Moment of inertia of liquid in a tank

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ABSTRACT: *In this study, the inertial properties of fully filled liquid in a tank were studied based on the potential theory. The analytic solution was obtained for the rectangular tank, and the numerical solutions using Green's 2nd identity were obtained for other shapes. The inertia of liquid behaves like solid in recti-linear acceleration. But under rotational acceleration, the moment of inertia of liquid becomes small compared to that of solid. The shapes of tank investigated in this study were ellipse, rectangle, hexagon, and octagon with various aspect ratios. The numerical solutions were compared with analytic solution, and an ad hoc semi-analytical approximate formula is proposed herein and this formula gives very good predictions for the moment of inertia of the liquid in a tank of several different geometrical shapes. The results of this study will be useful in analyzing of the motion of LNG/LPG tanker, liquid cargo ship, and damaged ship.*

KEY WORDS: Inertia of liquid; Sloshing; Roll moment of inertia; Liquid in a tank; Fully filled tank.

INTRODUCTION

If a ship has floodwater and/or liquid cargo inside, the liquid influences the ship motions. The mechanism of this influence depends on the existence of free surface in the tank containing the liquid. If the free surface exists, the motion of liquid is relatively free compared with when no free surface exists. In statics with free surface, the center of gravity can move and change the restoring force, which is called the free surface effect. However, when a ship motion problem is considered in a typical ocean environment, the motion of the ship induces the motion of the liquid in a tank, and at the same time the motion of the liquid in the tank also imparts the hydrodynamic excitations to the motion of the ship. This is fully coupled problem. When we further assume the tank to be completely filled, i.e., no free surface in the tank, the effect of the liquid in the tank can be treated as a solid for the recti-linear motions of the ship: this particular case reduces to be trivial since there is no liquid motion in the tank with respect to the ship motions. However, when a ship undergoes angular motions, then the liquid in the tank can no longer to be treated as a solid for the computation of the moment of inertia of that.

It is well known that the study of the liquid in a ship was started by W. Froude in 1874 as he conducted the model test to see the effectiveness of free surface tank as a ship stabilizer. In 1910, Frahm introduced U-tube as a stabilizing device and it was widely used (Bhattacharyya, 1978). But in 1950's the free surface tank was revived and used in many naval vessels, because it has the added advantage varying the natural frequency of the tank by changing the water level and thus accommodating the changes in ship's metacentric height. During that period and after, the characteristics of free surface tank were investigated by many researchers; Verhagen and van Wijngaarden (1965) studied the non-linear hydrodynamics when the average height of

free surface was low so the free surface elevation to water depth ratio was high, and Van Den Bosch et al. (1965) gave the results of their studies on the performance of free surface tank as a stabilizer.

With the progress of aeronautics, many studies on the movements of fuel have been done. Especially in 1950's and 1960's, a lot of researches have been done on the control of missiles and rockets to take into account the effect of the fuel, of which the amount was diminishing and the mass and inertia had been changing continuously (Graham and Rodriguez, 1952; Abramson, 1966; Roberts et al., 1966). The main focus was on finding the resonance mode of fuel tank that has free surface, and finding the equivalent mass-spring-damper system that has the same hydrodynamic properties. The linear analysis was sufficient in the dynamics for the control of missiles and rockets (Ibrahim, 2005).

Currently, the researches on the liquid cargo are mainly focused on the analysis of the sloshing phenomena. Sloshing phenomenon did not cause problems in crude oil takers, but caused problems in purified oil tankers and LNG(liquefied natural gas) tankers. It has effects on the ship motion, and also damages on the top of tank because of the excessive pressure made by the hydrodynamics of free surface in a restricted area (Kim et al., 2013; Ahn et al., 2013). Other researches include sloshing under micro-gravity in space to be used in the field of the analysis of the space vehicles and the control of them (Helder, 2005) and sloshing in seismic conditions as the earthquake often demolish the liquid tank on land (Housner, 1954; Dogangun and Livaoglu, 2008).

When the free surface exists, as mentioned previously, the effects of liquid cargo have been taken into account by using the method of coupling the ship motion dynamics and liquid hydrodynamics, which are solved by one of the proven methods such as the equivalent mass-spring-damper dynamics, potential flow hydrodynamics and computational fluid dynamics (CFD). In the case of fully filled liquid, the liquid is treated as solid and is included in the ship's mass in many studies for the motion dynamics of ships. However, the inertia of liquid has difference compared with that of solid, especially in rotational acceleration. The inertia of the liquid in a fully filled tank was studied in this paper. The formulations on the liquid dynamics and inertia properties were solved, and the analysis of inertia properties has been done. As a result, for the rotational acceleration, the moment of inertia of liquid turns out to be small compared to that of solid, while the liquid acts like solid in recti-linear acceleration. The numerical solution was compared with the analytic solution, and the formula that accurately estimates the moment of inertia has been proposed.

INERTIA OF LIQUID

The inertia of material in a ship must be treated in different ways according to whether it is moving with the ship or not. In the case of moving with the ship, the ship's dynamic can be analyzed under the condition that the mass of that material is included in ship's mass. But in the case that the motion of material is different with that of the ship, the dynamics of the material and the ship must be solved concurrently and they must be coupled together to interact with each other. For example as in Fig.1 the material is on the deck of a ship and not fixed, if the downward acceleration is less than the gravitational acceleration it moves with the ship, but if the downward acceleration is greater than gravity the material fly and the mass of material is not included in the ship's mass. For another example as in Fig. 2, when the circular cylinder accelerates in rotation, the inner liquid does not accelerate if the viscosity is neglected. In this case, the moment of inertia comes only from the circular tank and not from the liquid in it.
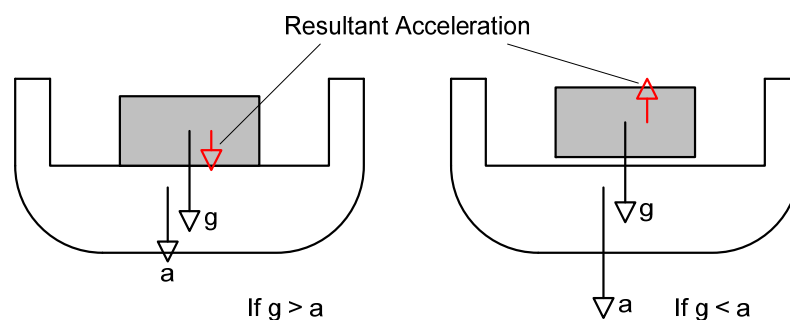

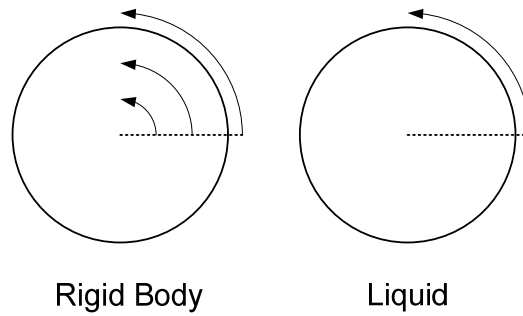Fig. 1 Resultant acceleration of rigid material on deck.

