\}$-*inverse* of A if G satisfies (1) and (4). Finally, if G satisfies from (1) to (4), then it is called a *Moore-Penrose inverse* of A. We note that if A has a g-inverse, then A always has a reflexive g-inverse: for, if G is a g-inverse of A, then we can easily show that GAG is a reflexive g-inverse of A.

There are many papers ([1]-[6]) on characterizing matrices having generalized inverses over various sets. Prasad [5] characterized regular matrices over

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commutative rings and Bapat [1] characterized generalized inverses of nonnegative real matrices.

In this paper, we investigate nonnegative integral regular matrices and their generalized inverses. Many of our results are counterparts to results on nonnegative real matrices ([1], [4]). A sectional summary is as follows: In Section 2, some definitions and preliminary results are presented. In Section 3, we characterize nonnegative integral idempotent matrices. In Section 4, we obtain the general form of nonnegative integral regular matrices applying the characterization of nonnegative integral idempotent matrices. Also, we define a space decomposition of a nonnegative integral matrix, and prove that a matrix is regular if and only if it has a space decomposition. Furthermore, using this decomposition, we characterize nonnegative integral matrices having reflexive g-inverses. In the final section, we establish necessary and sufficient conditions for an nonnegative integral regular matrix to possess various types of g-inverses including a Moore-Penrose inverse.

2. Preliminaries and some results

Let $\mathbb{Z}_+$ be the set of all nonnegative integers, and let $\mathcal{M}_{m,n}(\mathbb{Z}_+)$ denote the set of all $m \times n$ matrices with entries in $\mathbb{Z}_+$. If $m = n$, we use the notation $\mathcal{M}_n(\mathbb{Z}_+)$ instead of $\mathcal{M}_{n,n}(\mathbb{Z}_+)$. Addition, multiplication by scalars, and the product of matrices are defined as if $\mathbb{Z}_+$ were a field. Algebraic concepts such as zero matrix (denoted by O), identity matrix of order n (denoted by I_n), transpose and symmetry are defined in usual way.

If $\mathbb{V}$ is a nonempty subset of $\mathbb{Z}_+^n \equiv \mathcal{M}_{n,1}(\mathbb{Z}_+)$ that is closed under addition and multiplication by scalars, $\mathbb{V}$ is called a *vector space* over $\mathbb{Z}_+$. The notions of subspace and spanning sets are defined as if $\mathbb{Z}_+$ were a field. A subset Ω of a vector space $\mathbb{V}$ is called a *basis* if it spans $\mathbb{V}$ and no proper subset of Ω spans $\mathbb{V}$.

A set Φ of vectors over $\mathbb{Z}_+$ is *linearly dependent* if there is a vector $\boldsymbol{x} \in \Phi$ such that $\boldsymbol{x}$ is a linear combination of the vectors in $\Phi \setminus \{\boldsymbol{x}\}$. Otherwise Φ is *linearly independent*. Thus a linearly independent set cannot contain a zero vector. Also a basis of a vector space is linearly independent.

For matrices A and B in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, we use the notation $A \leq B$ or $B \geq A$ if $a_{i,j} \leq b_{i,j}$ for all i and j.

Theorem 2.1. *If $\Omega_1 = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_p\}$ and $\Omega_2 = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_q\}$ are bases of a vector space $\mathbb{V}$ over $\mathbb{Z}_+$, then $\Omega_1 = \Omega_2$.*

Proof. Let $\boldsymbol{x}_k$ be arbitrary in Ω_1. Then $\boldsymbol{x}_k$ is a linear combination of vectors in Ω_2, each of which is a linear combination of vectors in Ω_1. Thus we have

$$\boldsymbol{x}_k = \sum_{i=1}^{q} \alpha_i \boldsymbol{y}_i \quad \text{and} \quad \boldsymbol{y}_i = \sum_{j=1}^{p} \beta_{i,j} \boldsymbol{x}_j \tag{2.1}$$

for some scalars α_i and $\beta_{i,j}$ in $\mathbb{Z}_+$, equivalently

$$\boldsymbol{x}_k = \sum_{i=1}^{q} \alpha_i \left(\sum_{j=1}^{p} \beta_{i,j} \boldsymbol{x}_j \right) = \left(\sum_{i=1}^{q} \alpha_i \beta_{i,k} \right) \boldsymbol{x}_k + \sum_{j \neq k}^{p} \left(\sum_{i=1}^{q} \alpha_i \beta_{i,j} \right) \boldsymbol{x}_j.$$

Since Ω_1 is linearly independent, we have $0 \neq \sum_{i=1}^{q} \alpha_i \beta_{i,k}$ and hence $\alpha_h \beta_{h,k} \neq 0$ for some $h \in \{1, \dots, q\}$. By (2.1), we have $\boldsymbol{x}_k \geq \alpha_h \boldsymbol{y}_h$ and $\boldsymbol{y}_h \geq \beta_{h,k} \boldsymbol{x}_k$ so that $\boldsymbol{x}_k \geq \alpha_h \beta_{h,k} \boldsymbol{x}_k \geq \boldsymbol{x}_k$. Hence we have $\alpha_h = \beta_{h,k} = 1$. It follows that $\boldsymbol{x}_k = \boldsymbol{y}_h$.

