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Nil_*-COHERENT RINGS

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Abstract. Let R be a ring and $Nil_*(R)$ be the prime radical of R. In this paper, we say that a ring R is left Nil_*-coherent if $Nil_*(R)$ is coherent as a left R-module. The concept is introduced as the generalization of left J-coherent rings and semiprime rings. Some properties of Nil_*-coherent rings are also studied in terms of N-injective modules and N-flat modules.

1. Introduction

Throughout R is an associative ring with identity and all modules are unitary. ${}_R\mathcal{M}(\mathcal{M}_R)$ stands for the category of all left (right) R-modules. $\mathrm{Hom}(M,N)$ (resp. $\mathrm{Ext}^n(M,N)$) means $\mathrm{Hom}_R(M,N)$ (resp. $\mathrm{Ext}^n_R(M,N)$), and similarly $M\otimes N$ (resp. $\mathrm{Tor}_n(M,N)$) denotes $M\otimes_R N$ (resp. $\mathrm{Tor}^R_n(M,N)$). The character module M^+ is defined by $M^+=\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$. The Jacobson radical of R is denoted by $J(R)$. If X is a subset of R, the right (left) annihilator of X in R is denoted by $r(X)$ ($l(X)$). We will use the usual notations from [1, 8, 9, 13, 14, 22].

We first recall some known notions needed in the sequel.

Let $\mathcal{C}$ be the class of R-modules. For an R-module M, $C\in\mathcal{C}$ is called a $\mathcal{C}$-cover [8] of M if there is a homomorphism $g: C\to M$ such that the following hold: (1) For any homomorphism $g': C'\to M$ with $C'\in\mathcal{C}$, there exists a homomorphism $f: C'\to C$ with $g'=gf$. (2) If f is an endomorphism of C with $gf=g$, then f must be an automorphism. If (1) holds but (2) may not, $g: C\to M$ is called a $\mathcal{C}$-precover. Dually we have the definition of a $\mathcal{C}$-(pre)envelope. $\mathcal{C}$-covers and $\mathcal{C}$-envelopes may not exist in general, but if they exist, they are unique up to isomorphism. A homomorphism $g: M\to C$ with $C\in\mathcal{C}$ is said to a $\mathcal{C}$-envelope with the unique mapping property (see [6]) if for any homomorphism $g': M\to C'$ with $C'\in\mathcal{C}$, there is a unique homomorphism $f: C\to C'$ such that $fg=g'$. Dually, we have the definition of $\mathcal{C}$-cover with the unique mapping property.

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Let M be a left R-module. A right $\mathcal{C}$-resolution of M is a complex (need not be exact) $0 \to M \to F^0 \to F^1 \to \cdots$ with each $F^i \in \mathcal{C}$. Write

$$L^0 = M,\ L^1 = \mathrm{Coker}(M \to F^0),\ L^i = \mathrm{Coker}(F^{i-2} \to F^{i-1}) \ \text{ for } \ i \geq 2.$$

Here $M \to F^0, L^1 \to F^1, L^i \to F^i$ for $i \geq 2$ are $\mathcal{C}$-preenvelopes. The nth cokernel $L^n (n \geq 0)$ is called the nth $\mathcal{C}$-cosyzygy of M.

A left $\mathcal{C}$-resolution of M is a complex $\cdots \to I_1 \to I_0 \to M \to 0$ with each $I_i \in \mathcal{C}$. Write

$$K_0 = M,\ K_1 = \mathrm{Ker}(I_0 \to M),\ K_i = \mathrm{Ker}(I_{i-1} \to I_{i-2}) \ \text{ for } \ i \geq 2.$$

Here $I_0 \to M, I_1 \to K_1, I_i \to K_i$ for $i \geq 2$ are $\mathcal{C}$-precovers. The nth kernel $K_n (n \geq 0)$ is called the nth $\mathcal{C}$-syzygy of M.

A left $\mathcal{C}$-resolution $\cdots \to I_1 \to I_0 \to M \to 0$ is called minimal if every $I_i \to K_i$ is $\mathcal{C}$-cover for any $i \geq 0$.

Let R be a ring. A left R-module M is coherent if every finitely generated submodule of M is finitely presented. The ring R is said to be left coherent if R is a coherent as a left R-module. Since coherence of rings and modules first appeared in [2], their generalizations have been studied extensively by many authors (see, [3, 4, 7, 9, 11, 15, 17]). A ring R is called left J-coherent [7] if the Jacobson radical $J(R)$ of R is a coherent left R-module. R is said to be left P-coherent [17] (resp. left min-coherent [15]) if every principal (resp. minimal) left ideal of R is finitely presented.

Recall that the prime radical $Nil_*(R)$ [14] ($N(R)$ for short) of R is the intersection of all prime ideals of R. $N(R)$ contains all nilpotent one-side ideal of R. A ring R is semiprime if $N(R) = 0$. We say that a ring R is left Nil_*-coherent if the prime radical $N(R)$ of R is a coherent left R-module, or equivalently, any finitely generated left ideal in $N(R)$ is finitely presented. Nil_*-coherent rings are introduced, in this paper, as the generalization of J-coherent rings and semiprime rings. Some examples of left Nil_*-coherent rings are given, and some properties of left Nil_*-coherent rings are studied. We prove that if R is right perfect, then R is left Nil_*-coherent if and only if R is left coherent. To characterize left Nil_*-coherent rings, we introduce left N-injective modules and right N-flat modules. The class of left N-injective (resp. right N-flat) R-modules is denoted $\mathcal{NI}$ (resp. $\mathcal{NF}$). We also show that if R is left Nil_*-coherent, then every right R-module has an $\mathcal{NF}$-preenvelope and every left R-module has an $\mathcal{NI}$-cover.

In [8], Enochs and Jenda investigated the global dimension of a left Noetherian ring using the left injective resolutions of left R-modules. Mao recently generalized their results to left coherent rings (see [16]). In the third section of this paper, left strongly Nil_*-coherent rings and the N-injective dimensions are defined. We study the N-injective dimensions of modules and rings in terms of left $\mathcal{NI}$-resolutions and right $\mathcal{NF}$-resolutions of modules.

2. Nil_*-coherent rings

Definition 2.1. A ring R is said to be left Nil_*-coherent if the prime radical $N(R)$ of R is coherent left R-module, or equivalently, every finitely generated left ideal in $N(R)$ is finitely presented. Similarly, we have the concept of right Nil_*-coherent rings.

Remark 2.2. Here give some examples of Nil_*-coherent rings.

(1) Obviously, left J-coherent rings are left Nil_*-coherent because $N(R) \subseteq J(R)$.

(2) A semiprime ring is right and left Nil_*-coherent. Moreover, a domain is right and left Nil_*-coherent.