Fig. 2 Inner acceleration of circular cylinder.

Under recti-linear accelerations, the liquid moves with the tank as if the liquid is frozen as a solid attached to the ship , but for a rotational acceleration the liquid has a relative motion with respect to the tank, except for the trivial case of a circular cylinder discussed above. Therefore it is needed to investigate the inner flow of the liquid to obtain the inertial properties of the liquid.

Formulation

The potential flow model is assumed to analyze the inner flow of the tank. Fig. 3 shows the domain of the liquid and coordinates system.
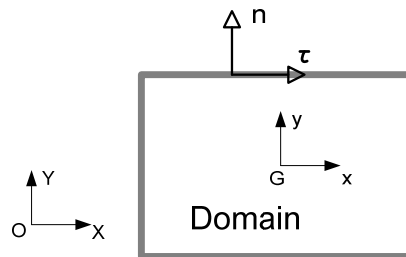

Fig. 3 Coordinates and domain.

The orientation of the normal vector is outward from the domain, and the governing equation that the flow should satisfy is the Laplace equation

$$\nabla^2 \phi = 0 \tag{1}$$

and the boundary conditions on the surface of the tank.

$$\frac{\partial \phi}{\partial n} = \vec{v}_B \cdot \vec{n} \tag{2}$$

The velocity potential is denoted by ϕ, $\vec{v}_B$ denotes the velocity of tank on the boundary, and $\vec{n} = (n_x, n_y)$ is the normal vector. The velocity of the tank can be written as follows.

$$\vec{v_B} = \vec{U} + \vec{\omega} \times (\vec{r_G} + \vec{r}) \tag{3}$$

where $\vec{U} = (U_x, U_y)$ is the recti-linear velocity, $\vec{\omega}$ the rotational velocity vector, $\vec{r}_G = (x_G, y_G)$ the position vector of the

center of gravity, and $\vec{r} = (x, y)$ the position vector from the center of gravity. The boundary condition in xy-plane can be written as follows.

$$\frac{\partial \phi}{\partial n} = U_x n_x + U_y n_y + \omega \left\{ x_G n_y - y_G n_x + \left(x n_y - y n_x \right) \right\}$$ (4)

The above problem is linear, so the linear superposition technique can be applied. Let us introduce the velocity potentials which satisfy the boundary conditions with unit velocity for each motion mode; ϕ_H for heave, ϕ_S for sway, ϕ_R for roll. The boundary conditions of heave, sway, and roll are:

$$\frac{\partial \phi_H}{\partial n} = n_y, \ \frac{\partial \phi_S}{\partial n} = n_x, \ \frac{\partial \phi_R}{\partial n} = x n_y - y n_x$$ (5)

Then the velocity potential that satisfies the boundary condition Eq. (4) can be represented as follows.

$$\phi = U_x \phi_S + U_y \phi_H + \omega \left(x_G \phi_H - y_G \phi_S + \phi_R \right)$$ (6)

The forces resulting from this flow can be expressed as in Newman (1977), chapter 4. Newman derived the formula of hydrodynamic forces to obtain added mass of a body in unbounded fluid. The same formula can be applied in the inner flow.

$$\vec{F} = -\rho \frac{d}{dt} \int_{S_B} \phi \vec{n} dS$$

$$\vec{M} = -\rho \frac{d}{dt} \int_{S_B} \phi \left(\vec{r} \times \vec{n} \right) dS$$ (7)

The above expressions can be rewritten into the forces in component.

$$F_x = \dot{U}_x a_{xS} - \dot{U}_y a_{xH} - \dot{\omega} \left(x_G a_{xH} - y_G a_{xS} + a_{xR} \right)$$

$$F_y = -\dot{U}_x a_{yS} - \dot{U}_y a_{yH} - \dot{\omega} \left(x_G a_{yH} - y_G a_{yS} + a_{yR} \right)$$ (8)

$$M_z = -\dot{U}_x \left(x_G a_{yS} - y_G a_{xS} + a_{RS} \right) - \dot{U}_y \left(x_G a_{yH} - y_G a_{xH} + a_{RH} \right)$$
$$- \dot{\omega} \left(x_G^2 a_{yH} - x_G y_G a_{xH} - x_G y_G a_{yS} + y_G^2 a_{xS} + x_G a_{yR} - y_G a_{xR} + x_G a_{RH} - y_G a_{RS} + a_{RR} \right)$$

The coefficients in the above equations are as follows.

$$a_{xS} = \rho \int_{S_B} \phi_S n_x dS , \quad a_{xH} = \rho \int_{S_B} \phi_H n_x dS , \quad a_{xR} = \rho \int_{S_B} \phi_R n_x dS$$

$$a_{yS} = \rho \int_{S_B} \phi_S n_y dS , \quad a_{yH} = \rho \int_{S_B} \phi_H n_y dS , \quad a_{yR} = \rho \int_{S_B} \phi_R n_y dS$$ (9)

$$a_{RS} = \rho \int_{S_B} \phi_S \left(x n_y - y n_x \right) dS , \quad a_{RH} = \rho \int_{S_B} \phi_H \left(x n_y - y n_x \right) dS , \quad a_{RR} = \rho \int_{S_B} \phi_R \left(x n_y - y n_x \right) dS$$

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The boundary conditions in Eq. (5) are all the normal boundary conditions, so the solution of the potential problem has an arbitrary constant. And if the potential is constant, the integrals above disappear. So the arbitrary constant can be omitted for the convenience.

Inertia forces

For the case of fully filled tank, the solutions of the above potential problem for heave and sway are the functions whose partial derivative in the direction of the motion mode are 1's. So the potentials are $\phi_H = y + C$ in heave, and $\phi_S = x + C$ in sway, where C is an arbitrary constant. Some of the coefficients in Eq. (9) can be integrated as follows.

$$a_{xS} = \rho \int_{S_B} \phi_S n_x dS = \rho \oint_{S_B} (x+C) dy = \rho S = m$$

$$a_{yH} = \rho \int_{S_B} \phi_H n_y dS = -\rho \oint_{S_B} (y+C) dx = \rho S = m \tag{10}$$

$$a_{xH} = a_{yS} = a_{RS} = a_{RH} = 0$$

where *m* is the mass of liquid in a tank. In obtaining the above results, Stokes theorem and the fact that the origin of the coordinates coincide with the center of gravity of the tank were used. The fact that ϕ_R is asymmetric in both directions *x* and *y*, and that the origin of the coordinates is the center of gravity gives the following properties.

$$a_{xR} = \rho \int_{S_B} \phi_R n_x dS = \rho \oint_{S_B} \phi_R dy = 0$$

$$a_{yR} = \rho \int_{S_B} \phi_R n_y dS = -\rho \oint_{S_B} \phi_R dx = 0 \tag{11}$$

Substitution of the above results of Eqs. (10) and (11) into Eq. (8) leads the following inertial force expression.

$$F_x = -m\dot{U}_x + my_G \dot{\omega}$$

$$F_y = -m\dot{U}_y - mx_G \dot{\omega} \tag{12}$$

$$M_z = my_G \dot{U}_x - mx_G \dot{U}_y - \left(mx_G^2 + my_G^2 + a_{RR} \right) \dot{\omega}$$

Eq. (12) shows that the mass matrix has a symmetric property, and it is identical with that of rigid dynamics except the roll moment of inertia a_{RR}. That is, except roll moment of inertia, the inertia of the liquid fully filled in a tank has the same property as rigid body.