A parallel argument shows that if $\boldsymbol{y}_k$ is arbitrary in Ω_2, then $\boldsymbol{y}_k = \boldsymbol{x}_h$ for some $h \in \{1, \dots, p\}$. Therefore we have established that $\Omega_1 = \Omega_2$. □

If $\mathbb{V}$ is a vector space over $\mathbb{Z}_+$ that has a finite spanning subset, then Theorem 2.1 shows that it has a unique basis. The cardinality of the basis is called the *dimension* of $\mathbb{V}$, and is denoted by $\dim(\mathbb{V})$.

The *column space* of a matrix $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ is the subspace spanned by its columns and is denoted by $\mathcal{C}(A)$; the *row space* of A is the subspace spanned by its rows and is denoted by $\mathcal{R}(A)$. The *column rank*, $c(A)$ is $\dim(\mathcal{C}(A))$; the *row rank*, $r(A)$ is $\dim(\mathcal{R}(A))$. In particular, we say that A has *rank* r if $c(A) = r(A) = r$. Generally the column rank and row rank of a matrix over $\mathbb{Z}_+$ need not be equal. For example, consider the 2×2 matrix $A = \left[\begin{smallmatrix} 2 & 3 \\ 4 & 6 \end{smallmatrix}\right]$ over $\mathbb{Z}_+$. Then $c(A) = 2$, while $r(A) = 1$.

For any matrix A in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, let $\boldsymbol{a}_{i*}$ and $\boldsymbol{a}_{*j}$ denote the ith row and the jth column of A, respectively.

Lemma 2.2. *For a matrix A in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$,*

(i) *if $c(A) = r$, then there are r columns $\boldsymbol{a}_{*j_1}, \dots, \boldsymbol{a}_{*j_r}$ of A such that $\{\boldsymbol{a}_{*j_1}, \dots, \boldsymbol{a}_{*j_r}\}$ is the basis of $\mathcal{C}(A)$;*

(ii) *if $r(A) = r$, then there are r rows $\boldsymbol{a}_{i_1*}, \dots, \boldsymbol{a}_{i_r*}$ of A such that $\{\boldsymbol{a}_{i_1*}, \dots, \boldsymbol{a}_{i_r*}\}$ is the basis of $\mathcal{R}(A)$.*

Proof. Assume that $c(A) = r$. Notice that $\{\boldsymbol{a}_{*1}, \dots, \boldsymbol{a}_{*n}\}$ spans $\mathcal{C}(A)$. Thus there are k linearly independent columns $\boldsymbol{a}_{*j_1}, \dots, \boldsymbol{a}_{*j_k}$ of A such that

$$\{\boldsymbol{a}_{*j_1}, \dots, \boldsymbol{a}_{*j_k}\} \text{ spans } \mathcal{C}(A). \tag{2.2}$$

Here we may assume that k is the minimum number satisfying (2.2). Thus, $\{\boldsymbol{a}_{*j_1}, \dots, \boldsymbol{a}_{*j_k}\}$ is the basis of $\mathcal{C}(A)$, and hence $k = r$ by Theorem 2.1. Thus (i) is satisfied. By a parallel argument, (ii) is satisfied. □

Corollary 2.3. *Let $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ and $B \in \mathcal{M}_{m,q}(\mathbb{Z}_+)$ be matrices with $\mathcal{C}(A) = \mathcal{C}(B)$. If $c(A) = r$ and the first r columns of A compose the basis of $\mathcal{C}(A)$, then there are a permutation matrix Q of order q, and matrices, $X \in \mathcal{M}_{r,q-r}(\mathbb{Z}_+)$ and $Y \in \mathcal{M}_{n-r,q-r}(\mathbb{Z}_+)$, such that $B = A \left[\begin{smallmatrix} I_r & X \\ O & Y \end{smallmatrix}\right] Q$.*

Proof. Since $c(B) = c(A) = r$, by Lemma 2.2, there are r columns $\boldsymbol{b}_{*j_1}, \dots, \boldsymbol{b}_{*j_r}$ of B such that $\{\boldsymbol{b}_{*j_1}, \dots, \boldsymbol{b}_{*j_r}\}$ is the basis of $\mathcal{C}(B)$. But then it follows from Theorem 2.1 that without loss of generality, we may assume that $\boldsymbol{b}_{*j_1} = \boldsymbol{a}_{*1}$,

