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PRECONDITIONED SPECTRAL COLLOCATION METHOD ON CURVED ELEMENT DOMAINS USING THE GORDON-HALL TRANSFORMATION

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Abstract. The spectral collocation method for a second order elliptic boundary value problem on a domain Ω with curved boundaries is studied using the Gordon and Hall transformation which enables us to have a transformed elliptic problem and a square domain $S = [0, h] \times [0, h]$, $h > 0$. The preconditioned system of the spectral collocation approximation based on Legendre-Gauss-Lobatto points by the matrix based on piecewise bilinear finite element discretizations is shown to have the high order accuracy of convergence and the efficiency of the finite element preconditioner.

1. Introduction

Since the spectral methods employ a high order polynomial interpolation for approximating solutions of differential equations, the approximated solutions usually are very accurate [2, 3]. Even though the spectral collocation method using collocation grids based on Gauss-Lobatto points is well suited for rectangle domains, it is not easy to use the spectral collocation method for a complex domain directly. Hence, one of the goals of this paper is to apply the Gordon-Hall transformation method [8, 9] (see also chapter 8 in [4]) which enables us to apply the spectral collocation method on a nonconvex domain Ω with a curved boundary $\partial\Omega$. In this paper, we are taking the target problem as a second order elliptic boundary value problem defined in Ω such that

$$(1) \qquad \begin{cases} -\Delta u + \mathbf{b} \cdot \nabla u + cu = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

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where f is a given continuous function, $\mathbf{b} = (b_1, b_2)$ and c are given constant vector and scalar, respectively.

To solve (1) with the spectral collocation method on a curved domain Ω, by the Gordon and Hall transformation [8, 9], we transform not only a domain Ω into a square $[0, h] \times [0, h]$ but also (1) in Ω into a corresponding second-order elliptic equation defined in the square domain $[0, h] \times [0, h]$ (see section 4). Then, the usual spectral collocation method [2, 19] will be employed for the transformed problem, so that the spectral convergence is shown for the transformed problem corresponding to (1).

One disadvantage of the transformed spectral collocation methods by the Gordon-Hall transformation is occurred from growing spectral condition numbers which makes the linear system ill-conditioned, so that it is not comfortable to use the well known iterative methods like conjugate gradient method and multigrid methods, for example. In the case that the Gordon and Hall transformation has some singularities, a large spectral condition number of the system can be expected. Therefore, it is necessary to have a linear system with a small condition number. Furthermore, it will be nice for such a transformed linear system arisen from the spectral collocation discretizations to keep a bounded condition number as the number of grid points get grows. Hence the other goal is to precondition such a transformed linear system by a finite element preconditioner [5, 16] which is used to get a condition number bounded uniformly. Note that such preconditioners are studied in a square domain in [1], [5] and [16]-[18] for example. Finally, the spectral element collocation method [7] is used on a nonconvex domain for a transformed linear system.

The outline of this paper is as follows. In the following section we provide some definitions and notations. In Section 3, we present the Gordon and Hall transformation, briefly. The transformed second order boundary value problems are presented in Section 4. This is followed by spectral collocation and some numerical examples including discretization errors in L^2 and H^1-norm in Section 5. In Section 6, we precondition the linear system arisen from discretization by the finite element preconditioner. We explain the spectral element scheme for a complex domain with curved boundaries including some numerical examples in Section 7. We finalize the paper by some concluding remarks.

2. Preliminaries

In this section, we give some preliminaries, definitions and notations which are useful in the sequel. The standard notations and definitions are used for the Sobolev spaces $H^s(\Omega)$ equipped with inner product $(\cdot, \cdot)_s$ and corresponding norms $\|\cdot\|_s, s \geq 0$. The space $H^0(\Omega)$ coincides with $L^2(\Omega)$, in which the norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot, \cdot)$, respectively.

Let $\mathcal{P}_N$ be the space of all polynomials of degree less than or equal to N and $\{\xi_i\}_{i=0}^N$ be the Legendre Gauss Lobatto (LGL) points on $[-1,1]$ such that

$$-1 := \xi_0 < \xi_1 < \cdots < \xi_{N-1} < \xi_N := 1.$$

Here, $\{\xi_i\}_{i=0}^N$ are the zeros of $(1-t^2)L_N'(t)$, where L_N is the N-th Legendre polynomial and the corresponding quadrature weights $\{\omega_i\}_{i=0}^N$ are given by

$$\begin{aligned} \omega_j &= \frac{2}{N(N+1)}\frac{1}{[L_N(\xi_j)]^2}, \quad 1 \le j \le N-1 \\ \omega_0 &= \omega_N = \frac{2}{N(N+1)}. \end{aligned} \tag{2}$$

Then, we have the following LGL quadrature formula such that

$$\int_{-1}^{1} p(t)dt = \sum_{i=0}^{N} \omega_i p(\xi_i), \quad \forall p \in \mathcal{P}_{2N-1}. \tag{3}$$

Let $\{\phi_i\}_{i=0}^N$ be the set of Lagrange polynomials of degree N with respect to LGL points $\{\xi_i\}_{i=0}^N$ which satisfy

$$\phi_i(\xi_j) = \delta_{ij} \quad \forall i,j = 0,1,\ldots,N,$$

where δ_{ij} denotes the Kronecker delta function. The two dimensional LGL nodes $\{x_{ij}\}$ and weights w_{ij} are given by

$$x_{ij} = (\xi_i, \xi_j), \quad w_{ij} = \omega_i\omega_j, \quad \forall i,j = 0,1,\ldots,N.$$

Let $\mathcal{Q}_N$ be the space of all polynomials of degree less than or equal N with respect to each single variable x and y. Define the basis for $\mathcal{Q}_N$ as

$$\psi_{ij}(x,y) = \phi_i(x)\phi_j(y), \quad i,j = 0,1,\ldots,N.$$

For any continuous functions p and q on $[-1,1]^2$, the associated discrete scalar product and norm are defined as

$$\langle p,q\rangle_{N^2} = \sum_{i,j=0}^{N} w_{ij}p(x_{ij})q(x_{ij}) \quad \text{and} \quad \|p\|_{N^2} = \langle p,q\rangle_{N^2}^{\frac{1}{2}}.$$

3. Gordon and Hall transformation

Before applying the spectral element collocation methods to solve elliptic problems defined in rectangular domain with a hole in Section 7, we briefly review the Gordon and Hall transformation on a simply connected domain Ω for convenience.

