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Quantile regression with errors in variables[†]

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Abstract

Quantile regression models with errors in variables have received a great deal of attention in the social and natural sciences. Some efforts have been devoted to develop effective estimation methods for such quantile regression models. In this paper we propose an orthogonal distance quantile regression model that effectively considers the errors on both input and response variables. The performance of the proposed method is evaluated through simulation studies.

Keywords: Check function, errors in variables, iteratively reweighted least squares procedure, orthogonal distance regression, quantile regression.

1. Introduction

Quantile regression (QR) introduced by Koenker and Bassett (1978) provides a more informative description of relationships between input variables and response variables. Just as classical linear regression methods based on minimizing sum of squared residuals enable us to estimate a wide variety of models for conditional mean functions, quantile regression methods offer a mechanism for estimating models for the full range of conditional quantile functions including the conditional median function. By supplementing the estimation of conditional mean functions with techniques for estimating an entire family of conditional quantile functions, quantile regression is capable of providing a better statistical analysis of the stochastic relationships among random variables. The introductions and applications of the quantile regression can be found in Koenker *et al.* (1994), Yu *et al.* (2003), Koenker (2005), Hwang (2010), Shim and Lee (2010), Lee (2012).

Most of this attention has been paid to data measured exactly without error but QR analysis with errors in variables (EIV) is evolving, albeit slowly. See, for example, Boggs *et al.* (1987), Fuller (1987), He and Liang (2000), Chesher (2001), Schennach (2008), Hu and Schennach (2008), Ioannidesa and Matzner-Lϕber (2009), Wei and Carroll (2009), Ma and Yin (2011), Wang *et al.* (2012). However, less attention has been paid to QR with EIV than to mean regression with EIV because of two main difficulties for correcting the bias in QR caused by EIV (Wang *et al.*, 2012). One is that a parametric regression-error likelihood is usually not specified in QR. The other is that the quantile of the sum of two random

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variables is not necessarily the sum of the two marginal quantiles. In addition, most literature has centered around the parametric approach in which the quantile regression function is assumed to take on a particular functional form.

In this paper we consider an orthogonal distance quantile regression model (ODQR) with EIV by applying check function of orthogonal residuals to the formulation of quantile regression of Koenker and Basset (1978). Like He and Liang (2000), the proposed ODQR assumes that the random errors in the response variable and the measurement errors in the input variables follow the same symmetric distribution but the input variable unobserved can be estimated in the proposed QR with EIV. We use the weighted squared loss function in EIV objective function and use the iteratively reweighted least squares procedure to estimate the regression quantiles and input variables unobserved simultaneously.

The rest of this paper is organized as follows. In Section 2 we briefly review the basic principle of orthogonal distance regression. In Section 3 we propose new estimation procedure of ODQR with the weighted squared loss function in EIV objective function. In Section 4 and 5 we present simulation studies and conclusion, respectively.

2. Orthogonal distance regression

We first briefly review the basic principle of orthogonal distance regression (ODR). A more detail description can be found in Boggs and Rogers (1990) and some other related references. Then we apply this principle to ODQR in the next section. We first consider the mean regression model with EIV which is defined as follows:

$$\begin{cases} y_i = f(\boldsymbol{x}_i^*) + e_i = \boldsymbol{\beta}'\boldsymbol{x}_i^* + b + e_i \\ \boldsymbol{x}_i = \boldsymbol{x}_i^* + \boldsymbol{u}_i, i = 1, \cdots, n, \end{cases} \tag{2.1}$$

where $\boldsymbol{x}_i^*$ is $d \times 1$ vector of unknown and unobservable input vector, $\boldsymbol{u}_i$ is $d \times 1$ vector of measurement errors and $\boldsymbol{x}_i$ is $d \times 1$ vector of observed input vector. Here, the random errors e_i's are independent and identically distributed with zero mean and finite variance σ^2. The measurement errors $\boldsymbol{u}_i$'s are independent and identically distributed with zero mean and covariance matrix Σ_u. We assume that e_i's and $\boldsymbol{u}_i$'s are uncorrelated. The recent literature has become aware of the inadequacy of assumption that e_i's and $\boldsymbol{u}_i$'s have a joint distribution that is spherically symmetric, and proposes the use of general orthogonal distance regression method that accounts for different uncertainties of the two types of errors. For basic ODR, it is usually assumed that Σ_u is a diagonal matrix, i.e., $\Sigma_u = \sigma^2 \mathrm{I}_{d \times d}$. For convenience of illustration, we assume that $\boldsymbol{x}_i^*$'s are fixed unknown. Then the ODR problem with EIV objective function can be expressed as

$$L = \sum_{i=1}^{n} (y_i - \boldsymbol{\beta}'\boldsymbol{x}_i^* - b)^2 + \sum_{i=1}^{n} \sum_{j=1}^{d} (x_{ij} - x_{ij}^*)^2 \tag{2.2}$$