$\boldsymbol{b}_{*j_2} = \boldsymbol{a}_{*2}$, …, $\boldsymbol{b}_{*j_r} = \boldsymbol{a}_{*r}$. Thus there is a permutation matrix Q_1 of order q such that

$$BQ_1 = \begin{bmatrix} \boldsymbol{a}_{*1} & \cdots & \boldsymbol{a}_{*r} & \boldsymbol{b}_{*j_{r+1}} & \cdots & \boldsymbol{b}_{*j_q} \end{bmatrix} = A \begin{bmatrix} I_r & X \\ O & Y \end{bmatrix},$$

where $X \in \mathcal{M}_{r,q-r}(\mathbb{Z}_+)$ and $Y \in \mathcal{M}_{n-r,q-r}(\mathbb{Z}_+)$. If we take $Q = Q_1^t$, the result follows. □

A matrix A in $\mathcal{M}_n(\mathbb{Z}_+)$ is said to be *idempotent* if $A^2 = A$.

Lemma 2.4. *For $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ and $G \in \mathcal{M}_{n,m}(\mathbb{Z}_+)$, the following are equivalent*:

(i) $AGA = A$;
(ii) *AG is idempotent in $\mathcal{M}_m(\mathbb{Z}_+)$ and $\mathcal{C}(A) = \mathcal{C}(AG)$*;
(iii) *GA is idempotent in $\mathcal{M}_n(\mathbb{Z}_+)$ and $\mathcal{R}(A) = \mathcal{R}(GA)$.*

Proof. (i)⇒(ii): Assume (i). Then $AGAG = AG$ and hence AG is idempotent. Now, we will show that $\mathcal{C}(A) = \mathcal{C}(AG)$. If $\boldsymbol{y}$ is an arbitrary vector in $\mathcal{C}(AG)$, then there is a vector $\boldsymbol{x}$ such that $\boldsymbol{y} = AG\boldsymbol{x}$ so that $\boldsymbol{y} = A(G\boldsymbol{x}) \in \mathcal{C}(A)$. Hence $\mathcal{C}(AG) \subseteq \mathcal{C}(A)$. Let $\boldsymbol{z}$ be arbitrary in $\mathcal{C}(A)$ so that $\boldsymbol{z} = A\boldsymbol{x}'$ for some vector $\boldsymbol{x}'$. It follows from $AGA = A$ that

$$\boldsymbol{z} = AGA\boldsymbol{x}' = (AG)(A\boldsymbol{x}') \in \mathcal{C}(AG),$$

and hence $\mathcal{C}(A) \subseteq \mathcal{C}(AG)$. Thus $\mathcal{C}(A) = \mathcal{C}(AG)$. Therefore (ii) is satisfied.

(ii)⇒(i): Assume (ii). Then $A = AGX$ for some $X \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ because $\mathcal{C}(A) = \mathcal{C}(AG)$. Since AG is idempotent, we have that $AGA = AGAGX = AGX = A$. Therefore (i) is satisfied.

By a parallel argument, we can easily show that (i)⇔(iii). □

3. Nonnegative integral idempotent matrices

In this section, we characterize nonnegative integral idempotent matrices. If A is an idempotent matrix in $\mathcal{M}_n(\mathbb{Z}_+)$, then we can easily check that all main diagonal entries of A are 0 or 1.

Proposition 3.1. *If $A \in \mathcal{M}_n(\mathbb{Z}_+)$ is nonzero and $a_{i,i} = 0$ for all i, then A is not idempotent.*

Proof. If A is any real idempotent matrix, and $a_{i,i} = 0$ for all i, then the trace of A is 0. Thus, the sum of the eigenvalues of A is 0, and hence, all eigenvalues of A are zero (idempotent matrices have only eigenvalues of 0 or 1). That is A is both nilpotent and idempotent. It follows that $A = O$. □

The following lemma is easily established by multiplying each side of $A^2 = A$ by A^{-1}.

Lemma 3.2. *If $A \in \mathcal{M}_n(\mathbb{Z}_+)$ is idempotent and has rank n, then $A = I_n$.*

Proposition 3.1 shows that if $A \in \mathcal{M}_n(\mathbb{Z}_+)$ is idempotent and nonzero, then A has at least one nonzero main diagonal entry.