Let $\mathbf{F}$ be a vector-valued function of two independent variables $\hat{x}$ and $\hat{y}$ over a domain $\mathcal{S} = [0,h]\times[0,h]$ in the $\hat{x}\hat{y}$-plane whose range is Ω in $\mathbb{R}^2$. We assume that $\mathbf{F}$ is a continuous one-to-one transformation which maps $\mathcal{S}$ onto a simply connected bounded region Ω in $\mathbb{R}^2$ such that $\mathbf{F} : \partial\mathcal{S} \to \partial\Omega$.

We would like to construct a one-to-one function $\mathbf{T} : \mathcal{S} \to \Omega$ which matches $\mathbf{F}$ on the boundaries of $\mathcal{S}$, so-called the boundary interpolant of F, such that

$$\left\{ \begin{array}{lll} \mathbf{T}(0,\hat{y}) = \mathbf{F}(0,\hat{y}), & \mathbf{T}(h,\hat{y}) = \mathbf{F}(h,\hat{y}), & 0 \le \hat{y} \le h, \\ \mathbf{T}(\hat{x},0) = \mathbf{F}(\hat{x},0), & \mathbf{T}(\hat{x},h) = \mathbf{F}(\hat{x},h), & 0 \le \hat{x} \le h. \end{array} \right. \tag{4}$$

By using the similar arguments given in [8, 9], we can choose the following simple transfinite bilinear Lagrange interpolant of $\mathbf{F}$:

$$\begin{aligned} \mathbf{T}(\hat{x},\hat{y}) = \begin{bmatrix} x(\hat{x},\hat{y}) \\ y(\hat{x},\hat{y}) \end{bmatrix} \\ := (1-\hat{x}/h)\mathbf{F}(0,\hat{y}) + (\hat{x}/h)\mathbf{F}(h,\hat{y}) \\ + (1-\hat{y}/h)\mathbf{F}(\hat{x},0) + (\hat{y}/h)\mathbf{F}(\hat{x},h) \\ - (1-\hat{x}/h)(1-\hat{y}/h)\mathbf{F}(0,0) - (1-\hat{x}/h)(\hat{y}/h)\mathbf{F}(0,h) \\ - (1-\hat{y}/h)(\hat{x}/h)\mathbf{F}(h,0) - (\hat{y}/h)(\hat{x}/h)\mathbf{F}(h,h). \end{aligned} \tag{5}$$

It is to be noted that in practice we do not need the function $\mathbf{F}$ in the transfinite interpolation $\mathbf{T}$, the only thing we need is the geometric description of Ω in terms of its boundary which is subdivided into four parametric curve segments. It is necessary that the transfinite interpolation $\mathbf{T}$ has to be one-to-one in the interior of Ω. If $\mathbf{T}$ is one-to-one, it is invertible [6]. By the Implicit Function Theorem [6], if the Jacobian of the transformation $\mathbf{T}$

$$\left| \frac{\partial(x,y)}{\partial(\hat{x},\hat{y})} \right| = \begin{vmatrix} \dfrac{\partial x}{\partial \hat{x}} & \dfrac{\partial x}{\partial \hat{y}} \\ \dfrac{\partial y}{\partial \hat{x}} & \dfrac{\partial y}{\partial \hat{y}} \end{vmatrix} \tag{6}$$

is non-zero and $\mathbf{T}$ is continuously differentiable in the interior of $\mathcal{S}$, then $\mathbf{T}$ has a local inverse at each point of Ω.

Example. Consider the square of length two in which a half of the unit disc is removed in left hand side of this square, as shown in Fig 1. In this case, the four parametric curves in the Gordon and Hall transformation are:

$$\begin{aligned} &\mathbf{F}(\hat{x},0) = \begin{pmatrix} \hat{x} \\ 0 \end{pmatrix}, \quad \mathbf{F}(\hat{x},h) = \begin{pmatrix} \hat{x} \\ 2 \end{pmatrix}, \quad 0 \le \hat{x} \le 2, \\ &\mathbf{F}(h,\hat{y}) = \begin{pmatrix} 2 \\ \hat{y} \end{pmatrix}, \quad \mathbf{F}(0,\hat{y}) = \begin{pmatrix} \cos(\pi\hat{y}/2 - \pi/2) \\ 1 + \sin(\pi\hat{y}/2 - \pi/2) \end{pmatrix}, \quad 0 \le \hat{y} \le 2. \end{aligned}$$

The explicit form of the transformation (5) reduces to

$$\mathbf{T}(\hat{x},\hat{y}) = \begin{bmatrix} (1-\hat{x}/2)\cos(\pi\hat{y}/2 - \pi/2) + \hat{x} \\ (1-\hat{x}/2)(1+\sin(\pi\hat{y}/2 - \pi/2)) + \hat{y}\hat{x}/2 \end{bmatrix}.$$

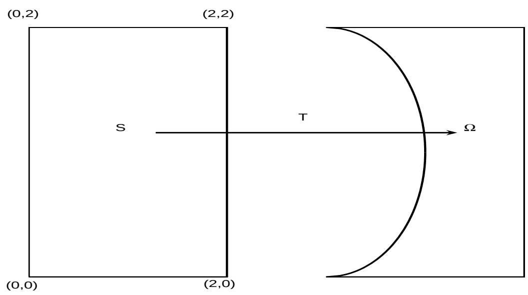

FIGURE 1. The transformation **T**

4. Second order elliptic boundary value problems on curved domain

Let Ω be a simply connected domain with a curved boundary $\partial\Omega$ for the second order elliptic boundary value problem (1). Now, with the help of the Gordon and Hall transformation in section 3 using the transfinite interpolation $(x,y)=\mathbf{T}(\hat{x},\hat{y})$ from Q into Ω, i.e., $x=x(\hat{x},\hat{y})$, $y=y(\hat{x},\hat{y})$, we transform the second order elliptic boundary value problem (1) defined in Ω into the problem defined in the rectangular domain $Q=(0,2)^2$.

A function u defined in Ω can be represented through the function $\hat{u}(\hat{x},\hat{y})=u\left(\mathbf{T}(\hat{x},\hat{y})\right)$ defined in Q. The partial derivatives of $\hat{u}$ in the variable $(\hat{x},\hat{y})$ can be computed by the chain rule:

$$\begin{cases} \dfrac{\partial \hat{u}}{\partial \hat{x}} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial \hat{x}} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial \hat{x}} \\[2ex] \dfrac{\partial \hat{u}}{\partial \hat{y}} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial \hat{y}} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial \hat{y}} \end{cases} \quad \text{or} \quad \hat{\nabla}\hat{u}(\hat{x},\hat{y}) = J^T\,\nabla u(x,y),$$

where $J=\frac{\partial(x,y)}{\partial(\hat{x},\hat{y})}$ denotes the Jacobian matrix of (x,y) with respect to $(\hat{x},\hat{y})$. In this paper we assume that the Jacobian matrix $J:=J(\hat{x},\hat{y})$ is regular for almost all $(\hat{x},\hat{y})\in\bar{Q}$, i.e., $\det(J)\neq 0$.