It is proven that this EIV objective function is less prone to bias effects when training with noisy inputs (Van Gorp *et al.*, 2000). However, since both $(\boldsymbol{\beta}, b)$ and x_{ij}'s form the unknown parameters, the parameter space enlarges and thus has $n \times d$ additional parameters. Moreover, since $\boldsymbol{x}_i^*$'s in a real world problem are usually not known, these must be estimated during the optimization process. Thus, ODR problem (2.2) is rewritten by replacing the above EIV objective function with that without $\boldsymbol{x}_i^*$ therein such as

$$L = \sum_{i=1}^{n} \frac{(y_i - \boldsymbol{\beta}'\boldsymbol{x}_i - b)^2}{1 + \boldsymbol{\beta}'\boldsymbol{\beta}} \tag{2.3}$$

In example of $d = 1$ the objective function (2.2) and (2.3) are equal to the squared distance between (x_i, y_i) and $(x_i^*, \beta x_i^* + b)$, which can be easily seen in Figure 2.1. Squared orthogonal distance between (x_i, y_i) and the line $y = \beta x + b$ is the sum of the squared distance between (x_i, y_i) and $(x_i, \beta x_i^* + b)$ and the squared distance between $(x_i, \beta x_i^* + b)$ and $(x_i^*, \beta x_i^* + b)$, which is $(y_i - \beta x_i^* - b)^2 + (x_i - x_i^*)^2$.

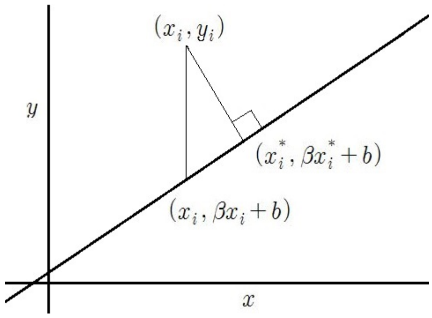

Figure 2.1 Orthogonal distance between (x_i, y_i) and the line $y = \beta x + b$

3. Orthogonal distance quantile regression

We set $\boldsymbol{x}_i^*$ is $d \times 1$ vector of unknown and unobservable input vector, $\boldsymbol{u}_i$ is $d \times 1$ vector of measurement errors and $\boldsymbol{x}_i$ is $d \times 1$ vector of observed input vector. Here, the random errors e_i's and the measurement errors u_{ij}'s are assumed to be independent and identically distributed with zero mean and finite variance σ^2.

He and Liang (2000) proposed the quantile regression in EIV models by using the check function ρ_θ instead of the squared loss function (2.3) as follows:

$$\min L = \sum_{i=1}^{n} \rho_\theta \left(\frac{y_i - \boldsymbol{\beta}'\boldsymbol{x}_i - b}{\sqrt{1 + \boldsymbol{\beta}^2}} \right), \tag{3.1}$$

where $\rho_\theta(r) = \theta r I_{(r>0)} + (1-\theta) r I_{(r \le 0)}$. The objective function (3.1) can be interpreted as the sum of check functions of orthogonal residuals $(y_i - \boldsymbol{\beta}'\boldsymbol{x}_i - b)/\sqrt{1 + \boldsymbol{\beta}^2}$, which is also equal to

$$\begin{aligned} \sum_{i=1}^{n} [\theta 1_{(y_i > \boldsymbol{\beta}'\boldsymbol{x}_i + b)} &\times \text{distance between } (\boldsymbol{x}_i, y_i) \text{ and } y = \boldsymbol{\beta}'\boldsymbol{x}_+ b \\ &+ (1-\theta) 1_{(y_i \le \boldsymbol{\beta}'\boldsymbol{x}_i + b)} \times \text{distance between } (\boldsymbol{x}_i, y_i) \text{ and } y = \boldsymbol{\beta}'\boldsymbol{x}_+ b] \end{aligned} \tag{3.2}$$