Theorem 3.3. *Suppose that $A \in \mathcal{M}_n(\mathbb{Z}_+)$ has $r(\geq 1)$ nonzero main diagonal entries. Then A is idempotent if and only if there is a permutation matrix P of order n such that $A = P \left[\begin{smallmatrix} I_r & C \\ D & DC \end{smallmatrix}\right] P^t$, where $C \in \mathcal{M}_{r,n-r}(\mathbb{Z}_+)$, $D \in \mathcal{M}_{n-r,r}(\mathbb{Z}_+)$ and $CD = O$.*

Proof. Let P be a permutation matrix such that $B = P^tAP$ has $b_{i,i} \neq 0$ for $1 = 1, \ldots, r$ and $b_{j,j} = 0$ for $j > r$. Partition B so that $B = \left[\begin{smallmatrix} M & C \\ D & Z \end{smallmatrix}\right]$ where M is $r \times r$ so that $z_{j,j} = 0$ for all j. Since A, and hence B, is idempotent,

$$\begin{bmatrix} M & C \\ D & Z \end{bmatrix} = \begin{bmatrix} M & C \\ D & Z \end{bmatrix}^2 = \begin{bmatrix} M^2 + CD & MC + CZ \\ DM + ZD & DC + Z^2 \end{bmatrix}. \tag{3.1}$$

Since $m_{i,i} \neq 0$ for all i, MC has nonzero entries everywhere that C does. Since $MC + CZ = C$ (from the (1,2) entry of (3.1)), it follows (since arithmetic is in $\mathbb{Z}_+$) that

$$MC = C \text{ and } CZ = O. \tag{3.2}$$

Similarly (from the (2,1) entry of (3.1)), we have

$$DM = D \text{ and } ZD = O. \tag{3.3}$$

Now consider the (1,1) entry of (3.1): $M^2 + CD = M$. Multiplying by M we get $M^3 + MCD = M^2$, and by (3.2) we have $M^3 + CD = M^2$. Adding CD we get $M^3 + 2CD = M^2 + CD = M$. Repeating this process we obtain for any $k \geq 1$ that

$$M^{k+1} + kCD = M.$$

If the (i, j) entry of CD is nonzero, then for sufficiently large k, the (i, j) entry of kCD is strictly greater than $m_{i,j}$, a contradiction. Thus, $CD = O$ and $M^2 = M$. Since all the diagonal entries of M are positive, the trace of M is at least r, and all eigenvalues of M are 0 or 1 (M is idempotent), so the trace of M is at most r. That is, M has rank r and by Lemma 3.2,

$$M = I_r. \tag{3.4}$$

Now consider the (2,2) entry of (3.1): $DC + Z^2 = Z$. Multiply by Z to get $ZDC + Z^3 = Z^2$, so that by (3.3) we have $Z^3 = Z^2$. Thus all eigenvalues of Z are 0 or 1. Since the trace of Z is zero, we have all eigenvalues of Z are 0, and since $Z^3 = Z^2$, we have that

$$Z^2 = O. \tag{3.5}$$

From (3.4) and (3.5) we have that

$$B = \begin{bmatrix} I_r & C \\ D & DC \end{bmatrix}.$$

The converse is obvious. $\square$

In Theorem 3.3, we note that $c(A) = r(A) = r$ and hence A has rank r. Thus the following Corollary is obtained:

Corollary 3.4. *If $A \in \mathcal{M}_n(\mathbb{Z}_+)$ is idempotent, then $c(A) = r(A)$.*

Theorem 3.5. *$A \in \mathcal{M}_n(\mathbb{Z}_+)$ is a symmetric idempotent matrix of rank r if and only if there is a permutation matrix P of order n such that $A = P \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} P^t$.*

Proof. Suppose that A is a symmetric idempotent matrix of rank r. By Theorem 3.3, there is a permutation matrix P of order n such that $A = \begin{bmatrix} I_r & C \\ C^t & C^tC \end{bmatrix} P^t$, where $C \in \mathcal{M}_{r,n-r}(\mathbb{Z}_+)$ and $CC^t = O$. It follows from $CC^t = O$ that $C = O$. Therefore we have $A = P \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} P^t$. The converse is obvious. □

4. Nonnegative integral regular matrices

In this section, we give a characterization of nonnegative integral regular matrices using Theorem 3.3. Also, we define a space decomposition of a nonnegative integral matrix, and prove that a nonnegative integral matrix A is regular if and only if it has a space decomposition. Furthermore, using this decomposition, we characterize nonnegative integral matrices having reflexive g-inverses.

Theorem 4.1. *Let A be a matrix in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$. Then A is regular if and only if there are permutation matrices P and Q of orders m and n, respectively such that*

$$A = P \begin{bmatrix} I_r & N \\ D & DN \end{bmatrix} Q,$$

where $r(\geq 0)$ is an integer; $N \in \mathcal{M}_{r,n-r}(\mathbb{Z}_+)$ and $D \in \mathcal{M}_{m-r,r}(\mathbb{Z}_+)$.