For the roll moment of inertia, if the shape of the tank is circular, the boundary condition in Eq. (5) is vanishing on the boundary, so the potential becomes the arbitrary constant.

$$a_{RR} = \rho \int_{S_B} \phi_R \left(xn_y - yn_x \right) dS = \rho C \int_{S_B} \left(xn_y - yn_x \right) dS$$

$$= -\rho C \oint_{S_B} \left(xdx + ydy \right) = -\rho C \iint_D (-yx + xy) dxdy = 0 \tag{13}$$

If the shape of tank is not circular, the non-trivial flow takes place and non-zero moment of inertia comes from that flow.

ROLL MOMENT OF INERTIA

As seen in the previous section, the liquid in a circular cylinder has no roll moment of inertia provided that the origin of coordinates coincides with the center of gravity of that tank. For other than circular shape tank, it exists. Consider the roll moment of inertia for a rectangular shape tank.
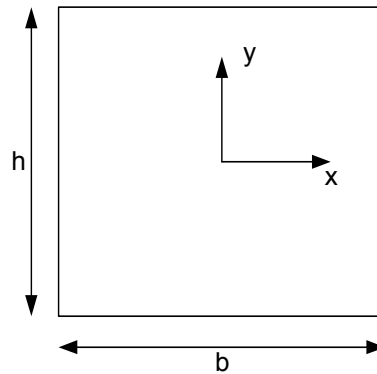
Rectangle

Fig. 4 Coordinates of rectangle.

When the rectangular tank undergoes roll motion, the boundary condition for the inner flow is as follows.

$$\frac{\partial \phi_R}{\partial n} = x n_y - y n_x$$

(5')

Condition on left boundary   : $-\partial \phi_R / \partial x = y$
Condition on right boundary   : $\partial \phi_R / \partial x = -y$
Condition on bottom boundary : $-\partial \phi_R / \partial y = -x$
Condition on top boundary   : $\partial \phi_R / \partial y = x$

From the above boundary condition, at least on the boundary, the potential has asymmetric properties. The analytic solution of this problem was obtained in Appendix A.

$$\phi_R = \sum_{n=0}^{\infty} \frac{4(-1)^n}{(2n+1)^3 \pi^3} \left\{ \frac{b^2 \sinh\left[(2n+1)\frac{\pi}{b}y\right]\sin\left[(2n+1)\frac{\pi}{b}x\right]}{\cosh\left[(2n+1)\frac{\pi}{2}\frac{h}{b}\right]} - \frac{h^2 \sinh\left[(2n+1)\frac{\pi}{h}x\right]\sin\left[(2n+1)\frac{\pi}{h}y\right]}{\cosh\left[(2n+1)\frac{\pi}{2}\frac{b}{h}\right]} \right\}$$

(14)

The moment of inertia was also obtained as follows.(see Appendix A.)

$$\frac{a_{RR}}{\rho} = -\frac{1}{12}\left(h^3 b + h b^3\right) + \sum_{n=0}^{\infty} \frac{32}{(2n+1)^5 \pi^5}\left\{ b^4 \tanh\left[(2n+1)\frac{\pi}{2}\frac{h}{b}\right] + h^4 \tanh\left[(2n+1)\frac{\pi}{2}\frac{b}{h}\right] \right\}$$

(15)

As seen in the above equation, the moment of inertia is symmetric with respect to height *h* and breadth *b*, i.e. it has the same value even if *h* and *b* are interchanged with each other. This symmetry relation with respect to *h* and *b* is not surprising since the

change of the coordinate (*x,y*) by (*y,x*) in the onset of the formulation does not change the problem at all. The ratio of moment of inertia between liquid and solid, C_R can be derived as follows.

$$a_{RR} = I_{Solid} C_R \tag{16}$$

$$C_R = -1 + \frac{32 \times 12}{\pi^5 (h/b)(1+(h/b)^2)} \times \sum_{n=0}^{\infty} \frac{1}{(2n+1)^5} \left\{ \tanh\left[(2n+1)\frac{\pi}{2}\frac{h}{b}\right] + (h/b)^4 \tanh\left[(2n+1)\frac{\pi}{2}\frac{b}{h}\right] \right\} \tag{17}$$

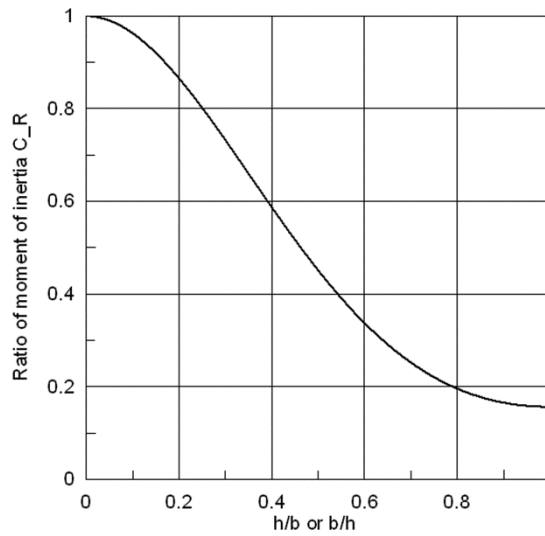Fig. 5 shows the ratio of moment of inertia C_R.

Fig. 5 Ratio of roll moment of inertia of liquid and solid rectangle.

From Fig. 5, it can be known that as the aspect ratio *h*/*b* or *b*/*h* goes to zero the ratio of moment of inertia goes to 1, that is, the moment of inertia of liquid has similar values to that of solid. And when *h*/*b* or *b*/*h* goes to 1, the moment of inertia of liquid becomes small. This means that the portion of liquid which moves together with the tank becomes small as rectangle goes square. This ratio of moment of inertia was also obtained by Graham and Rodriguez (1952) and cited in Roberts et al. (1966), Abramson (1966) as follows.

$$C_e = 1 - \frac{4}{r_1^2 + 1} + \frac{768}{\pi^5 r_1 (1+r_1^2)} \sum_{n=0}^{\infty} \frac{\tanh\left[(2n+1)\frac{\pi}{2} r_1\right]}{(2n+1)^5} \tag{18}$$

where $r_1 = h/b$. Even though expression is different with Eq. (17), but it gives exactly the same values.