Here the distance between $(\boldsymbol{x}_i, y_i)$ and $y = \boldsymbol{\beta}'\boldsymbol{x} + b$ is obtained in (2.2). We proposed the quantile regression in EIV models using (2.2) and (3.2) as follows:

$$\min L = \sum_{i=1}^{n} W_i(\theta)[(y_i - \boldsymbol{\beta}'\boldsymbol{x}_i^* - b)^2 + \sum_{j=1}^{d}(x_{ij} - x_{ij}^*)^2] \tag{3.3}$$

where

$$W_i(\theta) = \frac{\theta}{\sqrt{(y_i - \boldsymbol{\beta}'\boldsymbol{x}_i^* - b)^2 + \sum_{j=1}^{d}(x_{ij} - x_{ij}^*)}} 1_{(y_i > \boldsymbol{\beta}'\boldsymbol{x}_i + b)} + \frac{1-\theta}{\sqrt{(y_i - \boldsymbol{\beta}'\boldsymbol{x}_i^* - b)^2 + \sum_{j=1}^{d}(x_{ij} - x_{ij}^*)}} 1_{(y_i \le \boldsymbol{\beta}'\boldsymbol{x}_i + b)}.$$

The advantage of the proposed quantile regression in EIV models of (3.3) has advantage over (3.1) that not only the estimates of $(\boldsymbol{\beta}, b)$ but also the estimates of $\boldsymbol{x}_i^*$'s can be obtained, which leads to have the estimate of $\boldsymbol{\beta}'\boldsymbol{x}_i^* + b$, where $\boldsymbol{x}_i^*$ was not observed for training data.

Given $W_i(\theta)$ taking partial derivatives of (3.3) with regard to $(\boldsymbol{\beta}, b, x_{ij}^*)$ leads to the optimal values of $(\boldsymbol{\beta}, b, x_{ij}^*)$ to be the solution to

$$\frac{\partial L}{\partial \boldsymbol{\beta}} = \boldsymbol{x}'W(\theta)(\boldsymbol{y} - \boldsymbol{\beta}'\boldsymbol{x}^* - b) = \mathbf{0}_{d\times 1}, \ \frac{\partial L}{\partial \beta_0} = \mathbf{1}'W(\theta)(\boldsymbol{y} - \boldsymbol{\beta}'\boldsymbol{x}^* - b) = 0 \tag{3.4}$$

$$\frac{\partial L}{\partial x_{ij}^*} = W_i(\theta)(y_i - \boldsymbol{\beta}'\boldsymbol{x}_i^* - b)\beta_j + (x_{ij} - x_{ij}^*) = 0, \ i = 1, \cdots, n, \ j = 1, \cdots, d, \tag{3.5}$$

where $W(\theta)$ is a $n \times n$ diagonal matrix of $W_i(\theta)$'s.

The solution to (3.4) and (3.5) cannot be obtained in a single step since $W_i(\theta)$ contains $(\boldsymbol{\beta}, b, x_{ij}^*)$ and is unknown. Thus we need to apply the iteratively reweighted least squares (IRWLS) procedure as follows:

(i) With previous estimates of x_{ij}^* and initialized values of $(\boldsymbol{\beta}, b)$

- (i-1) Find $W(\theta)$ with $(\boldsymbol{\beta}, b)$.
- (i-2) Find $(\boldsymbol{\beta}, b)$ from $\begin{pmatrix}\boldsymbol{\beta}\\ b\end{pmatrix} = \begin{pmatrix}\boldsymbol{x}^{*\prime}W(\theta)\boldsymbol{x}^* & \boldsymbol{x}^*W(\theta)\mathbf{1}\\ \mathbf{1}'W(\theta)\boldsymbol{x}^* & \mathbf{1}'W(\theta)\mathbf{1}\end{pmatrix}^{-1}\begin{pmatrix}\boldsymbol{x}^{*\prime}W(\theta)\\ \mathbf{1}'W(\theta)\end{pmatrix}\boldsymbol{y}$.
- (i-3) Reiterate above steps until convergence such that $||\boldsymbol{\beta}^{(t)} - \boldsymbol{\beta}^{(t-1)}||^2 < \text{Tol}_1$.