Proof. Suppose that A is regular. Then A has a g-inverse $G \in \mathcal{M}_{n,m}(\mathbb{Z}_+)$ so that $AGA = A$. By Lemma 2.4, $AG \in \mathcal{M}_m(\mathbb{Z}_+)$ is idempotent and $\mathcal{C}(A) = \mathcal{C}(AG)$. Let AG have the rank r. By Theorem 3.3, there is a permutation matrix P of order m such that $AG = P \begin{bmatrix} I_r & C \\ D & DC \end{bmatrix} P^t$, where $C \in \mathcal{M}_{r,m-r}(\mathbb{Z}_+)$, $D \in \mathcal{M}_{m-r,r}(\mathbb{Z}_+)$ and $CD = O$. Notice that $\mathcal{C}(P^tA) = \mathcal{C}(P^tAG) = \mathcal{C}(P^tAGP)$ and the first r columns of P^tAGP compose the basis of $\mathcal{C}(P^tAGP)$. Thus by Corollary 2.3, there are a permutation matrix Q of order n, and matrices $X \in \mathcal{M}_{r,n-r}(\mathbb{Z}_+)$ and $Y \in \mathcal{M}_{m-r,n-r}(\mathbb{Z}_+)$ such that $P^tA = P^tAGP \begin{bmatrix} I_r & X \\ O & Y \end{bmatrix} Q$, equivalently $A = AGP \begin{bmatrix} I_r & X \\ O & Y \end{bmatrix} Q$. Since $AGP = P \begin{bmatrix} I_r & C \\ D & DC \end{bmatrix}$, we have

$$A = P \begin{bmatrix} I_r & C \\ D & DC \end{bmatrix} \begin{bmatrix} I_r & X \\ O & Y \end{bmatrix} Q = P \begin{bmatrix} I_r & N \\ D & DN \end{bmatrix} Q,$$

where $N = X + CY$.

The converse is obvious. In fact, $Q^t \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} P^t$ is a g-inverse of A. □

Furthermore, we have $c(A) = r(A) = r$ and hence A has rank r. Thus we obtain the following:

Corollary 4.2. *If $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ is regular, then $c(A) = r(A)$.*

For any $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$, we notice that A is regular if and only if A^t is regular because $G \in \mathcal{M}_{n,m}(\mathbb{Z}_+)$ is a g-inverse of A if and only if $G^t \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ is a g-inverse of A^t.

Corollary 4.3. *For matrices A and B in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, let A be regular. If $\mathcal{C}(A) = \mathcal{C}(B)$ or $\mathcal{R}(A) = \mathcal{R}(B)$, then B is regular.*

Proof. Suppose that $\mathcal{C}(A) = \mathcal{C}(B)$ with $c(A) = r$. Since A is regular, by Theorem 4.1, there are permutation matrices P and Q of orders m and n, respectively such that $A = P \left[\begin{smallmatrix} I_r & N \\ D & DN \end{smallmatrix}\right] Q$, where $N \in \mathcal{M}_{r,n-r}(\mathbb{Z}_+)$ and $D \in \mathcal{M}_{m-r,r}(\mathbb{Z}_+)$. Clearly $\mathcal{C}(P^tAQ^t) = \mathcal{C}(P^tA) = \mathcal{C}(P^tB)$ and hence by Corollary 2.3, there are a permutation matrix Q_1 of order n, and matrices $X \in \mathcal{M}_{r,n-r}(\mathbb{Z}_+)$ and $Y \in \mathcal{M}_{n-r,n-r}(\mathbb{Z}_+)$ such that $P^tB = P^tAQ^t \left[\begin{smallmatrix} I_r & X \\ O & Y \end{smallmatrix}\right] Q_1$, equivalently $B = AQ^t \left[\begin{smallmatrix} I_r & X \\ O & Y \end{smallmatrix}\right] Q_1$. It follows from $A = P \left[\begin{smallmatrix} I_r & N \\ D & DN \end{smallmatrix}\right] Q$ that $B = P \left[\begin{smallmatrix} I_r & M \\ D & DM \end{smallmatrix}\right] Q_1$, where $M = X + NY$. Thus by Theorem 4.1, B is regular.