The high-order partial derivatives can be easily evaluated in the similar way. The following formulation is also given in [10]:

$$\begin{bmatrix} \frac{\partial x}{\partial \hat{x}} & \frac{\partial y}{\partial \hat{x}} & 0 & 0 & 0 \\ \frac{\partial x}{\partial \hat{y}} & \frac{\partial y}{\partial \hat{y}} & 0 & 0 & 0 \\ \frac{\partial^2 x}{\partial \hat{x}^2} & \frac{\partial^2 y}{\partial \hat{x}^2} & \left(\frac{\partial x}{\partial \hat{x}}\right)^2 & 2\,\frac{\partial x}{\partial \hat{x}}\,\frac{\partial y}{\partial \hat{x}} & \left(\frac{\partial y}{\partial \hat{x}}\right)^2 \\ \frac{\partial^2 x}{\partial \hat{x}\partial \hat{y}} & \frac{\partial^2 y}{\partial \hat{x}\partial \hat{y}} & \frac{\partial x}{\partial \hat{x}}\,\frac{\partial x}{\partial \hat{y}} & \frac{\partial x}{\partial \hat{x}}\,\frac{\partial y}{\partial \hat{y}} + \frac{\partial x}{\partial \hat{y}}\,\frac{\partial y}{\partial \hat{x}} & \frac{\partial y}{\partial \hat{x}}\,\frac{\partial y}{\partial \hat{y}} \\ \frac{\partial^2 x}{\partial \hat{y}\partial \hat{y}} & \frac{\partial^2 y}{\partial \hat{y}\partial \hat{y}} & \left(\frac{\partial x}{\partial \hat{y}}\right)^2 & 2\,\frac{\partial x}{\partial \hat{y}}\,\frac{\partial y}{\partial \hat{y}} & \left(\frac{\partial y}{\partial \hat{y}}\right)^2 \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial \hat{u}}{\partial \hat{x}} \\ \frac{\partial \hat{u}}{\partial \hat{y}} \\ \frac{\partial^2 \hat{u}}{\partial \hat{x}^2} \\ \frac{\partial^2 \hat{u}}{\partial \hat{x}\partial \hat{y}} \\ \frac{\partial^2 \hat{u}}{\partial \hat{y}^2} \end{bmatrix}.$$

Now, the second order elliptic boundary value problem (1) can be equivalently transformed to the following equation defined in Q:

$$L\hat{u} := -a_1\frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + 2\,a_2\frac{\partial^2 \hat{u}}{\partial \hat{x}\partial \hat{y}} - a_3\frac{\partial^2 \hat{u}}{\partial \hat{y}^2} + a_4\frac{\partial \hat{u}}{\partial \hat{x}} + a_5\frac{\partial \hat{u}}{\partial \hat{y}} + c\,\hat{u} = \hat{f}, \tag{7}$$

where the coefficients of the operator L are given by

$$\begin{aligned}
a_1 &= \frac{1}{\sigma^2}\left[\left(\frac{\partial x}{\partial \hat{y}}\right)^2 + \left(\frac{\partial y}{\partial \hat{y}}\right)^2\right],\\
a_2 &= \frac{1}{\sigma^2}\left[\frac{\partial y}{\partial \hat{x}}\frac{\partial y}{\partial \hat{y}} + \frac{\partial x}{\partial \hat{x}}\frac{\partial x}{\partial \hat{y}}\right],\\
a_3 &= \frac{1}{\sigma^2}\left[\left(\frac{\partial x}{\partial \hat{x}}\right)^2 + \left(\frac{\partial y}{\partial \hat{x}}\right)^2\right],\\
a_4 &= -\frac{1}{\sigma^3}(\alpha_1+\alpha_2) + \frac{1}{\sigma}\left[b_1\frac{\partial y}{\partial \hat{y}} - b_2\frac{\partial x}{\partial \hat{y}}\right],\\
a_5 &= -\frac{1}{\sigma^3}(\alpha_3+\alpha_4) - \frac{1}{\sigma}\left[b_1\frac{\partial y}{\partial \hat{x}} - b_2\frac{\partial x}{\partial \hat{x}}\right],\\
\hat{f} &= f\big(\mathbf{T}(\hat{x},\hat{y})\big),
\end{aligned} \tag{8}$$

with the Jacobian of the transformation $\mathbf{T}$

$$\sigma = \det(J) = \frac{\partial x}{\partial \hat{x}}\frac{\partial y}{\partial \hat{y}} - \frac{\partial x}{\partial \hat{y}}\frac{\partial y}{\partial \hat{x}} \tag{9}$$