(ii) Find new estimates of x_{il}^* from (3.5) with $(\boldsymbol{\beta}, b)$.

(iii) Reiterate steps (i) and (ii) until convergence such that $||\boldsymbol{x}^{*(t)} - \boldsymbol{x}^{*(t-1)}||^2 < \text{Tol}_2$.

With estimates of $(\boldsymbol{\beta}, b, x_{ij}^*)$ from IRWLS procedure above, the estimator of the θ th quantile regression function given the input vector $\boldsymbol{x}_i^*$ (not observed, instead $\boldsymbol{x}_i$ was observed for training data) is obtained as follows:

$$\widehat{q}_\theta(y|\boldsymbol{x}_i^*) = \widehat{\boldsymbol{\beta}}'\widehat{\boldsymbol{x}}_i^* + \widehat{b}, \tag{3.6}$$

and the estimator of the θ th quantile regression function given new input vector $\boldsymbol{x}_t$ which was not used for the estimation of $(\boldsymbol{\beta}, b, x_{ij}^*)$ is obtained as follows:

$$\widehat{q}_\theta(y|\boldsymbol{x}_t) = \widehat{\boldsymbol{\beta}}'\boldsymbol{x}_t + \widehat{b}. \tag{3.7}$$

For 1-dimensional input variable we can extend the estimation procedure of ODQR explained above to the polynomial regression case. The squared orthogonal distance between (x_i, y_i) and the polynomial curve can be expressed as

$$(y_i - f(x_i^*))^2 + (x_i - x_i^*)^2 = \frac{(y_i - f(x_i^*) + f'(x_i^*)x_i^* - f'(x_i^*)x_i)^2}{1 + f'(x_i^*)^2} \tag{3.8}$$

where $f(x) = \sum_{k=1}^p \beta_k x^k + b$ and $(x_i^*, f(x_i^*))$ is the closest data point on the polynomial curve to the data point (x_i, y_i). In the polynomial regression case with order 2, $\boldsymbol{x}^*$ in (3.4) is replaced with $\boldsymbol{x}^* = \{x_i^*, x_i^{*2}\}$ and (3.5) is replaced with

$$\begin{aligned}\frac{\partial L}{\partial x_i^*} &= -f'(x_i^*)(y_i - f(x_i^*)) + x_i^* - x_i = 0 \\ &\to 2\beta_2^2 x_i^{*3} + 3\beta_1\beta_2 x_i^{*2} + (\beta_1^2 - 2\beta_2 y_i + 2b\beta_2 + 1)x_i^* + (-\beta_1 y_i + b\beta_1 - x_i) = 0. \end{aligned} \tag{3.9}$$

Using (3.9) the estimates of $(\boldsymbol{\beta}, b, x_i^*)$ for the polynomial regression are obtained by IRWLS procedure explained above.

4. Numerical studies

We illustrate the performance of proposed QR with EIV through the simulated data under different error distributions and quantile levels. We are concerned with the quantile regression in which the input variable is 1-dimensional and 2-dimensional. We use $\theta = 0.1$, 0.5, 0.9 for quantile levels. We compare the proposed QR with linear QR. In proposed QR the tolerance Tol_1 and Tol_2 are set to 0.001. In the linear QR the measurement errors in input variables are not considered. We are basically interested in estimating $q_\theta(\boldsymbol{x}^*)$ which is the θ th quantile of y conditional on $\boldsymbol{x}^*$ which is the unobserved value of the input vector. Thus, for comparison we calculate the average and standard error of mean squared errors (MSEs) for each estimated quantile regression function as follows:

$$MSE_q = \frac{1}{n}\sum_{i=1}^n (\widehat{q}_\theta(y_i) - q_\theta(y_i))^2,$$

where $\widehat{q}_\theta(y_i) = \widehat{q}_\theta(\widehat{\boldsymbol{x}}_i^*)$ for the proposed QR, $\widehat{q}_\theta(y_i) = \widehat{q}_\theta(\boldsymbol{x}_i)$ for the linear QR and n is the size of each data set.We also calculate the average and root mean squared errors (RMSEs) of estimates of regression quantiles as follows:

$$RMSE_{\ \beta} = \sqrt{\frac{1}{n_t}\sum_{t=1}^{n_t} (\widehat{\beta}(\theta)^{(t)} - \beta(\theta))^2}\ ,$$

where $\widehat{\beta}(\theta)^{(t)}$ is the estimates of $\beta(\theta)$ (regression quantile; Koenker and Basset, 1978) at tth iteration and n_t is the number of iterations. In each example 100 data sets are generated to present the performance of the proposed QR.