Next, suppose that $\mathcal{R}(A) = \mathcal{R}(B)$. Then $\mathcal{C}(A^t) = \mathcal{C}(B^t)$. Since A^t is regular, by the above argument, B^t is regular and hence B is regular. $\square$

Consider the matrix $A = \left[\begin{smallmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 0 & 2 \end{smallmatrix}\right]$ over $\mathbb{Z}_+$. By Theorem 4.1, A is regular with a g-inverse $G = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$. On the other hand, we can easily check that $G' = \left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$ is also a g-inverse of A. Generally the following holds:

Proposition 4.4. *Let $A = \left[\begin{smallmatrix} I_r & N \\ D & DN \end{smallmatrix}\right] \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$. Then $\left[\begin{smallmatrix} G_1 & G_2 \\ G_3 & G_4 \end{smallmatrix}\right] \in \mathcal{M}_{n,m}(\mathbb{Z}_+)$ is a g-inverse of A if and only if $I_r = G_1 + NG_3 + G_2D + NG_4D$.*

Theorem 4.5. *Let $G_1, G_2 \in \mathcal{M}_{n,m}(\mathbb{Z}_+)$ be any two g-inverses of a regular matrix $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$.*

(i) *If $r(A) = m$, then $AG_1 = AG_2$;*
(ii) *If $c(A) = n$, then $G_1A = G_2A$.*

Proof. (i) By Lemma 2.4, AG_1 and AG_2 are idempotent in $\mathcal{M}_m(\mathbb{Z}_+)$ and $\mathcal{C}(AG_1) = \mathcal{C}(A) = \mathcal{C}(AG_2)$. Furthermore both AG_1 and AG_2 have the rank m by Corollary 3.4. Thus by Lemma 3.2, $AG_1 = I_m = AG_2$.

(ii) Notice that G_1^t and G_2^t are g-inverses of A^t. If $c(A) = n$, then $r(A^t) = n$. Thus by (i), $A^tG_1^t = A^tG_2^t$ and hence $G_1A = G_2A$. $\square$

A nonzero matrix $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ is said to be *space decomposable* if for some r, there are two matrices $L \in \mathcal{M}_{m,r}(\mathbb{Z}_+)$ and $R \in \mathcal{M}_{r,n}(\mathbb{Z}_+)$ such that

$$A = LR, \quad \mathcal{C}(A) = \mathcal{C}(L) \quad \text{and} \quad \mathcal{R}(A) = \mathcal{R}(R). \tag{4.1}$$

The decomposition LR is called a *space decomposition* of A of order r.

Theorem 4.6. *For a nonzero matrix $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$, A is regular if and only if A has a space decomposition.*

Proof. Suppose that A is regular and has rank r. By Theorem 4.1, there are permutation matrices P and Q of orders m and n, respectively such that $A = P\left[\begin{smallmatrix} I_r & N \\ D & DN\end{smallmatrix}\right]Q$, where $N \in \mathcal{M}_{r,n-r}(\mathbb{Z}_+)$ and $D \in \mathcal{M}_{m-r,r}(\mathbb{Z}_+)$. Let

$$L_A = P\begin{bmatrix} I_r \\ D\end{bmatrix} \quad \text{and} \quad R_A = \begin{bmatrix} I_r & N\end{bmatrix} Q. \tag{4.2}$$

Then we can easily show that (4.2) satisfies the condition (4.1). Therefore A has a space decomposition.

Conversely assume that A has a space decomposition of order q. Then we can assume the condition (4.1) so that $L = AX$ and $R = YA$ for some matrices $X \in \mathcal{M}_{n,q}(\mathbb{Z}_+)$ and $Y \in \mathcal{M}_{q,m}(\mathbb{Z}_+)$. Then $A = LR = A(XY)A$ and hence A is regular. □

Theorem 4.6 implies that the rank of a regular matrix is the smallest integer r such that r can be taken in the definition of its space decomposition.

Theorem 4.7. *Let A be a nonzero matrix in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$ of rank r. Let L_AR_A be a space decomposition of order r of A as in* (4.2)*. Then LR is also a space decomposition of A of order r if and only if there is a permutation matrix P_1 of order r such that $L = L_AP_1$ and $R = P_1^tR_A$.*

Proof. Suppose that LR is a space decomposition of A of order r. By definition, we have $\mathcal{C}(L) = \mathcal{C}(A) = \mathcal{C}(L_A)$ and $\mathcal{R}(R) = \mathcal{R}(A) = \mathcal{R}(R_A)$. Thus there are permutation matrices P_1 and P_2 of order r such that $L = L_AP_1$ and $R = P_2R_A$. Then we have

$$A = P\begin{bmatrix} I_r & N \\ D & DN\end{bmatrix} Q = P\begin{bmatrix} I_r \\ D\end{bmatrix} P_1P_2\begin{bmatrix} I_r & N\end{bmatrix} Q,$$

and hence $I_r = I_rP_1P_2I_r$ so that $P_1P_2 = I_r$, equivalently $P_2 = P_1^t$.