The following examples show that Nil_*-coherent rings are non-trivial generalizations of J-coherent rings and semiprime rings.

Example 1. Let R be a valuation ring of $rankR > 1$. Then $R[[x]]$, the ring of power series in one variable x, is a commutative domain, and so it is Nil_*-coherent. But $R[[x]]$ is not a J-coherent ring by [7, Example 3.16].

Example 2. Let $R = \left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{smallmatrix}\right)$. Then R is a coherent ring, and hence it is a Nil_*-coherent ring. However, R is not semiprime because there is a nilpotent ideal $\left(\begin{smallmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{smallmatrix}\right) \neq 0$.

From the next example, we can see that the definition of Nil_*-coherent rings is not left-right symmetric.

Example 3. Let $L = \mathbb{Q}(x_2, x_3, \ldots)$ be a subfield of $K = \mathbb{Q}(x_1, x_2, \ldots)$ with $\mathbb{Q}$ the field of rational numbers, and there exists a field isomorphism $\varphi : K \to L$. We define a ring by taking $R = K \times K$ with multiplication

$$(x, y)(x', y') = (xx', \varphi(x)y' + yx'), \text{where } x, y, x', y' \in K.$$

It is easy to see that R has exactly three right ideals, (0), R, and $(0, K) = (0, 1)R$. So R is right Nil_*-coherent. Let $a = (0, 1)$. Note that $Ra \subseteq N(R)$ and $l(a)$ is not finitely generated. Then R is not left Nil_*-coherent.

Similar to [7, Proposition 2.10, Corollary 2.11 and Corollary 2.12], we have the following results.

Proposition 2.3. *Let $\varphi : R \to S$ be a ring homomorphism such that S is a finitely generated left R-module and $N(S)$ is a coherent left R-module. If R is a left Nil_*-coherent ring, then so is S.*

Proof. Let M be a finitely generated submodule of the left S-module $N(S)$. By assumption, M is a finitely generated submodule of the left R-module $N(S)$, and hence M is a finitely presented left R-module. So M is a finitely presented left S-module by [11, Theorem 1]. Therefore, S is a left Nil_*-coherent ring. □

Corollary 2.4. *Let R be a left Nil_*-coherent ring. Then $M_n(R)$, the ring of $n \times n$ matrices over R, is also a left Nil_*-coherent ring for every positive integer n.*

Proof. By [14, Theorem 10.21], $N(M_n(R)) = M_n(N(R)) \cong N(R)^{n^2}$. $N(R)^{n^2}$ is a coherent left R-module by assumption, so is $N(M_n(R))$. Then the result comes from Proposition 2.3. □

Corollary 2.5. *If R is a left Nil_*-coherent ring and a finitely generated left ideal $I \subseteq N(R)$, then the quotient ring R/I is also left Nil_*-coherent.*

Proof. We have $N(R/I) = N(R)/I$ in terms of [14, Exercise 10.20]. Now let X be a finitely generated submodule of the left R-module $N(R/I)$. Then there is a finitely generated left R-module J with $I \subseteq J \subseteq N(R)$ and $X = J/I$. Since R is left Nil_*-coherent, J is a finitely presented left R-module, so is X by [13, Lemma 4.54]. Thus $N(R/I)$ is a coherent left R-module. Therefore, R/I is a left Nil_*-coherent ring by Proposition 2.3. □

Proposition 2.6. *A direct product of rings $R = R_1 \times R_2 \times \cdots \times R_n$ is left Nil_*-coherent if and only if R_i is left Nil_*-coherent for $i = 1, \ldots, n$.*

Proof. Note that $N(R) = N(R_1) \times N(R_2) \times \cdots \times N(R_n)$. If R is left Nil_*-coherent, then $N(R)$ is coherent left R-module, so is $N(R_i)$ for all i. By Proposition 2.3, R_i is left Nil_*-coherent.

Conversely, it is enough to prove the assertion for $n = 2$. There exists an exact sequence $0 \to N(R_1) \to N(R) \to N(R_2) \to 0$. Hence $N(R_2) \cong N(R)/N(R_1)$ is a coherent R_2-module, thus, a coherent R-module by [9, Theorem 2.4.1]. Similarly, $N(R_1)$ is a coherent R-module. By [9, Theorem 2.2.1(2)], $N(R)$ is a coherent R-module, and hence R is left Nil_*-coherent. □

If R is the direct product of R_1 and R_2, where R_1 is a left J-coherent ring that is not semiprime and R_2 is a semiprime ring that is not left J-coherent, then R is a left Nil_*-coherent ring that is neither left J-coherent nor semiprime.

Let M be a bimodule over R. The trivial extension of R and M is $R \propto M = \{(a, x) | a \in R, x \in M\}$ with addition defined componentwise and multiplication defined by $(a, x)(b, y) = (ab, ay + xb)$. For convenience, we write $I \propto X = \{(a, x) | a \in I, x \in X\}$, where I is a subset of R and X is a subset of M. The below result is a generalization of [4, Theorem 12].

Proposition 2.7. *A ring R is left coherent if and only if $R \propto R$ is left Nil_*-coherent.*

Proof. ($\Rightarrow$). It follows from [4, Theorem 12] and Remark 2.2(1).

($\Leftarrow$). Set $S = R \propto R$. We first prove that R is left P-coherent. For any $a \in R$, $S(0, a) \subseteq N(S)$ and $l_S(0, a) = l_R(a) \propto R$. Since S is left Nil_*-coherent, $l_R(a) \propto R$ is a finitely generated left ideal of S. Write $l_R(a) \propto R = S(a_1, b_1) + \cdots + S(a_n, b_n)$ with all $(a_i, b_i) \in S$. It follows that $l_R(a) = Ra_1 + \cdots + Ra_n$. So R is left P-coherent.

Now since $R \propto R$ is left Nil_*-coherent, $M_n(R) \propto M_n(R) \cong M_n(R \propto R)$ is left Nil_*-coherent (for all $n > 0$) by Corollary 2.4. Thus, $M_n(R)$ is left P-coherent, and so R is left coherent by [17, Proposition 2.4]. □

Left Nil_*-coherent rings are left min-coherent. In fact, if Ra is a minimal left ideal of R, then we have either $(Ra)^2 = 0$, or $Ra = Re$ for some idempotent $e^2 = e \in R$ (see [14, Lemma 10.22]). The following example is constructed to show that min-coherent rings need not be Nil_*-coherent.

Example 4. Let R be a countable direct product of the polynomial ring $\mathbb{Q}[\mathrm{y}, \mathrm{z}]$ (see [13, Example 4.61(a)]). Then $R[x]$ is not a coherent ring. Note that $R[x] \propto R[x] \cong (R \propto R)[x]$, so $(R \propto R)[x]$ is not Nil_*-coherent by Proposition 2.7. But $(R \propto R)[x]$ is min-coherent because both socles are zero.