and

$$\begin{aligned}
\alpha_1 &= \left(\frac{\partial y}{\partial \hat{y}}\right)^3\frac{\partial^2 x}{\partial \hat{y}^2} + 2\frac{\partial y}{\partial \hat{x}}\left(\frac{\partial y}{\partial \hat{y}}\right)^2\frac{\partial^2 x}{\partial \hat{y}\partial \hat{x}} - \frac{\partial y}{\partial \hat{y}}\left(\frac{\partial y}{\partial \hat{x}}\right)^2\frac{\partial^2 x}{\partial \hat{y}^2}\\
&\quad + \frac{\partial x}{\partial \hat{y}}\left(\frac{\partial y}{\partial \hat{y}}\right)^2\frac{\partial^2 y}{\partial \hat{y}^2} - 2\frac{\partial x}{\partial \hat{y}}\frac{\partial y}{\partial \hat{x}}\frac{\partial y}{\partial \hat{y}}\frac{\partial^2 y}{\partial \hat{y}\partial \hat{x}} + \frac{\partial x}{\partial \hat{y}}\left(\frac{\partial y}{\partial \hat{x}}\right)^2\frac{\partial^2 y}{\partial \hat{y}^2},\\
\alpha_2 &= \left(\frac{\partial x}{\partial \hat{y}}\right)^3\frac{\partial^2 y}{\partial \hat{y}^2} - 2\frac{\partial x}{\partial \hat{x}}\left(\frac{\partial x}{\partial \hat{y}}\right)^2\frac{\partial^2 y}{\partial \hat{y}\partial \hat{x}} - \frac{\partial y}{\partial \hat{y}}\left(\frac{\partial x}{\partial \hat{y}}\right)^2\frac{\partial^2 x}{\partial \hat{y}^2}\\
&\quad + \frac{\partial x}{\partial \hat{y}}\left(\frac{\partial x}{\partial \hat{x}}\right)^2\frac{\partial^2 y}{\partial \hat{y}^2} + 2\frac{\partial y}{\partial \hat{y}}\frac{\partial x}{\partial \hat{x}}\frac{\partial x}{\partial \hat{y}}\frac{\partial^2 x}{\partial \hat{y}\partial \hat{x}} - \frac{\partial y}{\partial \hat{y}}\left(\frac{\partial x}{\partial \hat{x}}\right)^2\frac{\partial^2 x}{\partial \hat{y}^2},\\
\alpha_3 &= \left(\frac{\partial y}{\partial \hat{x}}\right)^3\frac{\partial^2 x}{\partial \hat{y}^2} - 2\frac{\partial y}{\partial \hat{y}}\left(\frac{\partial y}{\partial \hat{x}}\right)^2\frac{\partial^2 x}{\partial \hat{y}\partial \hat{x}} - \frac{\partial x}{\partial \hat{x}}\left(\frac{\partial y}{\partial \hat{y}}\right)^2\frac{\partial^2 y}{\partial \hat{y}^2}\\
&\quad - \frac{\partial x}{\partial \hat{x}}\left(\frac{\partial y}{\partial \hat{x}}\right)^2\frac{\partial^2 y}{\partial \hat{y}^2} + 2\frac{\partial x}{\partial \hat{x}}\frac{\partial y}{\partial \hat{x}}\frac{\partial y}{\partial \hat{y}}\frac{\partial^2 y}{\partial \hat{y}\partial \hat{x}} + \frac{\partial x}{\partial \hat{x}}\left(\frac{\partial y}{\partial \hat{x}}\right)^2\frac{\partial^2 y}{\partial \hat{y}^2},\\
\alpha_4 &= -\left(\frac{\partial x}{\partial \hat{x}}\right)^3\frac{\partial^2 y}{\partial \hat{y}^2} + 2\frac{\partial x}{\partial \hat{y}}\left(\frac{\partial x}{\partial \hat{x}}\right)^2\frac{\partial^2 y}{\partial \hat{y}\partial \hat{x}} - \frac{\partial x}{\partial \hat{x}}\left(\frac{\partial x}{\partial \hat{y}}\right)^2\frac{\partial^2 y}{\partial \hat{y}^2}
\end{aligned} \tag{10}$$

$$+\frac{\partial y}{\partial \hat{x}}\left(\frac{\partial x}{\partial \hat{y}}\right)^2\frac{\partial^2 x}{\partial \hat{y}^2}-2\frac{\partial y}{\partial \hat{x}}\frac{\partial x}{\partial \hat{x}}\frac{\partial x}{\partial \hat{y}}\frac{\partial^2 x}{\partial \hat{y}\partial \hat{x}}+\frac{\partial y}{\partial \hat{x}}\left(\frac{\partial x}{\partial \hat{x}}\right)^2\frac{\partial^2 x}{\partial \hat{y}^2}.$$

Furthermore the equation (7) can be written by

$$L\hat{u}=-\hat{\nabla}\cdot\left(A\,\hat{\nabla}\hat{u}\right)+\boldsymbol{\beta}\cdot\hat{\nabla}\hat{u}+c\,\hat{u}=\hat{f}, \tag{11}$$

where the coefficient matrix A is given by

$$A:=J^{-1}\,J^{-T}=\begin{bmatrix} a_1 & -a_2\\ -a_2 & a_3\end{bmatrix}, \tag{12}$$

and the coefficient of convection term $\boldsymbol{\beta}=(\beta_1,\beta_2)$ is given by

$$\begin{aligned}\beta_1(\hat{x},\hat{y})&=a_4+\frac{\partial a_1}{\partial \hat{x}}-\frac{\partial a_2}{\partial \hat{y}},\\ \beta_2(\hat{x},\hat{y})&=a_5-\frac{\partial a_2}{\partial \hat{x}}+\frac{\partial a_3}{\partial \hat{y}}.\end{aligned} \tag{13}$$

Lemma 4.1 ([20])**.** *Let $a,b,c\in\mathbb{R}$, $D=b^2-ac$, and $\phi(h,k)=ah^2+2bhk+ck^2$.*

(1) *If $D<0$, then a and $\phi(h,k)$ have the same sign for all $(h,k)\neq(0,0)$.*
(2) *If $D>0$, then $\phi(h,k)$ takes on both positive and negative values as (h,k) varies over $\mathbb{R}^2$.*

Theorem 4.1. *The matrix A in* (12) *is symmetric positive definite and uniformly elliptic.*

Proof. We show that the matrix A is uniformly elliptic, i.e., there are positive constants $0<\lambda\le\Lambda<\infty$ such that

$$0<\lambda\,\boldsymbol{\xi}^T\boldsymbol{\xi}\le\boldsymbol{\xi}^T A(\hat{x},\hat{y})\,\boldsymbol{\xi}\le\Lambda\,\boldsymbol{\xi}^T\boldsymbol{\xi}<\infty \tag{14}$$

for all $0\neq\boldsymbol{\xi}=(\xi_1,\xi_2)^T\in\mathbb{R}^2$ and almost all $(\hat{x},\hat{y})\in\bar{Q}$.

Since

$$\boldsymbol{\xi}^T A\boldsymbol{\xi}=a_1\xi_1^2-2a_2\xi_1\xi_2+a_3\xi_2^2, \tag{15}$$

it follows that

$$\begin{aligned}a_1\boldsymbol{\xi}^T A\boldsymbol{\xi}&=a_1^2\xi_1^2-2a_1a_2\xi_1\xi_2+a_1a_3\xi_2^2\\ &=(a_1\xi_1-a_2\xi_2)^2-(a_2^2-a_1a_3)\xi_2^2\\ &\ge-(a_2^2-a_1a_3)\xi_2^2.\end{aligned}$$

Therefore, we have

$$\frac{\boldsymbol{\xi}^T A\boldsymbol{\xi}}{2}\ge-\frac{(a_2^2-a_1a_3)}{2a_1}\xi_2^2=\frac{|D|}{2a_1}\xi_2^2,\quad D=a_2^2-a_1a_3=-\frac{1}{\sigma^2}. \tag{16}$$

Similarly, we have

$$\frac{\boldsymbol{\xi}^T A\boldsymbol{\xi}}{2}\ge-\frac{(a_2^2-a_1a_3)}{2a_3}\xi_1^2=\frac{|D|}{2a_3}\xi_1^2. \tag{17}$$