The converse is obvious. □

Corollary 4.8. *Let LR be an arbitrary space decomposition of order r of a regular matrix $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ of rank $r(\geq 1)$. Then L and R have g-inverses.*

Proof. By Theorem 4.7, we have $L = L_AP_1$ and $R = P_1^tR_A$ for some permutation matrix P_1 of order r, where L_A and R_A are of the form in (4.2). Then

$$L_G = P_1^t\begin{bmatrix} I_r & O\end{bmatrix} P^t \quad \text{and} \quad R_G = Q^t\begin{bmatrix} I_r \\ O\end{bmatrix} P_1 \tag{4.3}$$

are g-inverses of L and R, respectively. □

Theorem 4.9. *Let LR be a space decomposition of order r of a regular matrix $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ of rank $r(\geq 1)$, and $G \in \mathcal{M}_{n,m}(\mathbb{Z}_+)$ be a g-inverse of A. If L' and R' are arbitrary g-inverses of L and R, respectively, then*

(i) $L'L = RR' = I_r$;
(ii) $L'A = R$ *and* $AR' = L$;
(iii) $R'L'$ *is a g-inverse of* A;

(iv) *RG and GL are reflexive g-inverses of L and R, respectively.*

Proof. Since A is a regular matrix of rank r, we assume condition (4.2). Thus, by Theorem 4.7, we have $L = P\left[\begin{smallmatrix} I_r \\ D \end{smallmatrix}\right] P_1$ and $R = P_1^t \begin{bmatrix} I_r & N \end{bmatrix} Q$ for some permutation matrix P_1 of order r. Notice that L_G and R_G in (4.3) are g-inverses of L and R, respectively. Let L' and R' be arbitrary g-inverses of L and R, respectively. By Theorem 4.5, we have

$$L_G L = L'L \quad \text{and} \quad RR_G = RR'. \tag{4.4}$$

By a simple calculation, we have $L_G L = I_r = RR_G$ and hence $L'L = RR' = I_r$ by (4.4). Thus (i) holds. It follows that $L'A = L'LR = R$ and $AR' = LRR' = L$. Thus (ii) is satisfied. But then $A(R'L')A = AR'L'A = LR = A$ and hence (iii) holds.

Now, we will prove (iv). Since $AGA = A$, $LL'L = L$ and $RR'R = R$, it follows from (ii) that

$$L(RG)L = LL'AGAR' = LL'AR' = LL'L = L \tag{4.5}$$

and

$$(RG)L(RG) = (L'A)G(AR')(RG) = L'AR'RG = RR'RG = RG. \tag{4.6}$$

Therefore RG is a reflexive g-inverse of L by (4.5) and (4.6). Notice that R^tL^t is a space decomposition of A^t and G^t is a g-inverse of A^t. By the above result, L^tG^t is a reflexive g-inverse of R^t, equivalently GL is a reflexive g-inverse of R. Thus (iv) holds. □

The following is a characterization of reflexive g-inverses of a nonnegative integral regular matrix.

Theorem 4.10. *Let LR be a space decomposition of order r of a regular matrix $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ of rank $r(\geq 1)$. Then $G \in \mathcal{M}_{n,m}(\mathbb{Z}_+)$ is a reflexive g-inverse of A if and only if there are g-inverses L' and R' of L and R, respectively, such that $G = R'L'$.*

Proof. Suppose that G is a reflexive g-inverse of A. Let $L' = RG$ and $R' = GL$. Then L' and R' are reflexive g-inverses of L and R, respectively, by Theorem 4.9(iv). But then $G = GAG = GLRG = R'L'$.

Conversely, assume that there are g-inverses L' and R' of L and R, respectively such that $G = R'L'$. Then $G = R'L'$ is a g-inverse of A by Theorem 4.9(iii). Furthermore it follows from Theorem 4.9(i) that $GAG = R'L'LRR'L' = R'I_rI_rL' = R'L' = G$, and hence G is a reflexive g-inverse of A. □

5. Other types of g-inverses

In this section, we obtain necessary and sufficient conditions for a matrix $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ to have various types of g-inverses.

Recall that $G \in \mathcal{M}_{n,m}(\mathbb{Z}_+)$ is a $\{1,3\}$-inverse of $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ if and only if $AGA = A$ and $(AG)^t = AG$.