In order to characterize Nil_*-coherent rings, we introduce N-injective modules and N-flat modules as the following.

Definition 2.8. A left R-module M is said to be N-injective if $\mathrm{Ext}^1(R/I, M) = 0$ for every finitely generated left ideal I in $N(R)$. A right R-module F is called N-flat if $\mathrm{Tor}_1(F, R/I) = 0$ for every finitely generated left ideal I in $N(R)$. Dually, we can define right N-injective modules and left N-flat modules.

Remark 2.9. (1) In what follows, $\mathcal{NI}$ (resp. $\mathcal{NF}$) stands for the class of all N-injective left R-modules (resp. N-flat right R-modules). By the definition, it is clear that $\mathcal{NI}$ (resp. $\mathcal{NF}$) is closed under direct sums, direct summands, direct products (resp. direct limits) and extensions.

(2) A right R-module F is N-flat if and only if F^+ is N-injective by the standard isomorphism $\mathrm{Ext}^1(N, F^+) \cong \mathrm{Tor}_1(F, N)^+$ for every finitely generated left ideal I in $N(R)$.

(3) Recall that a left R-module M (resp. right R-module F) is J-injective (resp. J-flat) if $\mathrm{Ext}^1(R/I, M) = 0$ (resp. $\mathrm{Tor}_1(F, R/I) = 0$) for any finitely generated ideal I in $J(R)$ (see [7]). It is easy to see that left J-injective (resp. right J-flat) R-modules are left N-injective (resp. right N-flat). If R is left Artinian, then left J-injective (resp. right J-flat) R-modules coincide with left N-injective (resp. right N-flat).

Proposition 2.10. *Let R be a ring. Then the following are equivalent*:

(1) *R is a semiprime ring.*
(2) *Every left* (*or right*) *R-module is N-injective.*
(3) *Every left* (*or right*) *simple R-module is N-injective.*
(4) *Every principle left* (*or right*) *ideal in $N(R)$ is N-injective.*
(5) *Every right* (*or left*) *R-module is N-flat.*
(6) *Every finitely generated left* (*or right*) *ideal in $N(R)$ is a pure submodule of R.*

Proof. (1)⇒(2) is trivial since $N(R) = 0$. (2)⇒(3) and (2)⇒(4) are clear.

(2)⇒(5) holds by Remark 2.9(2).

(3)⇒(1). Let $a \in N(R)$. If $N(R) + l(a) \neq R$, then we take a maximal left ideal M of R such that $N(R) + l(a) \subseteq M$. Then R/M is N-injective by (3). Note that the homomorphism $f : Ra \to R/M$ given by $f(xa) = x + M,\ x \in R$ is well-defined. So there exists $c \in R$ such that $f = \cdot(c + M)$. Then $1 + M =$

$f(a) = a(c+M) = ac+M$, which implies that $1-ac \in M$. But $ac \in M$, which yields $1 \in M$, a contradiction. Therefore $N(R) + l(a) = R$ and so $l(a) = R$ because $N(R)$ is a small ideal of R. So $a = 0$. Hence $N(R) = 0$.

(5)⇒(6). For any finitely generated left ideal I in $N(R)$ and any right R-module M, $\mathrm{Tor}_1(M, R/I) = 0$ since M is N-flat. Then R/I is flat, and hence I is a pure submodule of R.

(6)⇒(2). Let I be a finitely generated left ideal in $N(R)$. Then R/I is flat by (6), and so it is projective. Thus every left R-module is N-injective.

(4)⇒(1). Suppose that $N(R) \neq 0$, then there exists an non-zero superfluous submodule Ra in $N(R)$. Thus $\mathrm{Ext}^1(R/Ra, Ra) = 0$ by (3), and so the exact sequence $0 \to Ra \to R \to R/Ra \to 0$ splits. Therefore Ra is a direct summand of R. Since Ra is superfluous, $Ra = 0$, a contradiction. Hence R is a semiprime ring. □

Let $R = \mathbb{Z}$, the integer ring. By the proposition above, any R-module is N-injective and N-flat. However, $\mathbb{Z}$ is not injective and $\mathbb{Z}/2\mathbb{Z}$ is not flat as R-module.

Similar to [7, Theorem 2.13], [15, Theorem 4.5] and [17, Theorem 2.7], we have the following theorem which characterize Nil_*-coherent rings in terms of, among others, N-injective modules, N-flat modules and N-flat preenvelope.

Theorem 2.11. *Let R be a ring. Then the following are equivalent*:

(1) *R is a left Nil_*-coherent ring.*
(2) *Any direct product of copies of R_R is N-flat.*
(3) *Any direct product of N-flat right R-modules is N-flat.*
(4) *Any direct limit of N-injective left R-modules is N-injective.*
(5) *For any finitely generated left ideal I in $N(R)$ and any family $\{M_i\}$ of right R-modules,* $\mathrm{Tor}_1(\prod M_i, R/I) \cong \prod \mathrm{Tor}_1(M_i, R/I)$.
(6) *A left R-module M is N-injective if and only if M^+ is N-flat.*
(7) *A left R-module M is N-injective if and only if M^{++} is N-injective.*
(8) *A right R-module P is N-flat if and only if P^{++} is N-flat.*
(9) *Every right R-module has an $\mathcal{NF}$-preenvelope.*

Corollary 2.12. *The following statements hold for any ring R* :

(1) *$\mathcal{NI}$ and $\mathcal{NF}$ are closed under pure submodules.*
(2) *If R is left Nil_*-coherent, then $\mathcal{NI}$ and $\mathcal{NF}$ are closed under pure quotient modules.*

Proof. (1). The proof is similar to that of [7, Lemma 2.4].

(2). For a pure exact sequence $0 \to A \to B \to C \to 0$ of left R-modules with B N-injective, there is a split exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$. By Theorem 2.11, B^+ is N-flat, so is C^+. Thus C is N-injective by Theorem 2.11 again. The $\mathcal{NF}$ case is similar. □

The following result will consider the existence of $\mathcal{NI}$-covers over a left Nil_*-coherent ring.

Proposition 2.13. *Let R be a left Nil_*-coherent ring. Then every left R-module has an $\mathcal{NI}$-cover.*

Proof. By Corollary 2.12(2), $\mathcal{NI}$ is closed under pure quotient modules. By Remark 2.9(1), $\mathcal{NI}$ is closed under direct sums. Then, in view of [12, Theorem 2.5], every left R-module has an $\mathcal{NI}$-cover. □

Remark 2.14. If R is a left Nil_*-coherent ring, then every right R-module has a right $\mathcal{NF}$-resolution by Theorem 2.11, and every right R-module has a left $\mathcal{NI}$-resolution by Proposition 2.13.