Now, from (16) and (17) we get

$$\begin{aligned} \boldsymbol{\xi}^T A \boldsymbol{\xi} &\geq \frac{|D|}{2a_3}\xi_1^2 + \frac{|D|}{2a_1}\xi_2^2 \\ &\geq \min\{\frac{|D|}{2a_3}, \frac{|D|}{2a_1}\}(\xi_1^2 + \xi_2^2), \end{aligned} \tag{18}$$

which implies

$$\boldsymbol{\xi}^T A \boldsymbol{\xi} \geq \lambda \boldsymbol{\xi}^T \boldsymbol{\xi}, \quad \lambda = \min\{\frac{|D|}{2a_3}, \frac{|D|}{2a_1}\}. \tag{19}$$

For the upper bound, we have

$$\begin{aligned} \boldsymbol{\xi}^T A \boldsymbol{\xi} &= a_1\xi_1^2 - 2a_2\xi_1\xi_2 + a_3\xi_2^2 \\ &= 2a_1\xi_1^2 + 2a_3\xi_2^2 - (a_1\xi_1^2 + 2a_1a_2\xi_1\xi_2 + a_3\xi_2^2) \\ &\leq 2a_1\xi_1^2 + 2a_3\xi_2^2 \\ &\leq 2\max\{a_1, a_3\}(\xi_1^2 + \xi_2^2). \end{aligned} \tag{20}$$

Since, by Lemma 4.1,

$$a_1\xi_1^2 + 2a_1a_2\xi_1\xi_2 + a_3\xi_2^2 > 0,$$

it follows that

$$\boldsymbol{\xi}^T A \boldsymbol{\xi} \leq \Lambda \boldsymbol{\xi}^T \boldsymbol{\xi}, \quad \Lambda = 2\max\{a_1, a_3\}. \tag{21}$$

Hence, from (19) and (21) we obtain ellipticity (14) of the matrix A. Again the positive definite property of the symmetric matrix A is obtained from (14) for almost all $(\hat{x}, \hat{y}) \in \bar{Q}$. This completes the proof. □

In the following, we mention the constants λ and Λ in (14)

$$\lambda = \min\{\frac{|D|}{2a_3}, \frac{|D|}{2a_1}\}, \quad \Lambda = 2\max\{a_1, a_3\} \tag{22}$$

for usages later. Now, we are able to apply the spectral collocation method on the transformed equation (11).

5. Spectral collocation and numerical results

The computations for problem (11) can be implemented by using one-dimensional pseudospectral matrix D_N associated with the $N+1$ values $\{\hat{u}(\xi_j)\}_{j=0}^N$, and $N+1$ values $\{(\partial_N \hat{u})(\xi_j)\}_{j=0}^N$ of the pseudospectral derivative of $\hat{u}$ at Legendre-Gauss-Lobatto (LGL) points (see [2], [19]). The entries of D_N can be computed by differentiating the Legendre polynomials ϕ_j which $\phi_j(\xi_i) = \delta_{ij}$. First we reorder the LGL points from down to up and then left to right such that $x_{k(N+1)+l} := x_{kl} = (\xi_k, \xi_l)$ for $k, l = 0, 1, \ldots, N$. Two dimensional basis functions $\psi_{k(N+1)+l}(x, y) := \psi_{kl}(x, y) = \phi_k(x)\phi_l(y)$ and quadrature weights

$\omega_{k(N+1)+l} := \omega_{kl} = \omega_k \omega_l$ are reordered accordingly. Then two dimensional pseudo-spectral matrices S_x and S_y related to

$$\{(\partial_x \hat{u})(x_j)\}_{j=0}^{(N+1)^2-1} \quad \text{and} \quad \{(\partial_y \hat{u})(x_j)\}_{j=0}^{(N+1)^2-1}$$

of the pseudo-spectral partial derivatives of $\hat{u}$, respectively, are given by the tensor product of the identity matrix I_N and one-dimensional pseudo-spectral matrix D_N such that

$$S_x = D_N \otimes I_N \qquad \text{and} \qquad S_y = I_N \otimes D_N.$$

In fact, (i,j)-entries of S_x and S_y are given by $\partial_x \psi_j(x_i)$ and $\partial_y \psi_j(x_i)$, respectively. Let $W = \text{diag}\{\omega_i\}$ be the diagonal weight matrix. We denote $\hat{\mathbf{s}}$ the vector containing the nodal values of the continuous function s, that is,

$$\hat{\mathbf{s}} = (s(x_0), \ldots, s(x_{(N+1)^2-1})^T.$$

We have the collocation derivative at the nodes through matrix multiplication

$$(\partial_{t,N})(x_i) = \sum_{j=0}^{(N+1)^2-1} (S_t)_{ij} p(x_j) = (S_t)\hat{\mathbf{p}}_i \qquad \text{for } t = x \text{ or } y,$$

and for any $p, q \in Q_N$,

$$\langle p, q \rangle = \hat{\mathbf{q}}^T W \hat{\mathbf{p}} \qquad \text{and} \qquad \langle \partial_{t1} p, \partial_{t2} q \rangle = (S_{t2}\hat{\mathbf{q}})^T W (S_{t1}\hat{\mathbf{p}}),$$

where t_1 and t_2 are x or y. Now, it is easy task to assemble problem (11).

Now, we present some numerical experiments for the second order elliptic boundary value problem (1) on the single domain Ω with a curved boundary as in Fig. 2. In this numerical example, we show the distribution of LGL points in Ω and the spectral convergence in the sense of L^2 and H^1 norms.

First, note that the Gordon-Hall transformation is given by

$$\mathbf{F}(\hat{x}, 0) = \begin{pmatrix} \hat{x} \\ -\frac{\hat{x}}{4} \end{pmatrix}, \qquad \mathbf{F}(\hat{x}, h) = \begin{pmatrix} \hat{x} \\ \frac{\hat{x}}{4} + 2 \end{pmatrix}, \quad 0 \le \hat{x} \le 2,$$

$$\mathbf{F}(h, \hat{y}) = \begin{pmatrix} 2 \\ \frac{3}{2}\hat{y} - \frac{1}{2} \end{pmatrix}, \quad \mathbf{F}(0, \hat{y}) = \begin{pmatrix} \cos((\hat{y}-1)\pi/2) \\ 1 + \sin((\hat{y}-1)\pi/2) \end{pmatrix}, \quad 0 \le \hat{y} \le 2,$$

and the explicit form of the transformation in (5) reduces to

$$\mathbf{T}(\hat{x}, \hat{y}) = \begin{bmatrix} (1 - \hat{x}/2)\cos((\hat{y}-1)\pi/2) + \hat{x} \\ (1 - \hat{x}/2)\sin((\hat{y}-1)\pi/2) + 1 - \frac{3}{4}\hat{x} + \frac{3}{4}\hat{y}\hat{x} \end{bmatrix}.$$