Theorem 5.1. *For a matrix A in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, the following are equivalent*:

(i) *A has a $\{1,3\}$-inverse*;

(ii) *There are permutation matrices P and Q of orders m and n, respectively such that $A = P\left[\begin{smallmatrix} I_r & X \\ O & O\end{smallmatrix}\right]Q$, where $r(\geq 0)$ is an integer and $X \in \mathcal{M}_{r,n-r}(\mathbb{Z}_+)$*;

(iii) *A is regular and $\mathcal{R}(A) = \mathcal{R}(A^tA)$*;

(iv) *There is a matrix G in $\mathcal{M}_{n,m}(\mathbb{Z}_+)$ such that $A^tAG = A^t$.*

Proof. (i)⇒(ii): Suppose that the rank of A is r and $G \in \mathcal{M}_{n,m}(\mathbb{Z}_+)$ is a $\{1,3\}$-inverse of A. Then $AG \in \mathcal{M}_m(\mathbb{Z}_+)$ is symmetric. Further, $\mathcal{C}(A) = \mathcal{C}(AG)$ and AG is idempotent by Lemma 2.4. Thus, by Theorem 3.5, there is a permutation matrix P of order m such that $AG = P\left[\begin{smallmatrix} I_r & O \\ O & O\end{smallmatrix}\right]P^t$. Notice that $\mathcal{C}(P^tA) = \mathcal{C}(P^tAG) = \mathcal{C}(P^tAGP)$. Hence by Corollary 2.3, there are a permutation matrix Q of order n, and matrices $X \in \mathcal{M}_{r,n-r}(\mathbb{Z}_+)$ and $Y \in \mathcal{M}_{m-r,n-r}(\mathbb{Z}_+)$ such that $P^tA = P^tAGP\left[\begin{smallmatrix} I_r & X \\ O & Y\end{smallmatrix}\right]Q$, equivalently $A = AGP\left[\begin{smallmatrix} I_r & X \\ O & Y\end{smallmatrix}\right]Q$. Thus we have

$$A = AGP\begin{bmatrix} I_r & X \\ O & Y\end{bmatrix}Q = P\begin{bmatrix} I_r & O \\ O & O\end{bmatrix}\begin{bmatrix} I_r & X \\ O & Y\end{bmatrix}Q = P\begin{bmatrix} I_r & X \\ O & O\end{bmatrix}Q.$$

(ii)⇒(iii): Clear because $A^tA = Q^t\left[\begin{smallmatrix} I_r & X \\ X^t & X^tX\end{smallmatrix}\right]Q$.

(iii)⇒(iv): Assume that A is regular and $\mathcal{R}(A) = \mathcal{R}(A^tA)$. Then $A = YA^tA$ for some $Y \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$. Furthermore it follows from Corollary 4.3 that $A^tA \in \mathcal{M}_n(\mathbb{Z}_+)$ is also regular and hence $(A^tA)G_1(A^tA) = A^tA$ for some $G_1 \in \mathcal{M}_n(\mathbb{Z}_+)$. Take $G = G_1A^t$. Then it follows from $A^t = A^tAY^t$ that $A^tAG = A^tAG_1A^t = (A^tA)G_1(A^tA)Y^t = A^tAY^t = A^t$.

(iv)⇒(i): Suppose that $A^tAG = A^t$ for some $G \in \mathcal{M}_{n,m}(\mathbb{Z}_+)$, equivalently $A = G^tA^tA$. Then we have $(AG)^t = G^tA^t = G^tA^tAG = AG$ and $AGA = (AG)^tA = G^tA^tA = A$. Therefore G is a $\{1,3\}$-inverse of A. □

Recall that $G \in \mathcal{M}_{n,m}(\mathbb{Z}_+)$ is a $\{1,4\}$-inverse of $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ if and only if $AGA = A$ and $(GA)^t = GA$. For any matrix A in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, we note that A has a $\{1,3\}$-inverse G if and only if A^t has a $\{1,4\}$-inverse G^t. Thus, by applying this result in Theorem 5.1, we obtain the following Theorem, and omit the proof:

Theorem 5.2. *For a matrix A in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, the following are equivalent*:

(i) *A has a $\{1,4\}$-inverse*;

(ii) *There are permutation matrices P and Q of orders m and n, respectively such that $A = P\left[\begin{smallmatrix} I_r & O \\ X & O\end{smallmatrix}\right]Q$, where $r(\geq 0)$ is an integer and $X \in \mathcal{M}_{m-r,r}(\mathbb{Z}_+)$*;

(iii) *A is regular and $\mathcal{C}(A) = \mathcal{C}(AA^t)$*;

(iv) *There is a matrix G in $\mathcal{M}_{n,m}(\mathbb{Z}_+)$ such that $GAA^t = A^t$.*

Finally, in the following, we characterize nonnegative integral matrices having Moore-Penrose inverses. The proof only depends on the above two Theorems, and we omit the proof:

Theorem 5.3. *For a matrix A in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, it has a Moore-Penrose inverse if and only if there are permutation matrices P and Q of orders m and n, respectively such that*

$$A = P \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} Q,$$

where $r(\geq 0)$ is an integer.

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