In general, an $\mathcal{NI}$-cover need not be an epimorphism and an $\mathcal{NF}$-preenvelope need not be a monomorphism. Now we consider when every left R-module has an epic $\mathcal{NI}$-cover and when every right R-module has a monic $\mathcal{NF}$-preenvelope.

Proposition 2.15. *Let R be left Nil_*-coherent. Then the following are equivalent*:

(1) *R is N-injective as left R-module.*
(2) *For any left R-module, there is an epimorphic $\mathcal{NI}$-cover.*
(3) *For any right R-module, there is a monomorphic $\mathcal{NF}$-preenvelope.*
(4) *Every $(FP$-$)$injective right R-module is N-flat.*
(5) *Every flat left R-module is N-injective.*

Proof. (1) ⇒ (3). Let M be any right R-module. Then M has an $\mathcal{NF}$-preenvelope $f : M \to F$ by Theorem 2.11. Since $({}_RR)^+$ is a cogenerator in the category of right R-modules, there is an exact sequence $0 \to M \xrightarrow{i} \prod({}_RR)^+$. By Theorem 2.11, $\prod({}_RR)^+$ is N-flat. So there exists a homomorphism $g : F \to \prod({}_RR)^+$ such that $gf = i$. Since i is a monomorphism, so is f.

(3) ⇒ (4). Note that the FP-injective right R-module E embeds in a N-flat right R-module by (3). Thus E is N-flat by Corollary 2.12.

(4) ⇒ (5). For any flat left R-module F, F^+ is injective. Then F^+ is N-flat by assumption, and hence F is N-injective by Theorem 2.11.

(5) ⇒ (2). For any left R-module M, in view of Proposition 2.13, there is an $\mathcal{NI}$-cover $f : C \to M$. Note that R is also N-injective by hypothesis, so f is an epimorphic.

(2) ⇒ (1). By assumption, R has an epimorphic $\mathcal{NI}$-cover $\varphi : D \to R$, then we have an exact sequence $0 \to K \to D \xrightarrow{\varphi} R \to 0$ with $K = \mathrm{Ker}\varphi$ and D N-injective. Note that R is projective, so the sequence is split, then R is N-injective as left R-module by Remark 2.9 (1). □

Corollary 2.16. *The following are equivalent for a ring R.*

(1) *R is semiprime.*
(2) *R is left N-injective and every finitely generated left ideal in $N(R)$ is projective.*

Proof. (1) ⇒ (2) is clear.

(2) $\Rightarrow$ (1). We firstly prove that every quotient module of a N-injective left R-module is N-injective. Let B be any N-injective left R-module and $A \subseteq B$. For any finitely generated left ideal I in $N(R)$ and a homomorphism $f : I \to B/A$, I is projective, so there is a homomorphism $g : I \to B$ such that $\pi g = f$, where $\pi : B \to B/A$ is the canonical epimorphism. Then there is a homomorphism $h : R \to B$ such that $hi = g$ since B is N-injective, where $i : I \to R$ is an inclusion. Thus, $f = \pi hi$, and hence B/A is N-injective. Thus, for any left R-module M, there is a monomorphic N-injective cover $\alpha : E \to M$ by [20, Proposition 4]. Since R is left N-injective, then α is epimorphic by Proposition 2.15, whence M is left N-injective. By Proposition 2.10, R is semiprime. □

Remark 2.17. The ring R in Example 2 is left hereditary, and hence every finitely generated left ideal in $N(R)$ is projective. But it is not semiprime, so R is not left N-injective by Corollary 2.16. Thus, there exists a ring whose every left R-module has an $\mathcal{NI}$- cover but need not be an epimorphism and every right R-module has an $\mathcal{NF}$-preenvelope but need not be a monomorphism.

Recall that a ring R is right perfect [18] if $R/J(R)$ is semisimple and $J(R)$ is right T-nilpotent. It was shown that if R is right perfect, then R is left J-coherent if and only if R is left coherent (see [7]). At the end of this section, we extend this result onto left Nil_*-coherent rings.

Proposition 2.18. *If R is right perfect, then R is left Nil_*-coherent if and only if R is left coherent.*

Proof. ($\Leftarrow$) is clear.

($\Rightarrow$). We first prove that every N-flat right R-module is flat. Let F be right N-flat. Note that $N(R) \cong \varinjlim I_i$, where I_i range over all finitely generated submodules of $N(R)$. Then

$$\mathrm{Tor}_1(F, R/N(R)) = \mathrm{Tor}_1(F, \varinjlim R/I_i) = \varinjlim \mathrm{Tor}_1(F, R/I_i) = 0.$$

Since $N(R) \subseteq J(R)$ is also right T-nilpotent, F is right flat by [23, Theorem 5.2].

Now let M be any N-injective left R-module. Then M^+ is N-flat by Theorem 2.11, and hence M^+ is flat by the preceding result. Thus M^{++} is FP-injective, whence M is FP-injective because M is a pure submodule of M^{++}. By Theorem 2.11 again, any direct limit of FP-injective left R-modules is FP-injective, which implies R is left coherent. □

3. Strongly Nil_*-coherent rings

A class $\mathcal{C}$ of left R-modules is said to be coresolving [19] if $E \in \mathcal{C}$ for all injective left R-modules E, if $\mathcal{C}$ is closed under extensions, and if given an exact sequence of left R-modules $0 \to A \to B \to C \to 0$, $C \in \mathcal{C}$ whenever $A, B \in \mathcal{C}$. Dually, we have the definition of resolving.

In the present section, we study the ring that $\mathcal{NI}$ is coresolving.

Lemma 3.1. *Let R be a ring. Then the following are equivalent*:

(1) $\mathcal{NI}$ *is coresolving.*

(2) $\mathrm{Ext}^k(R/I, M) = 0$ *for any N-injective left R-module M and any finitely generated left ideal I in $N(R)$, $k \geq 1$.*

(3) *R is left Nil_*-coherent and $\mathcal{NF}$ is resolving.*

(4) *R is left Nil_*-coherent and* $\mathrm{Tor}_k(N, R/I) = 0$ *for any N-flat right R-module N and any finitely generated left ideal I in $N(R)$, $k \geq 1$.*

Proof. The proof is similar to that of [7, Lemma 3.4]. □

Definition 3.2. We call the ring satisfying the equivalent conditions in Lemma 3.1 left strongly Nil_*-coherent. Dually, the notion of right strongly Nil_*-coherent rings can be defined.

Example 5. (1) By Proposition 2.10, a semiprime ring is left and right strongly Nil_*-coherent.

(2) If a ring R satisfies the condition that every finitely generated left ideal in $N(R)$ is projective, then R is left strongly Nil_*-coherent by the proof of Corollary 2.16.