Using this transformation, the transformed collocation nodes of this domain is shown in Figure 3. Denote by u_N the discrete solution to (1), and by $\mathbf{e} = u - u_N$, the errors. We present the discretization errors along with some coefficients $\mathbf{b}$ and c with the exact solutions:

$$u(x, y) = \sin(\frac{\pi x}{5})\sin(\frac{\pi y}{5}) + \frac{\exp(-x+y)}{\exp(4)} \tag{23}$$

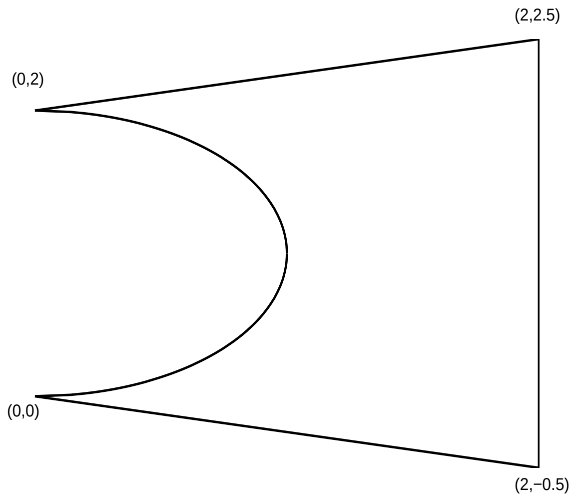

FIGURE 2. A curved domain.

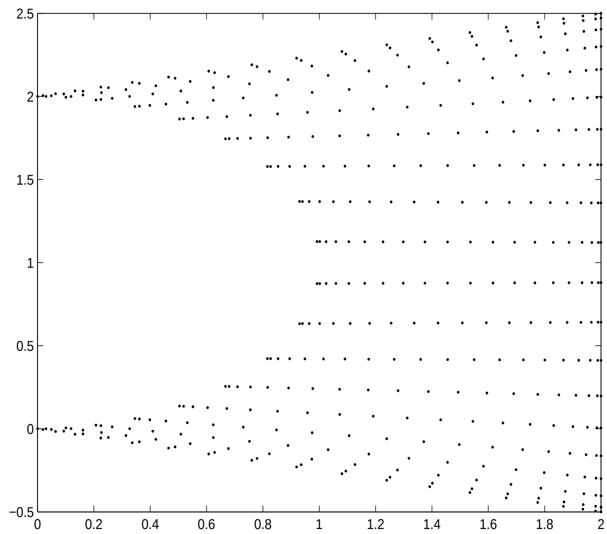

FIGURE 3. Collocation nodes for $N = 20$.

By substituting the above exact solution in the (1), we have the right hand side f along with different coefficients $\mathbf{b}$ and c. The errors measured in L^2 and H^1-norm which are presented in Table 1 for $\mathbf{b} = (0, 0)$, $c = 0$ and $\mathbf{b} = (3, 6)$, $c = 1$ of example (23). We see that errors decay exponentially with respect to

N, independent of the coefficient $\mathbf{b}$ and c. One may easily compute that the constants λ and Λ in (22) for this transformation are about 0.202642 and 32, respectively.

TABLE 1. Discretization errors for the example (23)

$\mathbf{b}$	c	N	$\|e\|$	$\|e\|_1$
(0,0)	0	4	1.0194e-002	3.9540e-002
		8	6.8200e-005	5.5796e-004
		12	3.7582e-007	4.5142e-006
		16	3.7268e-009	6.8528e-008
		20	1.6757e-011	4.1573e-010
		24	1.8636e-014	5.5109e-013
(3,6)	1	4	3.8496e-002	1.6171e-001
		8	1.6817e-004	1.6963e-003
		12	7.8620e-007	1.1312e-005
		16	7.3397e-009	1.5137e-007
		20	3.0454e-011	8.1981e-010
		24	3.4158e-014	1.0873e-012

6. Preconditioning

The matrices arisen from the pseudo-spectral or spectral method have large condition numbers dependent on both mesh sizes and degrees of polynomials which make the linear system to be ill conditioned. The cure for getting rid of ill conditioning is to resort preconditioning techniques. Following an early suggestion of [14, 15], several authors ([5, 16, 17, 18]) have investigated both finite difference and finite element preconditioning. Here, we use the finite element preconditioning for the operator (11). We consider a uniformly elliptic operator (11), given by

$$L\hat{u} = -\hat{\nabla} \cdot \left(A\,\hat{\nabla}\hat{u}\right) + \boldsymbol{\beta} \cdot \hat{\nabla}\hat{u} + c\,\hat{u} = \hat{f}, \qquad \text{in } \hat{\Omega} = [-1,1]^2, \tag{24}$$

with boundary condition

$$\hat{u} = 0 \qquad \text{on } \partial\hat{\Omega}.$$

Note that the elements of the matrix A and the vector $\boldsymbol{\beta}$ are described in (8) and (13) respectively. The domain of L is defined by

$$D(L) = \{u \in H^2(\Omega) : \hat{u} = 0 \text{ on } \partial\hat{\Omega}\}. \tag{25}$$

Now, we employ the following operator as a preconditioner

$$M\hat{u} = -\hat{\nabla} \cdot \left(A\,\hat{\nabla}\hat{u}\right) + \boldsymbol{\beta}' \cdot \hat{\nabla}\hat{u} + c'\,\hat{u} = \hat{f} \qquad \text{in } \hat{\Omega}, \tag{26}$$

with boundary condition

$$\hat{u} = 0 \qquad \text{on } \partial\hat{\Omega}.$$

Here, we consider that the coefficient of reaction term c' is an non-negative constant and the coefficient of convection term $\boldsymbol{\beta}' = (\beta_1', \beta_2')$ is a constant vector.