Example 4.1 Here x^*'s are generated from a uniform distribution $U(0,1)$ and x's are generated from a normal distribution $N(x^*, 0.1)$. y's are generated from a normal distribution $N(1+2x^*, 0.1)$. The true θth quantile regression function is given as

$$q_{0.1}(x^*) = 0.5749 + 2x^*,\ q_{0.5}(x^*) = 1 + 2x^*,\ q_{0.9}(x^*) = 1.4053 + 2x^*.$$

Example 4.2 Here (x_1^*, x_2^*)'s are generated independently from a uniform distribution $U(0,1)$ and x_1's and x_2's are generated from a normal distribution $N(x_1^*, 0.1)$ and $N(x_2^*, 0.1)$, respectively. y's are generated from a normal distribution $N(1+x_1^*-2x_2^*, 0.1)$. The true θth quantile regression function is given as

$$q_{0.1}(x_1^*, x_2^*) = 0.5749 + x_1^* - 2x_2^*,\ q_{0.5}(x_1^*, x_2^*) = 1 + x_1^* - 2x_2^*,\ q_{0.9}(x_1^*, x_2^*) = 1.4053 + x_1^* - 2x_2^*.$$

Example 4.3 Here x^*'s and x's from a uniform distribution $U(0,1)$ and a normal distribution $N(x^*, 0.1)$, respectively. y's are generated from a normal distribution $N(1+0.5x^* + x^{*2}, 0.25)$. The true θth quantile regression function is given as

$$q_{0.1}(x^*) = 0.3592 + 0.5x^* + x^{*2},\ q_{0.5}(x^*) = 1 + 0.5x^* + x^{*2},\ q_{0.9}(x^*) = 1.6408 + 0.5x^* + x^{*2}.$$

Figure 4.1 shows true quantile regression functions (solid lines), the estimated quantile regression functions $\widehat{q}_\theta(x_i^*)$'s by linear QR (left; dotted lines) and the proposed QR (right; dotted lines), respectively, which are superimposed on the scatter plots of x versus y. Here $\widehat{q}_\theta(x_i^*)$ implies the predicted quantile regression function of response given new input variable x_i^* based on 100 x's and 100 y's. We can see that quantile regression functions by the proposed QR are closer to the true quantile regression functions. Table 4.1 shows the average and standard error of 100 MSEs for each estimated quantile regression function. Standard error of MSEs is given in parentheses. As seen from Table 4.1, the proposed QR with EIV yields the smaller averages of MSEs and smaller standard deviations of MSEs for all cases. Table 4.2, 4.3 and 4.4 show the averages and RMSEs for each estimated regression quantiles. From those tables we can see that proposed QR with EIV overal yields the closer estimates of regression quantiles than the linear quantile regression. In tables Boldfaced value indicates the smallest/closest result in each quantile level. From tables we can see that proposed QR with EIV performs better than the linear quantile regression in estimation of quantile regression function and regression quantiles.

Table 4.1 The average of 100 MSEs of estimated quantile regression functions for θ=0.1, 0.5 and 0.9 (standard error is in parenthesis)

θ	Ex1		Ex2		Ex3	
	LinQR	ODQR	LinQR	ODQR	LinQR	ODQR
0.1	0.2858	**0.1390**	0.3549	**0.1491**	1.3157	**0.4163**
	(0.0063)	(0.0030)	(0.0076	(0.0027)	(0.0531)	(0.0156)
0.5	0.1950	**0.0811**	0.2347	**0.0821**	0.8378	**0.2220**
	(0.0027	(0.0013)	(0.0033)	(0.0014)	(0.0252)	(0.0083)
0.9	0.2920	**0.1346**	0.3687	**0.1452**	1.6599	**0.3037**
	(0.0077)	(0.0024)	(0.0091)	(0.0028)	(0.0794)	(0.0085)