(3) A right perfect and left Nil_*-coherent ring is left strongly Nil_*-coherent by Proposition 2.18 and Lemma 3.1.

Remark 3.3. We claim that the definition of strongly Nil_*-coherent rings is also not left-right symmetric. Indeed, the ring R in Example 3 is right Nil_*-coherent but not left Nil_*-coherent. Note that it has only three right ideals, 0, $(0, K) = (0, 1)R$ and R. Thus R is left prefect by [18, Theorem B.39], and hence R is right strongly Nil_*-coherent ring but not left strongly Nil_*-coherent.

Definition 3.4. The left N-injective dimension of a left R-module M, denoted by $l.N - Id(M)$, is defined as the least nonnegative integer n such that $\mathrm{Ext}^{n+1}(R/I, M) = 0$ for any finitely generated left ideal I in $N(R)$. If no such n exists, then $l.N - Id(M) = \infty$. Set $l.N - I.dim(R) = \sup\{l.N - Id(M) : M \in_R \mathcal{M}\}$ and call $l.N - I.dim(R)$ the left N-injective dimension of R.

By Proposition 2.10, $l.N - I.dim(R) = 0$ if and only if R is a semiprime ring. Then the N-injective dimension of R can measure how far away a ring is from being a semiprime ring.

Proposition 3.5. *Let R be a left strongly Nil_*-coherent ring. Then the following are equivalent for a left R-module M :*

(1) $l.N - Id(M) \leq n$.

(2) $\mathrm{Ext}^{n+1}(R/I, M) = 0$ *for every finitely generated left ideal I in $N(R)$.*

(3) $\mathrm{Ext}^{n+k}(R/I, M) = 0$ *for every finitely generated left ideal I in $N(R)$, and $k \geq 1$.*

(4) *For every exact sequence* $0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to L_n \to 0$ *with each E_i N-injective, L_n is N-injective.*

Proof. The proof is similar to that of [7, Lemma 3.6]. □

Proposition 3.6. *Let R be a strongly Nil_*-coherent ring and $0 \to A \to B \to C \to 0$ be an exact sequence of left R-modules. Then*:

(1) $l.N - Id(B) \leq \sup\{l.N - Id(A), l.N - Id(C)\}$.
(2) $l.N - Id(A) \leq \sup\{l.N - Id(B), l.N - Id(C) + 1\}$.
(3) $l.N - Id(C) \leq \sup\{l.N - Id(B), l.N - Id(A) - 1\}$.

Proof. (1). For any finitely generated left ideal I in $N(R)$, we have the following exact sequence

$$\begin{aligned}\operatorname{Ext}^n(R/I, A) \to \operatorname{Ext}^n(R/I, B) \to \operatorname{Ext}^n(R/I, C)\\ \to \operatorname{Ext}^{n+1}(R/I, A) \to \operatorname{Ext}^{n+1}(R/I, B).\end{aligned}$$

Let $l.N - Id(B) = n$. If $l.N - Id(C) \leq n-1$, by Proposition 3.5, $\operatorname{Ext}^n(R/I, C) = \operatorname{Ext}^{n+1}(R/I, B) = 0$. Then $\operatorname{Ext}^{n+1}(R/I, A) = 0$, and hence $l.N - Id(A) \leq n$ by Proposition 3.5 again. If $l.N - Id(A) < n$, then $\operatorname{Ext}^n(R/I, A) = 0$, so $\operatorname{Ext}^n(R/I, B) = 0$, and hence $l.N - Id(B) < n$, contradicting with assumption. Thus $l.N - Id(A) = n$, and (1) follows. If $l.N - Id(C) \geq n$, it is clear that (1) hold.

Similarly, we can prove (2) and (3). □

By Proposition 3.6, we immediately deduce the following corollary.

Corollary 3.7. *Let R be a strongly Nil_*-coherent ring and $0 \to A \to B \to C \to 0$ be an exact sequence of left R-modules with B N-injective. If $0 < l.N - Id(A) < \infty$, then $l.N - Id(A) = l.N - Id(C) + 1$.*

Lemma 3.8. *Let R be a ring and M a left R-module. There is an exact sequence $0 \to M \to I \to N \to 0$ with I N-injective and $\operatorname{Ext}^1(N, I') = 0$ for all N-injective left R-modules I'. Moreover, $\operatorname{Tor}_1(F, N) = 0$ for all N-flat right R-modules F.*

Proof. In view of [10, Theorem 4.1.6] and [21, Corollary 3.5], left R-module M has a special $\mathcal{NI}$-preenvelope $f : M \to I$, that is, there is an exact sequence $0 \to M \to I \to N \to 0$, where I is N-injective and $\operatorname{Ext}^1(N, I') = 0$ for all N-injective left R-modules I'.

For any N-flat right R-module F, F^+ is N-injective by Remark 2.9(2). Thus $(\operatorname{Tor}_1(F, N))^+ \simeq \operatorname{Ext}^1(N, F^+) = 0$, and hence $\operatorname{Tor}_1(F, N) = 0$. □

Proposition 3.9. *Let R be a left strongly Nil_*-coherent ring and M a left R-module. Then $l.N - Id(M) \leq n (n \geq 0)$ if and only if for every left $\mathcal{NI}$-resolution $\cdots \to I_n \to I_{n-1} \to \cdots \to I_1 \to I_0 \to N \to 0$ of any right R-module N, $\operatorname{Hom}(M, I_n) \to \operatorname{Hom}(M, K_n)$ is an epimorphism, where K_n is the nth $\mathcal{NI}$-syzygy of N.*

Proof. We proceed by induction on n. For $n \geq 1$, by Lemma 3.8, there is an exact sequence $0 \to M \to I \to N \to 0$, where I is N-injective and $\operatorname{Ext}^1(N, I') = 0$ for all N-injective left R-modules I'. Then we have the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}(I,I_n) & \to & \mathrm{Hom}(I,K_n) & \to 0 \\ \downarrow & & \downarrow & \\ \mathrm{Hom}(M,I_n) & \to & \mathrm{Hom}(M,K_n) & \\ \downarrow & & & \\ 0. & & & \end{array}$$

Since $I_n \to K_n$ is an $\mathcal{NI}$-precover of K_n, the first arrow is exact. In addition, the first column is exact since $\mathrm{Ext}^1(N, I_n) = 0$. Furthermore, there is commutative diagram

$$\begin{array}{cccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathrm{Hom}(N,K_n) & \to & \mathrm{Hom}(N,I_{n-1}) & \to & \mathrm{Hom}(N,K_{n-1}) & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathrm{Hom}(I,K_n) & \to & \mathrm{Hom}(I,I_{n-1}) & \to & \mathrm{Hom}(I,K_{n-1}) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathrm{Hom}(M,K_n) & \to & \mathrm{Hom}(M,I_{n-1}) & \to & \mathrm{Hom}(M,K_{n-1}) & \\ & & & \downarrow & & & \\ & & & 0. & & & \end{array}$$

$l.N - Id(M) \leq n$ if and only if $l.N - Id(N) \leq n-1$ by Corollary 3.7 if and only if $\mathrm{Hom}(N, I_{n-1}) \to \mathrm{Hom}(N, K_{n-1})$ is an epimorphism by induction if and only if $\mathrm{Hom}(I, K_n) \to \mathrm{Hom}(M, K_n)$ is an epimorphism by the second diagram if and only if $\mathrm{Hom}(M, I_n) \to \mathrm{Hom}(M, K_n)$ is an epimorphism by the first diagram.