The inner flow can also be obtained by using Green's 2nd identity.

$$\phi(P) = \int_S \left\{ \phi(Q) \frac{\partial G}{\partial n_Q}(P,Q) - \frac{\partial \phi}{\partial n}(Q) G(P,Q) \right\} dS(Q) \tag{19}$$

where *P* is the field point, and *Q* is the integral variable. $G(P,Q)$ is the fundamental solution of Laplace equation(often called

as a source potential in the field of hydrodynamics), i.e. natural logarithmic function of the distance between the field and source points $\log(|P-Q|)/2\pi$. Calculation procedure is; first, divide the boundary into line segments; second, take the integral over the line segment assuming that the potential value is constant along the line segment and can be represented by the value at the collocation point; third, obtain potential values on boundary by solving matrix equation. The moment of inertia can then be calculated by integrating it on the boundary. The details of the procedure are omitted.
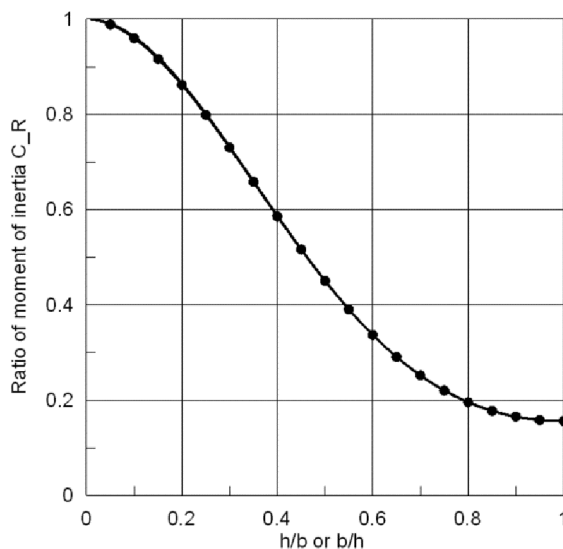

Fig. 6 Analytic value and numerical value of C_R.

The numerically obtained values using Eq. (19) are plotted as points in Fig.6, these values agree very well with the solid line, the analytic values. The number of line segments is about 100. This agreement validates the numerical method using Eq. (19). The analytic solution can be obtained only for the special shape such as rectangle. For the other shapes, there are no analytical solutions available; the numerical solutions are presented for these cases.
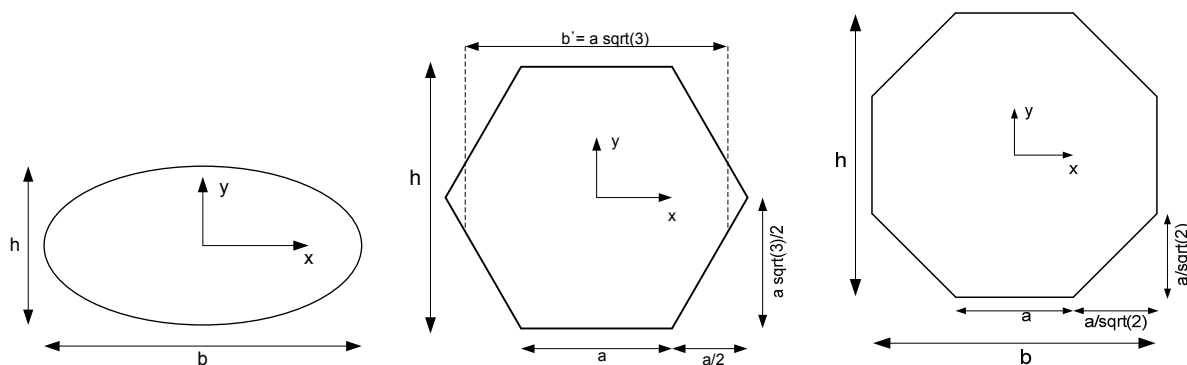
Other shapes

Fig. 7 Coordinates of ellipse, hexagon, and octagon.

Fig. 7 shows the shapes of ellipse, hexagon, and octagon and the coordinates. In the above figures, hexagon and octagon are regular shape, i.e. equiangular and equilateral shape. In this study, the shape was modified by scaling the *x* and *y* axis to fit the height *h* and breadth *b*. For hexagon, the effective breadth b' was introduced, so that the height and the effective breadth equal to each other if the shape is regular, i.e. if the aspect ratio is one. The areas and moment of inertias of solid were obtained as in Appendix C. Table 1 shows the results.

Table 1 Area and moment of inertia of solid for various shapes.

shape	Number of edge (n)	Area	Moment of inertia for roll
Rectangle	$n = 4$	$A = hb$	$\dfrac{I_{Solid}}{\rho} = \dfrac{1}{12} A\left(h^2 + b^2\right)$
Hexagon	$n = 6$	$A = \dfrac{\sqrt{3}}{2} hb'$	$\dfrac{I_{Solid}}{\rho} = \dfrac{5}{72} A\left(h^2 + b^2\right)$
Octagon	$n = 8$	$A = 2\left(\sqrt{2}-1\right)hb$	$\dfrac{I_{Solid}}{\rho} = \dfrac{3-\sqrt{2}}{24} A\left(h^2 + b^2\right)$
Ellipse	$n = \infty$	$A = \dfrac{\pi}{4} hb$	$\dfrac{I_{Solid}}{\rho} = \dfrac{1}{16} A\left(h^2 + b^2\right)$

Among these shapes, the moment of inertia of rectangle and ellipse are well known, but those of hexagon and octagon cannot be found even in famous textbooks, so the above formulas have been induced directly in Appendix C.

The ratios of moment of inertia for the above shapes were obtained numerically and plotted in Fig. 8.
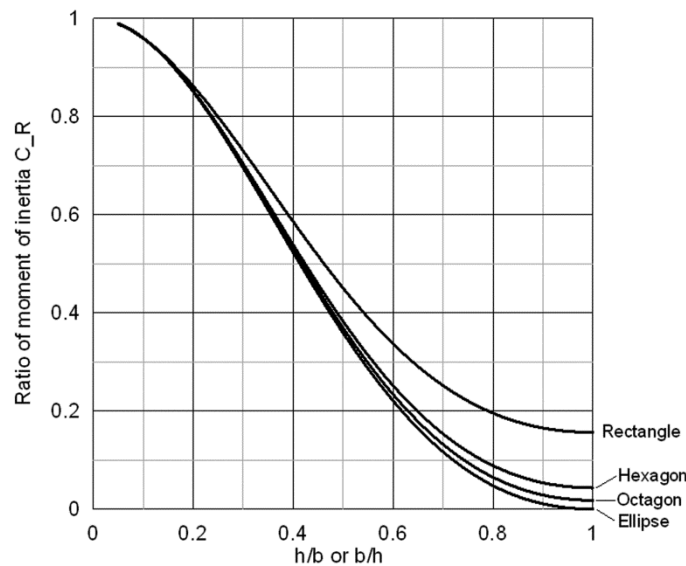

Fig. 8 Calculated C_R of various shapes.