The adjoint operator $L^*$of L is defined by

$$L^*\hat{v} = -\hat{\nabla} \cdot \left(A\,\hat{\nabla}\hat{v}\right) - \hat{\nabla}\,(\boldsymbol{\beta} \cdot \hat{v}) + c\,\hat{v}, \tag{27}$$

with boundary condition

$$\hat{v} = 0 \qquad \text{on } \partial\hat{\Omega}.$$

Hence, we have

$$D(L^*) = \{v \in H^2(\Omega) : \hat{v} = 0 \text{ on } \partial\hat{\Omega}\}. \tag{28}$$

The adjoint operator $M^*$of M is defined by

$$M^*\hat{v} = -\hat{\nabla} \cdot \left(A\,\hat{\nabla}\hat{v}\right) - \hat{\nabla}\left(\boldsymbol{\beta}' \cdot \hat{v}\right) + c'\,\hat{v}, \tag{29}$$

with boundary condition

$$\hat{v} = 0 \qquad \text{on } \partial\hat{\Omega}.$$

Hence, we have

$$D(M^*) = \{v \in H^2(\Omega) : \hat{v} = 0 \text{ on } \partial\hat{\Omega}\}. \tag{30}$$

Manteuffel and Parter ([13], Theorem 3.1) showed that L^{-1} and M^{-1} are L_2 norm equivalent if and only if their adjoint operators have the same domain i.e., $D(L^*) = D(M^*)$. In other words, $D(L^*) = D(M^*)$ if and only if there is a constant $C > 0$ such that

$$\|M^{-1}L\| \le C, \qquad \|L^{-1}M\| \le C,$$

which means the L_2 condition number of $M^{-1}L$ is uniformly bounded. By (28) and (30), we have

$$D(L^*) = D(M^*),$$

which shows that in our case the L_2 condition number of $M^{-1}L$ is uniformly bounded.

The bilinear form of L is defined by

$$a(u, v) = \int\int_{\hat{\Omega}} A \cdot \hat{\nabla}\hat{u} \cdot \hat{\nabla}\hat{v} + \int\int_{\hat{\Omega}} \beta \cdot \hat{\nabla}\hat{u} \cdot \hat{v} + \int\int_{\hat{\Omega}} c\hat{u}\hat{v},$$

where $\hat{v}$ is a smooth function defined on $\hat{\Omega}$. The weak form of the boundary value problem (24) is

$$a(u, v) = (f, v). \tag{31}$$

For each fixed $u \in H^1$, the value $a(u, \cdot)$ is a bounded linear functional on H^1. The weak form L_w of L is the mapping taking u into $a(u, \cdot)$. That is

$$(L_w u)v = a(u, v). \tag{32}$$

Thus, $(L_w u) \in L_2(\hat{\Omega})$ and

$$(L_w u)(v) = (f, v).$$

Manteuffel and Parter ([13], Theorem 3.2) showed that L_w^{-1} and M_w^{-1} are H_1 norm equivalent on $L_2(\hat{\Omega})$ if and only if the operator L and M have the same Dirichlet boundary conditions. In other words, there is a constant $C > 0$ such that

$$\|M_w^{-1} L_w\|_1 \le C, \qquad \|L_w^{-1} M_w\|_1 \le C,$$

which means the H^1 condition number of $M_w^{-1} L_w$ is uniformly bounded if and only if the operator L and M have the same Dirichlet boundary conditions.

Since the operators L and M have the same Dirichlet boundary conditions and fulfill the assumptions of Theorem 3.2 in [13], we have the uniformly boundedness of the H_1 condition number of $M_w^{-1} L_w$. According to the above statements, it is possible to choose a preconditioning operator M in (26) for a target operator L in (24). In particular, in the construction of the finite element preconditioner matrix $\tilde{P}$ corresponding to M for the pseudospectral discretization matrix A_{N^2}corresponding to L, we let

$$\beta_1' = \beta_2' = 0 \quad \text{and} \quad c' = \gamma(\beta_M + c_M), \tag{33}$$

where γ is a constant and

$$\beta_M = \sup_{(\hat{x},\hat{y})\in\hat{\Omega}} \sqrt{\beta_1(\hat{x},\hat{y})^2 + \beta_2(\hat{x},\hat{y})^2} \quad \text{and} \quad c_M = \sup_{(\hat{x},\hat{y})\in\hat{\Omega}} |c(\hat{x},\hat{y})|.$$

Such a choice was explained in [16]-[18] for Legendre case and in [11] for Chebyshev case and the constant γ can be decided to reduce the condition number of the preconditioned system following the idea in [12]. We confirmed these results with our numerical experiments. In Tables 2 we give the condition number of WA_{N^2} and $\tilde{P}^{-1}WA_{N^2}$ for $\mathbf{b} = (0,0)$, $c = 0$ and $\mathbf{b} = (3,6)$, $c = 1$ for the example (23) defined on the single domain described in section 5 where A_{N^2} is the collocation matrix of the operator L and W is the diagonal matrix of the weights ω_i associated with Gauss-Lobatto quadrature. $\tilde{P}$ is the stiffness matrix of the finite element discretization of the operator M. The table shows that the spectral condition numbers of WA_{N^2} behave like $O(N^3)$ for all cases, while the spectral condition number of $\tilde{P}^{-1}WA_{N^2}$ are bounded regardless of the degree N of polynomials.

Remark. For the domain given in Figure 1, the transformation has two singular points at the corners $(0,0)$ and $(0,2)$ and we computed the constants $\lambda = 0.202642$ and $\Lambda = 5.33419 \times 10^{32}$. Table 3 shows that the non-precontioned matrix WA_{N^2} has bigger condition numbers than those for the domain given in Figure 2 but the condition number of the preconditioned matrix $\tilde{P}^{-1}WA_{N^2}$ seems to be bounded. This means such singularities do not seem to affect the condition numbers.

TABLE 2. Spectral condition number for the domain in Figure 2.

b	c	N	WA_{N^2}	$\tilde{P}^{-1}WA_{N^2}$
(0,0)	0	4	2.93361	2.5611
		8	26.9652	5.53756
		12	87.7584	6.48813
		16	208.552	7.03472
		20	409.946	7.46392
		24	712.685	8.56317
(3,6)	1	4	3.07274	2.7753
		8	20.8543	5.51029
		12	67.2798	6.47112
		16	159.862	7.02038
		20	314.288	7.45279
		24	546.445	8.5841

TABLE 3. Spectral condition number for the domain in Figure 1.

b	c	N	WA_{N^2}	$\tilde{P}^{-1}WA_{N^2}$
(0,0)	0	4	3.11432	2.42966
		8	49.0696	5.52105
		12	268.741	6.61631
		16	887.048	7.92221
		20	2222.96	10.6873
		24	4689.88	14.0763
(3,6)	1	4	3.05642	2.56057
		8	38.4967	5.50079
		12	210.198	6.59846
		16	693.837	7.99082
		20	1738.91	10.7803
		24	3668.84	14.1868

7. Spectral element collocation scheme

In this section, we extend the discussions on a single computational domain in previous sections to multiple computational domains. That is, we will divide the computational domain into some elements which include quadrilaterals with curved boundaries. To apply the pseudo-spectral collocation method to such a quadrilateral with curved boundary, we will use the ideas given by Gordon and Hall [8, 9].