For $n = 0$, let $K_0 = M$. Then $\mathrm{Hom}(M, I_0) \to \mathrm{Hom}(M, M)$ is an epimorphism means $\mathrm{Hom}(I, M) \to \mathrm{Hom}(M, M)$ is an epimorphism. Thus $0 \to M \to I \to N \to 0$ splits, and hence M is N-injective. Conversely, if M is N-injective, then it is clear that $\mathrm{Hom}(M, I_0) \to \mathrm{Hom}(M, K_0)$ is an epimorphism. □

Let $\mathfrak{C}, \mathfrak{D}$ and $\mathfrak{E}$ be categories of modules and $T : \mathfrak{C} \times \mathfrak{D} \to \mathfrak{E}$ be an additive functor contravariant in the first variable and covariant in the second. Let $\mathcal{I}$ and $\mathcal{F}$ be the classes of modules of $\mathfrak{C}$ and $\mathfrak{D}$ respectively. Then T is said to be right balanced by $\mathcal{I} \times \mathcal{F}$ if for each module M of $\mathfrak{C}$, there is a $T(-, \mathcal{F})$ exact complex $\cdots \to I_1 \to I_0 \to M \to 0$ with each $I_i \in \mathcal{I}$, and for each module N of $\mathfrak{D}$, there is a $T(\mathcal{I}, -)$ exact complex $0 \to N \to F^0 \to F^1 \to \cdots$ with $F^i \in \mathcal{F}$. Similarly, we have the definition of left balance. If T is covariant in both variables, then we would postulate the existence of complexes $\cdots \to I_1 \to I_0 \to M \to 0$ and $\cdots \to F_1 \to F_0 \to N \to 0$ or $0 \to M \to I^0 \to I^1 \to \cdots$ and $0 \to N \to F^0 \to F^1 \to \cdots$ to define the left or right balance functors relative to $\mathcal{I} \times \mathcal{F}$, respectively.

Lemma 3.10. *If R is left strongly Nil_*-coherent, then $- \otimes -$ on $\mathcal{M}_R \times_R \mathcal{M}$ is right balanced by $\mathcal{NF} \times \mathcal{NI}$.*

Proof. Let M be any right R-module. By Remark 2.14, there is a right $\mathcal{NF}$-resolution $0 \to M \to F^0 \to F^1 \to \cdots$. For any N-injective left R-module N,

N^+ is N-flat by Theorem 2.11. Thus we have an exact sequence

$$\cdots \to \mathrm{Hom}(F^1, N^+) \to \mathrm{Hom}(F^0, N^+) \to \mathrm{Hom}(M, N^+) \to 0.$$

Hence

$$\cdots \to (N \otimes F^1)^+ \to (N \otimes F^0)^+ \to (N \otimes M)^+ \to 0$$

is exact. Then $0 \to N \otimes M \to N \otimes F^0 \to N \otimes F^1 \to \cdots$ is exact. In addition, by Lemma 3.8, the right $\mathcal{NI}$-resolution $0 \to G \to I^0 \to I^1 \to \cdots$ of any left R-module G is exact, so the sequence $0 \to G \otimes F \to I^0 \otimes F \to I^1 \otimes F \to \cdots$ is exact for any $F \in \mathcal{NF}$ by Lemma 3.8 again, as desired. □

Remark 3.11. (1) $\mathrm{Tor}^n(-,-)$ denotes the nth right derived functor of $- \otimes -$ with respect to the pair $\mathcal{NF} \times \mathcal{NI}$. If R is a left strongly Nil_*-coherent ring, for any right R-module M and left R-module N, $\mathrm{Tor}^n(M, N)$ can be computed using either the right $\mathcal{NF}$-resolution of M or the right $\mathcal{NI}$-resolution of N by Lemma 3.10.

(2) If R is a left strongly Nil_*-coherent ring, by the proof of Lemma 3.8, every left R-module has a right $\mathcal{NI}$-resolution. So $\mathrm{Hom}(-,-)$ is left balanced on ${}_R\mathcal{M} \times_R \mathcal{M}$ by $\mathcal{NI} \times \mathcal{NI}$. Let $\mathrm{Ext}_n(-,-)$ be the nth left derived functor of $\mathrm{Hom}(-,-)$ with respect to the pair $\mathcal{NI} \times \mathcal{NI}$. Then, for two left R-modules M and N, $\mathrm{Ext}_n(M, N)$ can be computed using the right $\mathcal{NI}$-resolution of M or the left $\mathcal{NI}$-resolution of N.

We are now in a position to prove the following theorem.

Theorem 3.12. *If R is left strongly Nil_*-coherent and $n \geq 0$, then the following are equivalent*:

(1) $l.N - Id(R) \leq n$.
(2) *If $0 \to M \to F^0 \to F^1 \to \cdots$ is a right $\mathcal{NF}$-resolution of right R-module M, then the sequence is exact at F^k for $k \geq n-1$, where $F^{-1} = M$.*
(3) *For every flat left R-module F, there is an exact sequence $0 \to F \to A^0 \to A^1 \to \cdots \to A^n \to 0$ with each $A^i \in \mathcal{NI}$.*
(4) *For every injective right R-module A, there is an exact sequence $0 \to F_n \to \cdots \to F_1 \to F_0 \to A \to 0$ with each $F_i \in \mathcal{NF}$.*
(5) *If $\cdots \to I_1 \to I_0 \to M \to 0$ is a left $\mathcal{NI}$-resolution of a left R-module M, then the sequence is exact at I_k for $k \geq n-1$, where $I_{-1} = M$.*

Proof. (3)⇒(1) is trivial.

(1)⇒(2). By Remark 3.11 (1), the right derived functor $\mathrm{Tor}^n(R, M)$ can be computed using either a right $\mathcal{NF}$-resolution of M or a right $\mathcal{NI}$-resolution of R.