Fig. 8 shows that the ratio of moment of inertia approaches to that of ellipse as the number of edges of tank goes large, i.e. the shape becomes ellipse. And as the aspect ratio h/b or b/h goes 1, the part of liquid moving with a tank becomes smaller and has the minimum value at the aspect ratio of 1. This is not surprising since our problem is symmetrical with respect to h and b as discussed before.

Estimation formula

In this section we would like to derive an ad hoc semi-analytical formula by introducing the equivalent inner circle in the various shapes we considered here. Our finding of the final approximate formula is obtained not by a rigorous analytical procedure but through numerical experiment based on the semi-analytical results. This is why we name the present formula by an ad hoc semi-analytical approximate formula. So let's think about an effective circle with radius r_e in which the liquid will not moving with the tank.

$$\pi r_e^2 = \text{some portion of } A$$

where A is the area under consideration. Fig. 8 shows several facts that the area of effective circle should satisfy: (1) the effective areas have a similar pattern except only in magnitude, (2) it is greatest for the shape of ellipse and have the same area with ellipse when h and b are the same, (3) its ratio with the total area start from 0 when $h/b = 0$ and to 1 when $h/b = 1$, (4) the slope of it tends to zero as h/b goes to 0 or 1. Let us consider the case of ellipse, and then the other shape to introduce a correlation factor with shape. For the shape of ellipse, the portion of effective area that reflects the above facts well may be represented by the factor $\sqrt{2hb/\left(h^2 + b^2\right)}$. So let us represent the effective area as follows.

$$\pi r_e^2 = A \times \sqrt{\frac{2hb}{h^2 + b^2}} \tag{20}$$

The moment of inertia of this circle that does not move with tank can be estimated as follows.

$$\frac{I_e}{\rho} = k_e \frac{\pi}{2} r_e^4 = k_e \frac{A^2}{\pi}\left(\frac{hb}{h^2 + b^2}\right) \tag{21}$$

where k_e is introduced as the correlation coefficient to fit the values to the calculated values, and/or to include the shape effect.

$$k_e = \left(\frac{A_{ellipse}}{A}\right)^{2/n} = \left(\frac{\pi hb}{4A}\right)^{2/n} \tag{22}$$

where n is the number of edges that the shape has. If we put the areas in Table 1 into the above formula, k_e turns out to be independent on h and b, and it is dependent only on the shape.

$$k_e = \begin{cases} \left(\pi/4\right)^{1/2} & \text{for rectangle} \\ \left(\pi/2\sqrt{3}\right)^{1/3} & \text{for hexagon} \\ \left(\dfrac{\pi}{8\left(\sqrt{2}-1\right)}\right)^{1/4} & \text{for octagon} \\ 1 & \text{for ellipse} \end{cases} \tag{23}$$

Therefore the moment of inertia of liquid can be estimated by following formula.

$$a'_{RR} = I_{Solid} - I_e = I_{Solid} - \rho k_e \frac{A^2}{\pi}\left(\frac{hb}{h^2 + b^2}\right) \tag{24}$$

And the ratio of moment of inertia is as follows.

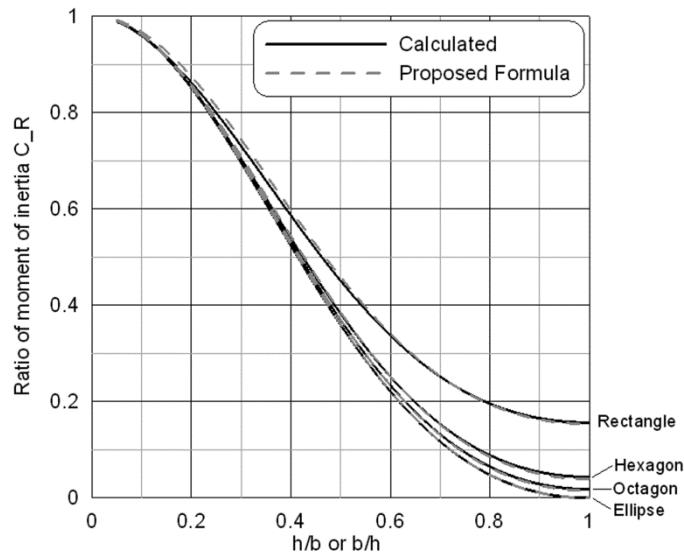$$C_R = \frac{a'_{RR}}{I_{Solid}} = 1 - \frac{I_e}{I_{Solid}} \tag{25}$$

Fig. 9 Calculated C_R and estimation formula.

Fig. 9 shows the calculated values and the estimation value by Eq. (24). Except small difference for rectangle, the proposed estimation formula gives the very accurate moment of inertia of liquid.
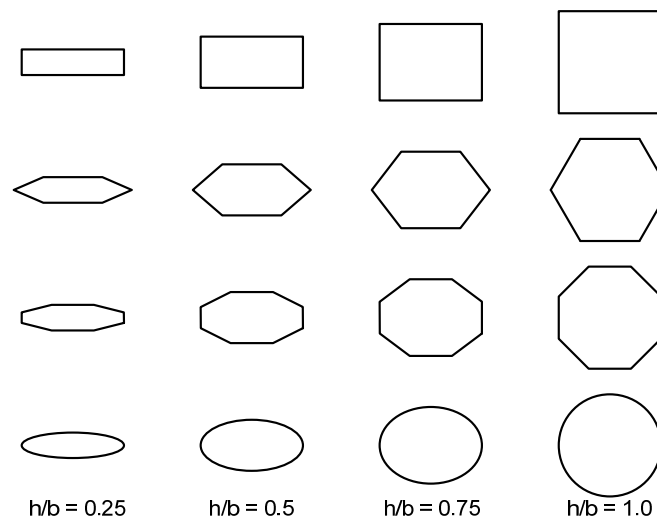

Fig. 10 Various shapes of tank useful for application.

The various shapes of tank that are useful for real application are shown in Fig. 10. The area, solid moment of inertia and moment of inertia of liquid can be calculated easily by using Table 1 and estimation formula Eq. (24). It might be useful in the application for the real ship.

CONCLUSIONS

In this paper, the inertia of the liquid fully filled in a tank is analyzed.

The inertia for the recti-linear acceleration is shown to be the mass of liquid. However for the rotational acceleration, the moment of inertia is smaller than that of solid. And the liquid in a tank whose aspect ratio of height and breadth is unity has the minimum moment of inertia compared to that of solid.