The first step in the spectral element collocation method is to divide the domain Ω into m non-overlapping elements $\{\Omega_i\}_{i=1}^m$, such that $\bar{\Omega} = \bigcup_{i=1}^m \bar{\Omega}_i$. Although each subdomain may be discretized independently of the others, without loss of generality we will consider that all subdomains have the same discretizations. Suppose that u_i is the restriction of u in the subdomain Ω_i. Then we have the following problem for each subdomain, $\Omega_i, 1 \le i \le m$,

$$\begin{cases} -\Delta u_i + \mathbf{b} \cdot \nabla u_i + c u_i = f & \text{in} \quad \Omega_i, \\ u_i = 0 & \text{on} \quad \partial\Omega_i \cap \partial\Omega. \end{cases} \tag{34}$$

Moreover, we need the continuity of solutions and normal derivatives across the interfaces. To do this, let Ω_l and Ω_k, $k \neq l$ be two neighboring subdomain sharing with a common side Γ, and $n = \overrightarrow{n}^{(l)} = -\overrightarrow{n}^{(k)}$ on Γ, where $\overrightarrow{n}^{(i)}$ is the unit outward normal vector to the boundary of Ω_i. The continuity of solution is

$$u_l = u_k \qquad \text{on} \qquad \Gamma \tag{35}$$

and continuity of normal derivative is

$$\frac{\partial}{\partial \overrightarrow{n}} u_l = \frac{\partial}{\partial \overrightarrow{n}} u_k \qquad \text{on} \qquad \Gamma. \tag{36}$$

We will transform every subdomain $\bar{\Omega}_i, 1 \le i \le m$, into the square $[0,h] \times [0,h]$ by Gordon and Hall transformation and approximate each function $u_i, 1 \le i \le m$ by a polynomial of degree N. The global approximation u is then the patchwork of m function $u_i, 1 \le i \le m$.

Here, we consider elliptic problem (1) on the domain Ω, where Ω is a 4×8 rectangular with a unit circle removed inside, so called obstacle domain. First, we decompose this domain into sixteen subdomain in which some of these subdomain has curve boundary. The subdivision of domain Ω in the spectral element method should be in such a way that all elements do not have singularities. We plot the collocation nodes of this domain in Figure 4. We present L^2 and H^1-errors for the exact solution (23) in Table 4 for $\mathbf{b} = (0,0)$, $c = 0$ and $\mathbf{b} = (3,6)$, $c = 1$ which shows the exponential decay of errors with respect to N, independent of the coefficient $\mathbf{b}$ and c. Again the finite element preconditioner matrix $\tilde{P}$ is constructed under the condition (33). In Table 5 we present the condition numbers of WA_{N^2} and $\tilde{P}^{-1}WA_{N^2}$ for $\mathbf{b} = (0,0)$, $c = 0$ and $\mathbf{b} = (3,6)$, $c = 1$ in the obstacle domain. Table 5 shows that the spectral condition numbers of WA_{N^2} behave like $O(N^3)$ for all cases, while the spectral condition number of $\tilde{P}^{-1}WA_{N^2}$ are bounded regardless of the degree N of polynomials.

8. Concluding remarks

The second order elliptic boundary value problems on a domain with a curved boundary are solved by using the Gordon and Hall transformation. Further, such elliptic problems on a domain with a hole are also approximated

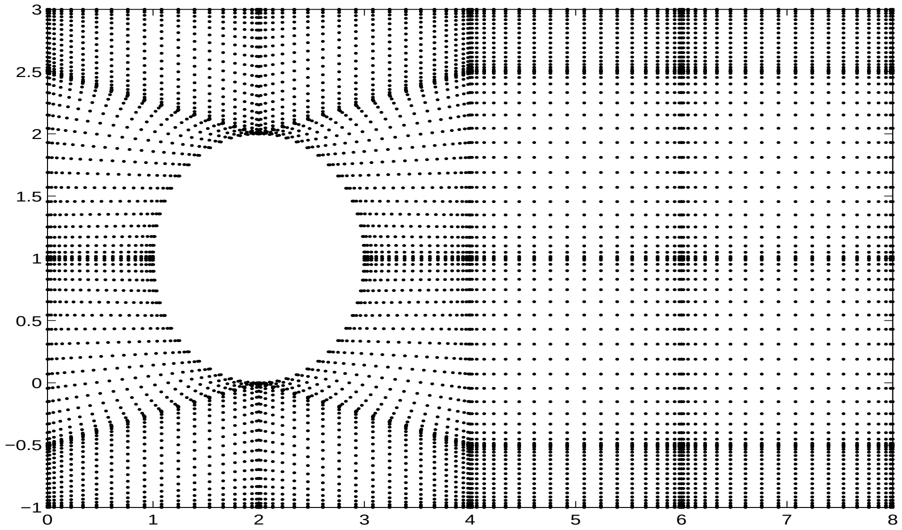

FIGURE 4. Collocation nodes of obstacle domain for $N = 20$.

TABLE 4. Error discretization for the spectral element method for the obstacle domain.

b	c	N	$\|e\|$	$\|e\|_1$
(0,0)	0	4	8.5721e-004	4.8329e-003
		8	7.7555e-007	1.2805e-005
		12	3.3809e-010	9.9912e-009
		16	1.1530e-013	3.9507e-012
		20	4.2768e-014	1.3698e-013
(3,6)	1	4	3.9506e-003	2.5376e-002
		8	1.3244e-006	2.1687e-005
		12	7.4788e-010	2.0763e-008
		16	1.7631e-013	6.8420e-012
		20	1.7564e-014	1.0700e-013

by the pseudo-spectral element method. The spectral convergence of solutions and the efficiency of finite element preconditioner are provided numerically. It was noted with numerical evidences that the Gordon and Hall transformation works quite well for a domain with corner singularities when an elliptic problem is solved by the pseudo-spectral method.

TABLE 5. Spectral condition numbers for the obstacle domain.

b	c	N	WA_{N^2}	$\tilde{P}^{-1}WA_{N^2}$
(0,0)	0	4	86.7417	5.57846
		8	881.917	7.55644
		12	3239.32	8.6007
		16	8056.9	9.38229
		20	16213.7	11.6041
(3,6)	1	4	29.3084	5.77838
		8	286.814	7.63215
		12	1054.23	8.66804
		16	2626.07	9.43196
		20	5291.05	11.8033

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