If $n \geq 2$, we have the exact sequence $0 \to R \to A^0 \to \cdots \to A^n \to 0$ with $A^i \in \mathcal{NI}$, so $\mathrm{Tor}^k(R, M) = 0$ for $k \geq n-1$. Computing using $0 \to M \to F^0 \to F^1 \to \cdots$ in (2), we see that the sequence $\cdots \to R \otimes F^{n-2} \to R \otimes F^{n-1} \to R \otimes F^n \to \cdots$ is exact at $R \otimes F^k$ for $k \geq n-1$, so $0 \to M \to F^0 \to F^1 \to \cdots$ is exact at F^k for $k \geq n-1$.

If $n = 1$, $0 \to R \to A^0 \to A^1 \to 0$ is exact, where A^i is N-injective. So $\mathrm{Tor}^1(R, M) = 0$ as above, $F^0 \to F^1 \to F^2$ is exact and $R \otimes M \to \mathrm{Tor}^0(R, M)$ is epic. Computing the latter morphism using $0 \to M \to F^0 \to F^1$, we have $M \to F^0 \to F^1 \to \cdots$ is exact.

If $n = 0$, then R is N-injective as a right R-module. But the balance of $- \otimes -$ then gives $0 \to R \otimes M \to R \otimes F^0 \to R \otimes F^1 \to \cdots$ is exact. Thus $0 \to M \to F^0 \to F^1 \to \cdots$ is exact.

(2)⇒(3). Let $0 \to M \to F^0 \to F^1 \to \cdots$ be a right $\mathcal{NF}$-resolution of a finitely presented left R-module M. By assumption, the sequence is exact at F^k for $k \geq n-1$. Let $0 \to F \to A^0 \to A^1 \to \cdots$ be exact with F flat and each A^i N-injective. If $n \geq 2$, we get $\mathrm{Tor}^k(F, M) = 0$ for $k \geq n-1$ since F is flat. Computing using $0 \to F \to A^0 \to A^1 \to \cdots$, then $A^{n-2} \otimes M \to A^{n-1} \otimes M \to A^n \otimes M \to A^{n+1} \otimes M$ is exact. By [8, Lemma 8.4.23], $K = \mathrm{Ker}(A^n \to A^{n+1})$ is a pure submodule of A^n, hence K is also N-injective by Corollary 2.12. Then $0 \to F \to A^0 \to A^1 \to \cdots \to A^{n-1} \to K \to 0$ gives the desired exact sequence.

If $n = 1$, then $M \to F^0 \to F^1 \to \cdots$ is exact. Thus $\mathrm{Tor}^k(F, M) = 0$ for $k \geq 1$ and $F \otimes M \to \mathrm{Tor}^0(F, M)$ is epic. So $F \otimes M \to A^0 \otimes M \to A^1 \otimes M \to A^2 \otimes M$ is exact. By [8, Lemma 8.4.23] again, we get the exact sequence $0 \to F \to A^0 \to K \to 0$ with $K = \mathrm{Ker}(A^1 \to A^2)$ N-injective.

If $n = 0$, then $0 \to M \to F^0 \to F^1 \to \cdots$ is exact, so $\mathrm{Tor}^k(F, M) = 0$ for $k \geq 0$ and $F \otimes M \to \mathrm{Tor}^0(F, M)$ is an isomorphism. This gives that $0 \to F \otimes M \to A^0 \otimes M \to A^1 \otimes M$ is exact, which implies F is a pure submodule of A^0, hence F is N-injective.

(5)⇒(1). By assumption, $I_n \to I_{n-1} \to I_{n-2}$ is exact at I_{n-1}. Thus $I_n \to K_n$ is epic, where $K_n = \mathrm{Ker}(I_{n-1} \to I_{n-2})$. Hence $\mathrm{Hom}(R, I_n) \to \mathrm{Hom}(R, K_n)$ is epic. By Proposition 3.9, $l.N - Id(R) \leq n$.

(1)⇒(5). If $n \geq 2$. Let $0 \to R \to A^0 \to \cdots \to A^n \to 0$ be a right $\mathcal{NI}$-resolution of a right R-module M, then $\mathrm{Ext}_k(R, M) = 0$ for $k \geq n-1$. By Remark 3.11 (2), we can compute $\mathrm{Ext}_k(R, M) = 0$ using a left $\mathcal{NI}$-resolution of M $\cdots \to I_1 \to I_0 \to M \to 0$, so $\cdots \to \mathrm{Hom}(R, I_n) \to \mathrm{Hom}(R, I_{n-1}) \to \cdots \to \mathrm{Hom}(R, I_1) \to \mathrm{Hom}(R, I_0) \to \mathrm{Hom}(R, M) \to 0$ is exact at $\mathrm{Hom}(R, I_k)$ for $k \geq n-1$. Hence $\cdots \to I_1 \to I_0 \to M \to 0$ is exact at I_k for $k \geq n-1$.

If $n = 1$, then there is an exact sequence $0 \to R \to A^0 \to A^1 \to 0$ with $A^i \in \mathcal{NI}$. So $0 \to \mathrm{Hom}(A^1, M) \to \mathrm{Hom}(A^0, M) \to \mathrm{Hom}(R, M)$ is exact. Thus $\mathrm{Ext}_k(R, M) = 0$ for $k \geq 1$ and $\mathrm{Ext}_0(R, M) \to \mathrm{Hom}(R, M)$ is a monomorphism. But computing $\mathrm{Ext}_0(R, M)$ using a left $\mathcal{NI}$-resolution of M, we see that $I_1 \to I_0 \to M$ is exact at I_0, so $\cdots \to I_1 \to I_0 \to M \to 0$ is exact at I_k for $k \geq 0$.

If $n = 0$, then R is N-injective as a left R-module. So every $\mathcal{NI}$-precover is epic, and hence $\cdots \to I_1 \to I_0 \to M \to 0$ is exact.

The proof of (4)⇔(5) is dual to that of (2)⇔(3). □

Proposition 3.13. *Let R be a left strongly Nil_*-coherent ring and $wD(R) < \infty$, where $wD(R)$ is the weak global dimension of R. Then $l.N - Id(R) = l.N - I.dim(R) \leq wD(R)$.*

Proof. We first prove the right inequality. By the definitions of left N-injective dimensions of modules and rings, we have

$$l.N-I.dim(R)=\sup\{l.pd(R/I)\mid I \text{ is finitely generated left ideal in } N(R)\},$$

where $l.pd(R/I)$ is the left projective dimension of R/I. Then $l.N-I.dim(R)\le wD(R)$. We suppose that $l.N-I.dim(R)=n<\infty$.