The flow and the inertia of liquid in a rectangular tank were derived analytically, and for other shape the numerical solutions were calculated. The shapes of tank investigated are ellipse, rectangle, hexagon, and octagon. And the results were given for

various aspect ratios of h/b or b/h, so they can be applied practically for the problem of liquid tank in a ship.

Also presented is ad hoc semi-analytical approximate formula for practical applications to compute the moment of inertia of the liquid in a tank. This formula gives extremely accurate predictions for the moment of inertia for several tank shapes considered in the paper.

The results of this study will be useful for the calculation of inertia of a ship which has large liquid tanks, especially for the ships like LNG/LPG carriers and purified oil tankers.

ACKNOWLEDGEMENTS

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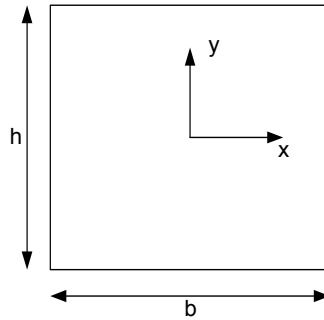APPENDIX A - Flow and moment of inertia of liquid in a rectangular tank

Fig. 4 Coordinates of Rectangle.

Let's introduce velocity potential to obtain the flow and moment of inertia of liquid in a tank shown above. The governing equation of the inner flow is Laplace equation.

$$\frac{\partial^2 \phi_R}{\partial x^2} + \frac{\partial^2 \phi_R}{\partial y^2} = 0 \qquad \text{(A-1)}$$

The boundary condition is

$$\frac{\partial \phi_R}{\partial n} = x n_y - y n_x \qquad \text{(A-2)}$$

Condition on left boundary   : $-\partial \phi_R / \partial x = y$
Condition on right boundary   : $\partial \phi_R / \partial x = -y$
Condition on bottom boundary   : $-\partial \phi_R / \partial y = -x$
Condition on top boundary   : $\partial \phi_R / \partial y = x$

From the above boundary condition, at least on the boundary, the potential has asymmetric properties. Fourier series that has asymmetry and is continuous at boundary is introduced as follows in Appendix B.

$$\phi_R = \sum_{n=0}^{\infty} \left(a_{1n} e^{k_{bn} y} + a_{2n} e^{-k_{bn} y} \right) \sin k_{bn} x + \sum_{m=0}^{\infty} \left(b_{1m} e^{k_{hm} x} + b_{2m} e^{-k_{hm} x} \right) \sin k_{hm} y$$

$$k_{bn} = \left(2n + 1 \right) \frac{\pi}{b} \qquad \text{(A-3)}$$

$$k_{hm} = \left(2m + 1 \right) \frac{\pi}{h}$$

At the top and bottom boundary, the second term vanishes. The boundary condition at top and bottom tells that the derivative of the first term with respect to *y* should be symmetric. And from the left and right boundary condition, the derivative of the second term w.r.t *x* axis should be symmetric, therefore the coefficients should satisfy the following condition.

$$a_{2n} = -a_{1n}, \quad b_{2n} = -b_{1n}$$

And the Fourier series turns out as follows.

$$\phi_R = \sum_{n=0}^{\infty} a_n \sinh k_{bn} y \sin k_{bn} x + \sum_{m=0}^{\infty} b_m \sinh k_{hm} x \sin k_{hm} y \tag{A-4}$$

At the top and bottom boundary, the potential should satisfy the following condition.

$$\frac{\partial \phi_R}{\partial y} = \sum_{n=0}^{\infty} a_n k_{bn} \cosh k_{bn} h / 2 \sin k_{bn} x = x \tag{A-5}$$

And at the left and right boundary, the potential should satisfy the following condition.

$$\frac{\partial \phi_R}{\partial x} = \sum_{m=0}^{\infty} b_m k_{hm} \cosh k_{hm} b / 2 \sin k_{hm} y = -y \tag{A-6}$$

From the above two conditions, the coefficients in eq.(A-4) can be obtained as follows by using the orthogonal property of the series.

$$\frac{b}{2} a_n k_{bn} \cosh k_{bn} h / 2 = \int_{-b/2}^{b/2} x \sin k_{bn} x dx = (-1)^n \frac{2}{k_{bn}^2}$$

$$\therefore a_n = (-1)^n \frac{4}{bk_{bn}^3 \cosh k_{bn} h / 2} = (-1)^n \frac{4b^2}{(2n+1)^3 \pi^3 \cosh\left[(2n+1)\frac{\pi h}{2b}\right]} \tag{A-7}$$

$$\frac{h}{2} b_m k_{hm} \cosh k_{hm} b / 2 = -\int_{-h/2}^{h/2} y \sin k_{hm} y dy = -(-1)^m \frac{2}{k_{hm}^2}$$

$$\therefore b_m = -(-1)^m \frac{4}{hk_{hm}^3 \cosh k_{hm} b / 2} = -(-1)^m \frac{4h^2}{(2m+1)^3 \pi^3 \cosh\left[(2m+1)\frac{\pi b}{2h}\right]} \tag{A-8}$$

The velocity potential we seek turns out to be

$$\phi_R = \sum_{n=0}^{\infty} \frac{4(-1)^n}{(2n+1)^3 \pi^3} \left\{ \frac{b^2 \sinh\left[(2n+1)\frac{\pi}{b} y\right] \sin\left[(2n+1)\frac{\pi}{b} x\right]}{\cosh\left[(2n+1)\frac{\pi h}{2b}\right]} - \frac{h^2 \sinh\left[(2n+1)\frac{\pi}{h} x\right] \sin\left[(2n+1)\frac{\pi}{h} y\right]}{\cosh\left[(2n+1)\frac{\pi b}{2h}\right]} \right\} \tag{A-9}$$

The moment of inertia can be obtained.

$$\frac{a_{RR}}{\rho} = \int_{S_B} \phi_R \left(x n_y - y n_x\right) dS = \int_{-b/2}^{b/2} \left(\phi_{y=h/2} - \phi_{y=-h/2}\right) x dx - \int_{-h/2}^{h/2} \left(\phi_{x=b/2} - \phi_{x=-b/2}\right) y dy$$

Substitute Eq. (A-9) into above equation,

$$\frac{a_{RR}}{\rho} = \sum_{n=0}^{\infty} \frac{4(-1)^n}{(2n+1)^3 \pi^3} \left\{ \begin{array}{l} 2b^2 \tanh\left[(2n+1)\frac{\pi}{2}\frac{h}{b}\right]\int_{-b/2}^{b/2} x\sin\left[(2n+1)\frac{\pi}{b}x\right]dx - \frac{(-1)^n 2h^2}{\cosh\left[(2n+1)\frac{\pi}{2}\frac{b}{h}\right]}\int_{-b/2}^{b/2} x\sinh\left[(2n+1)\frac{\pi}{h}x\right]dx \\ -\frac{(-1)^n 2b^2}{\cosh\left[(2n+1)\frac{\pi}{2}\frac{h}{b}\right]}\int_{-h/2}^{h/2} y\sinh\left[(2n+1)\frac{\pi}{b}y\right]dy + 2h^2 \tanh\left[(2n+1)\frac{\pi}{2}\frac{b}{h}\right]\int_{-h/2}^{h/2} y\sin\left[(2n+1)\frac{\pi}{h}y\right]dy \end{array} \right\}$$