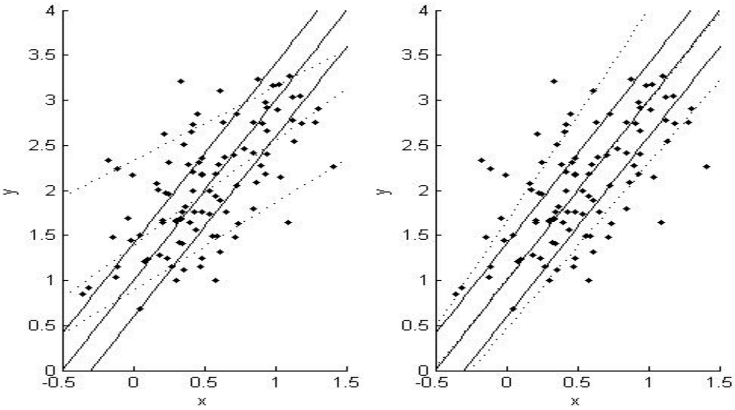

Figure 4.1 True quantile regression functions (solid lines) and the estimated quantile regression functions (dotted lines) superimposed on the scatter plots of and in Example 4.1

Table 4.2 The average of 100 estimates of $b(\theta)$ and β for θ =0.1, 0.5, 0.9 in example 4.1 (RMSE is in parenthesis)

θ	$b(\theta)$			β		
	true	LinQR	ODQR	true	LinQR	ODQR
0.1	0.5749	**0.8797** (0.3137)	0.2295 (0.5343)	2	0.8576 (1.1641)	**1.9409** (0.4208)
0.5	1	1.5450 (0.5554)	**1.0060** (0.1811)	2	0.9221 (1.0881)	**1.9766** (0.3119)
0.9	1.4053	2.2949 (0.9050)	**1.8057** (0.4181)	2	0.8180 (1.2016)	**1.9048** (0.3950)

Table 4.3 The average of 100 estimates of $b(\theta)$, β_1 and β_2 for θ=0.1, 0.5, 0.9 in example 4.2 (RMSE is in parenthesis)

θ	$b(\theta)$			β_1			β_2		
	true	LinQR	ODQR	true	LinQR	ODQR	true	LinQR	ODQR
0.1	0.5749	-0.0204 (0.6505)	**0.1090** (0.6439)	-2	-0.8729 (1.1521)	**-1.9134** (0.4306)	1	0.4531 (0.6024)	**0.9627** (0.5378)
0.5	1	0.7430 (0.2891)	**0.9989** (0.3003)	-2	-0.9436 (1.0653)	**-1.9279** (0.3137)	1	0.4677 (0.5557)	**0.9438** (0.4048)
0.9	1.4053	**1.4436** (0.1997)	1.8344 (0.6362)	-2	-0.8516 (1.1724)	**-1.8798** (0.4609)	1	0.4645 (0.5942)	**0.9548** (0.5381)

Table 4.4 The average of 100 estimates of $b(\theta)$, β_1 and β_2 for θ=0.1, 0.5, 0.9 in example 4.3 (RMSE is in parenthesis)

θ	$b(\theta)$			β_1			β_2		
	true	LinQR	ODQR	true	LinQR	ODQR	true	LinQR	ODQR
0.1	0.3592	0.4186 (0.1556)	**0.3728** (0.2377)	0.5	0.2340 (0.2992)	**0.4971** (0.2558)	1	0.4139 (0.6001)	**1.0036** (0.4481)
0.5	1	**1.1405** (0.1895)	0.7872 (0.2735)	0.5	0.3820 (0.1957)	**0.4423** (0.2057)	1	0.6332 (0.3855)	**0.9996** (0.1777)
0.9	1.6408	2.0886 (0.4876)	**1.6352** (0.3359)	0.5	0.5824 (0.2872)	**0.5115** (0.4805)	1	0.9561 (0.1952)	**0.9955** (0.4220)

5. Conclusions

In this paper, we dealt with estimating regression quantiles and quantile regression function in EIV models by using the weighted squared loss function in EIV objective function. We found that the proposed QR provides satisfying results in estimating regression quantiles and quantile regression functions in the given examples. The proposed QR can be extended to the nonlinear quantile regression in EIV models by introducing support vector machine (Vapnik, 1998) and support vector quantile regression (Takeuchi *et al.*, 2006), which can be applied easily and effectively to the nonlinear quantile regression in EIV models and the high dimensional input vector.

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