For the left equality, it suffices to prove $l.N-I.dim(R)\le l.N-Id(R)$. Assume that $l.N-Id(R)=m<\infty$. By the similar proof of [7, Proposition 3.10], it can be proven that $l.N-Id(F)\le m$ for any free left R-module F. Note that, for any left R-module M, there exists an exact sequence $0\to K_n\to F_{n-1}\to F_{n-2}\to\cdots\to F_0\to M\to 0$ with each F_i free. Then $l.N-Id(K_n)=n$ and $l.N-Id(F_i)\le m$. By Proposition 3.5, $\mathrm{Ext}^{m+1}(R/I,M)\cong\mathrm{Ext}^{m+n+1}(R/I,K_n)=0$ for every finitely generated left ideal I in $N(R)$, and hence $l.N-Id(M)\le m$. Therefore, $l.N-Id(R)=l.N-I.dim(R)$. □

Example 6. Let $\mathbb{F}[x]$ be a polynomial ring over a field $\mathbb{F}$. Then $\mathbb{F}[x]$ is semiprime, and hence $l.N-Id(R)=l.N-I.dim(R)=0$. It is easy to verify that $wD(R)=1$.

Lemma 3.14. *Let R be a left strongly Nil_*-coherent ring and M a left R-module. If $\mathrm{Ext}^1(E,M)=0$ for all N-injective left R-modules E, then M has an $\mathcal{NI}$-cover $L\to M$ with L injective.*

Proof. In view of Proposition 2.13, M has an $\mathcal{NI}$-cover $f:L\to M$. For the exact sequence $0\to L\xrightarrow{i}E\to L'\to 0$ with E injective, L' is N-injective. Thus $\mathrm{Hom}(E,M)\to\mathrm{Hom}(L,M)\to 0$ is exact since $\mathrm{Ext}^1(L',M)=0$, and hence there is $g\in\mathrm{Hom}(E,M)$ such that $f=gi$. Then there exists $h:E\to L$ such that $g=fh$ since $f:L\to M$ is an $\mathcal{NI}$-cover of M. So $f=fhi$, implies hi is isomorphism. Therefore, L is injective. □

Theorem 3.15. *If R is left strongly Nil_*-coherent and $n\ge 1$, then the following are equivalent*:

(1) $l.N-I.dim(R)\le n$.
(2) *Every nth $\mathcal{NI}$-syzygy of a left R-module is N-injective.*
(3) *Every $(n-1)$th $\mathcal{NI}$-syzygy of a right R-module has an $\mathcal{NI}$-cover which is a monomorphism.*
Moreover, if $n\ge 2$, then the above conditions are equivalent to:
(4) *Every $(n-2)$th $\mathcal{NI}$-syzygy in a minimal left $\mathcal{NI}$-resolution of a left R-module has an $\mathcal{NI}$-cover with the unique mapping property.*

Proof. (1)⇒(2). Let K_n be nth $\mathcal{NI}$-syzygy of a left R-module. Then $l.N-Id(K_n)\le n$. So $\mathrm{Hom}(K_n,I_n)\to\mathrm{Hom}(K_n,K_n)$ is an epimorphism by Proposition 3.9, whence K_n is N-injective.

(2)⇒(3). Let $f:I_{n-1}\to K_{n-1}$ be an $\mathcal{NI}$-precover of the $(n-1)$th $\mathcal{NI}$-syzygy K_{n-1}, and $K_n=\mathrm{Ker}(f)$. Then we have the exact sequence $0\to K_n\to$

$I_{n-1} \to \mathrm{im}(f) \to 0$. By assumption, K_n is N-injective, so is $\mathrm{im}(f)$. Thus the inclusion $\mathrm{im}(f) \to K_{n-1}$ is an $\mathcal{NI}$-cover which is a monomorphism.

(3)⇒(2). Let $\cdots \to I_n \to I_{n-1} \to \cdots \to I_1 \to I_0 \to N \to 0$ be any left $\mathcal{NI}$-resolution of a left R-module N and $K_n = \mathrm{Ker}(I_{n-1} \to I_{n-2}), K_{n-1} = \mathrm{Ker}(I_{n-2} \to I_{n-3})$. K_{n-1} has a monomorphic $\mathcal{NI}$-cover $I \to K_{n-1}$ by assumption. Thus $K_n \oplus I \simeq I_{n-1}$ in terms of [8, Lemma 8.6.3]. So K_n is N-injective by Remark 2.9(1).

(2)⇒(1). Let M be a left R-module. For a left $\mathcal{NI}$-resolution $\cdots \to I_n \to I_{n-1} \to \cdots \to I_1 \to I_0 \to N \to 0$ of a left R-module N, $I_n \to K_n$ is a split epimorphism since K_n is N-injective. Thus $\mathrm{Hom}(M, I_n) \to \mathrm{Hom}(M, K_n)$ is epimorphic, hence $l.N-Id(M) \leq n$ by Proposition 3.9. Then $l.N-I.dim(R) \leq n$.

(3)⇒(4). Let $\cdots \to I_{n-3} \to I_{n-4} \to \cdots \to I_1 \to I_0 \to M \to 0$ be a minimal $\mathcal{NI}$-resolution of a left R-module M with $K_{n-2} = \mathrm{Ker}(I_{n-3} \to I_{n-4})$. By assumption, $K_{n-1} = \mathrm{Ker}(I_{n-2} \to I_{n-3})$ has a monomorphic $\mathcal{NI}$-cover $i : I_{n-1} \to K_{n-1}$. Note $\mathrm{Ext}^1(I, K_{n-1}) = 0$ for all N-injective right R-modules I by Wakamatsu's Lemma. Thus I_{n-1} is injective by Lemma 3.14. But K_{n-1} has no nonzero injective submodule by [15, Corollary 1.2.8]. Thus $I_{n-1} = 0$, and hence $\mathrm{Hom}(I, K_{n-1}) = \mathrm{Hom}(I, I_{n-1}) = 0$ for any N-injective left R-module I. So we have the exact sequence $0 \to \mathrm{Hom}(I, I_{n-2}) \to \mathrm{Hom}(I, K_{n-2}) \to 0$ for any N-injective left R-module I, as desired.

(4)⇒(2). Let $\cdots \to I_n \to I_{n-1} \to \cdots \to I_1 \to I_0 \to M \to 0$ be an $\mathcal{NI}$-resolution of a left R-module M with $K_n = \mathrm{Ker}(I_{n-1} \to I_{n-2})$. By assumption, M has a minimal $\mathcal{NI}$-resolution of the form $0 \to I'_{n-2} \to I'_{n-3} \to \cdots \to I'_1 \to I'_0 \to M \to 0$. In view of [8, Corollary 8.6.4], $K_n \oplus I_{n-2} \oplus I'_{n-3} \oplus \cdots \cong I_{n-1} \oplus I'_{n-2} \oplus I_{n-3} \oplus \cdots$. Thus K_n is N-injective. □

Corollary 3.16. *If R is left strongly Nil_*-coherent, then the following are equivalent*:

(1) $l.N - I.dim(R) \leq 2$.
(2) *Every left R-module has an $\mathcal{NI}$-cover with the unique mapping property.*

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