Perform the integration gives

$$\frac{a_{RR}}{\rho} = -\frac{1}{12}\left(h^3 b + hb^3\right) + \sum_{n=0}^{\infty} \frac{32}{(2n+1)^5 \pi^5} \left\{ b^4 \tanh\left[(2n+1)\frac{\pi}{2}\frac{h}{b}\right] + h^4 \tanh\left[(2n+1)\frac{\pi}{2}\frac{b}{h}\right] \right\} \qquad \text{(A-10)}$$

APPENDIX B - Complete Fourier series for asymmetry

Let's consider a function defined in the interval L.

$$f = f(x), \quad -L/2 < x < L/2 \qquad \text{(B-1)}$$

The Fourier series representation of this function is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos k_n x + b_n \sin k_n x\right)$$

$$\text{(B-2)}$$

$$k_n = n\frac{2\pi}{L}$$

The function set $(1, \cos k_n x, \sin k_n x)$ used above is orthogonal set in the interval $(-L/2 < x < L/2)$. So the coefficients can be obtained as follows.

$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos k_n x\, dx, \quad b_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin k_n x\, dx \qquad \text{(B-3)}$$

This Fourier series i.e. set $(1, \cos k_n x, \sin k_n x)$ is the complete orthogonal set in the interval $(-L/2 < x < L/2)$. The term "complete" means that the Fourier series approaches unlimitedly to $f(x)$, if the number of terms goes large to infinity. And the term "orthogonal" means that the multiple of one item and another item vanishes if we integrate it in the interval $(-L/2 < x < L/2)$. Therefore the above formulas of the coefficients can be derived.

If $f(x)$ is symmetric, all b_n's vanish and the series consists of a constant and cosine functions, i.e. Fourier cosine series turns out. And if $f(x)$ is asymmetric, the series turns out Fourier sine series.

However, at the boundary, the values of all sine functions vanish while cosine function does not. And the Fourier series have repeat property, so the value of the series has a jump at boundary. If $f(x)$ is asymmetric, the Fourier sine series has a value of zero at boundary. If one want to have a series of asymmetric function whose value have no discontinuity at boundary, the above series is not adequate.

Orthogonal series for asymmetry

Assume that we treat the asymmetric function, and that we want series that is continuous at boundary. The main reason of discontinuity is the fact that all the above sine functions have value zero at boundary. So stretch the interval twice from *L* to *2l*, then the Fourier sine series comes out as follows.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin k_n x$$

$$k_n = n\frac{\pi}{l}$$

(B-4)

In order that this series to have non-zero values at boundary and become orthogonal in the interval ($-l < x < l$), n should be odd number. Therefore the complete orthogonal Fourier sine series in the interval ($-l < x < l$) becomes

$$f(x) = \sum_{n=0}^{\infty} a_n \sin(2n+1)kx$$

$$a_n = \frac{2}{l} \int_{-l/2}^{l/2} f(x)\sin(2n+1)kx\,dx$$

(B-5)

$$k = \frac{\pi}{l}$$

APPENDIX C - Area and moment of inertia for some shapes

The area and polar moment of inertia can be obtained by the following expression.

$$A = \int dx\,dy = \oint x\,dy$$

$$I_P = \int (x^2 + y^2)\,dA = \int x^2\,dA + \int y^2\,dA = I_{xx} + I_{yy}$$

(C-1)

$$I_{xx} = \int x^2\,dx\,dy = \frac{1}{3}\oint x^3\,dy, \quad I_{yy} = \int y^2\,dx\,dy = \oint xy^2\,dy$$

where the integration $\int dx\,dy$ should be carried out for the inner domain, and the line integral should be carried out along the boundary counter clockwise.

For the case of axis scaling, the area and moment of inertia should be changed as follows.

x-axis $\qquad r_x A, \ r_x^3 I_{xx}, \ r_x I_{yy}$

y-axis $\qquad r_y A, \ r_y I_{xx}, \ r_y^3 I_{yy}$

where r_x and r_y are the scale ratios of x and y axis. If we set total breadth as b, the total height as h, then the area and moment of inertia could be obtained as follows.

Ellipse

$$A = \pi a_1 a_2 = \frac{\pi}{4} hb$$

$$(C\text{-}2)$$

$$I_P = \frac{1}{4} A \left(a_1^2 + a_2^2 \right) = \frac{1}{16} A \left(h^2 + b^2 \right)$$

Rectangle

$$A = \oint x\,dy = \int_{=h/2}^{h/2} \frac{h}{2}\,dy + \int_{h/2}^{-h/2} \left(-\frac{h}{2} \right) dy = h^2$$

$$I_{xx} = \frac{1}{3} \oint x^3\,dy = \frac{1}{3} \left[\int_{-h/2}^{h/2} \frac{h^3}{8}\,dy + \int_{h/2}^{-h/2} \left(-\frac{h^3}{8} \right) dy \right] = \frac{1}{3} \frac{h^3}{8} 2h = \frac{1}{12} h^4$$

$$I_{yy} = \oint xy^2\,dy = \int_{-h/2}^{h/2} \frac{h}{2} y^2\,dy + \int_{h/2}^{-h/2} \left(-\frac{h}{2} \right) y^2\,dy = \frac{1}{3} \left[\frac{h}{2} \cdot \frac{h^3}{8} \cdot 2 \cdot 2 \right] = \frac{1}{12} h^4$$
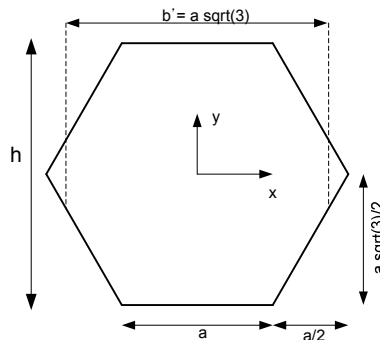
The breadth is b, so the x axis scale ratio is b/h.

$$A = hb$$

$$I_{xx} = \frac{1}{12} hb^3, \quad I_{yy} = \frac{1}{12} h^3 b$$

$$(C\text{-}3)$$

$$I_P = \frac{1}{12} hb \left(h^2 + b^2 \right) = \frac{1}{12} A \left(h^2 + b^2 \right)$$

Hexagon

The area and moment of inertia can be calculated as follows for regular hexagon.

$$A = \oint x\,dy = 2\left[\int_{-a\sqrt{3}/2}^{0}\left(a + \frac{1}{\sqrt{3}y}\right)dy + \int_{0}^{a\sqrt{3}/2}\left(a - \frac{1}{\sqrt{3}y}\right)dy\right] = 2\left[\frac{3a}{2}\cdot a \cdot \frac{\sqrt{3}}{2}\right] = \frac{3\sqrt{3}}{2}a^2$$

$$I_{xx} = \frac{1}{3}\oint x^3\,dy = \frac{2}{3}\left[\int_{-a\sqrt{3}/2}^{0}\left(a + \frac{1}{\sqrt{3}}y\right)^3 dy + \int_{0}^{a\sqrt{3}/2}\left(a - \frac{1}{\sqrt{3}y}\right)^3 dy\right]$$

$$= \frac{2}{3}\left[\int_{0}^{a\sqrt{3}/2}\left(\frac{9}{8}a^3 - \frac{9}{4\sqrt{3}}a^2 y + \frac{3}{2}ay^2\right)dy\right] = \frac{5\sqrt{3}}{16}a^4$$

$$I_{yy} = \oint x y^2\,dy = 2\left[\int_{-a\sqrt{3}/2}^{0}\left(a + \frac{1}{\sqrt{3}}y\right)y^2 dy + \int_{0}^{a\sqrt{3}/2}\left(a - \frac{1}{\sqrt{3}}y\right)y^2 dy\right]$$

$$= 2\left[\int_{0}^{a\sqrt{3}/2}\left(\frac{3}{8}a^3 - \frac{\sqrt{3}}{4}a^2 y + \frac{1}{2}ay^2\right)dy\right] = \frac{5\sqrt{3}}{16}a^4$$

In terms of *h* and *b*, the area and moment of inertia can be written as,

$$A = \frac{3\sqrt{3}}{2}a^2 = \frac{3}{4}hb$$

$$I_{xx} = \frac{5\sqrt{3}}{16}a^4 = \frac{5}{128}hb^3, \quad I_{yy} = \frac{5\sqrt{3}}{16}a^4 = \frac{5}{96}h^3 b$$

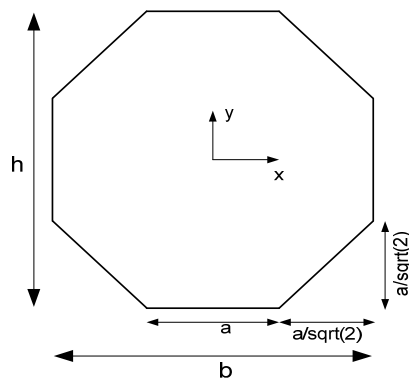For effective breadth,

$$b' \equiv \sqrt{3}a = \frac{\sqrt{3}}{2}b$$

$$A = \frac{3\sqrt{3}}{2}a^2 = \frac{\sqrt{3}}{2}hb'$$

(C-4)

$$I_{xx} = \frac{5\sqrt{3}}{16}a^4 = \frac{5\sqrt{3}}{144}hb'^3 = \frac{5}{72}Ab'^2, \quad I_{yy} = \frac{5\sqrt{3}}{16}a^4 = \frac{5\sqrt{3}}{144}h^3 b' = \frac{5}{72}Ah^2$$

$$I_P = \frac{5}{72}A\left(h^2 + b'^2\right)$$

Octagon

The area and moment of inertia can be calculated as follows for regular octagon.

$$A = \oint x\,dy$$

$$= 2\left[\int_{-a(1+\sqrt{2})/2}^{-a/2} \left(a + \frac{a}{\sqrt{2}} + y \right) dy + \int_{-a/2}^{a/2} \left(\frac{a}{2} + \frac{a}{\sqrt{2}} \right) dy + \int_{a/2}^{a(1+\sqrt{2})/2} \left(a + \frac{a}{\sqrt{2}} - y \right) dy \right]$$

$$= 2\left(1 + \sqrt{2}\right)a^2$$

$$I_{xx} = \frac{1}{3} \oint x^3\,dy$$

$$= \frac{2}{3}\left[\int_{-a(1+\sqrt{2})/2}^{-a/2} \left(a + \frac{a}{\sqrt{2}} + y \right)^3 dy + \int_{-a/2}^{a/2} \left(\frac{a}{2} + \frac{a}{\sqrt{2}} \right)^3 dy + \int_{a/2}^{a(1+\sqrt{2})/2} \left(a + \frac{a}{\sqrt{2}} - y \right)^3 dy \right]$$

$$= \frac{2}{3}\left[\frac{1}{2}\left\{ \left(\frac{1}{2}\left(1+\sqrt{2}\right) \right)^4 - \frac{1}{2^4} \right\} + \frac{1}{2^3}\left(1+\sqrt{2}\right)^3 \right]a^4 = \frac{1}{12}\left(11 + 8\sqrt{2}\right)a^4$$

$$I_{yy} = \oint x y^2\,dy$$

$$= 2\left[\int_{-a(1+\sqrt{2})/2}^{-a/2} \left(a + \frac{a}{\sqrt{2}} + y \right) y^2\,dy + \int_{-a/2}^{a/2} \left(\frac{a}{2} + \frac{a}{\sqrt{2}} \right) y^2\,dy + \int_{a/2}^{a(1+\sqrt{2})2} \left(a + \frac{a}{\sqrt{2}} - y \right) y^2\,dy \right]$$

$$= 2\left[\int_{-a/2}^{a/2} \left(\frac{a}{2} + \frac{a}{\sqrt{2}} \right) y^2\,dy + 2\int_{a/2}^{a(1+\sqrt{2})/2} \left(a + \frac{a}{\sqrt{2}} - y \right) y^2\,dy \right] = \frac{1}{12}\left(11 + 8\sqrt{2}\right)a^4$$

In terms of *h* and *b*, the area and moment of inertia can be written as,

$$A = 2\left(1+\sqrt{2}\right)a^2 = 2\left(\sqrt{2}-1\right)hb$$

$$I_{xx} = \frac{1}{12}\left(11+8\sqrt{2}\right)a^4 = \frac{1}{12}\frac{\left(11+8\sqrt{2}\right)}{\left(1+\sqrt{2}\right)^4}b^3 h = \frac{A}{24}\left(3-\sqrt{2}\right)b^2$$

(C-5)

$$I_{yy} = \frac{1}{12}\left(11+8\sqrt{2}\right)a^4 = \frac{1}{12}\frac{\left(11+8\sqrt{2}\right)}{\left(1+\sqrt{2}\right)^4}h^3 b = \frac{A}{24}\left(3-\sqrt{2}\right)h^2$$

$$I_P = \frac{A}{24}\left(3-\sqrt{2}\right)\left(h^2+b^2\right)$$