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A CHARACTERIZATION OF CONCENTRIC HYPERSPHERES IN $\mathbb{R}^n$

Dong-Soo Kim and Young Ho Kim

Abstract. Concentric hyperspheres in the n-dimensional Euclidean space $\mathbb{R}^n$ are the level hypersurfaces of a radial function $f : \mathbb{R}^n \to \mathbb{R}$. The magnitude $||\nabla f||$ of the gradient of such a radial function $f : \mathbb{R}^n \to \mathbb{R}$ is a function of the function f. We are interested in the converse problem. As a result, we show that if the magnitude of the gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$ with isolated critical points is a function of f itself, then f is either a radial function or a function of a linear function. That is, the level hypersurfaces are either concentric hyperspheres or parallel hyperplanes. As a corollary, we see that if the magnitude of a conservative vector field with isolated singularities on $\mathbb{R}^n$ is a function of its scalar potential, then either it is a central vector field or it has constant direction.

1. Introduction

Consider a radial function $f : \mathbb{R}^n \to \mathbb{R}$, that is, f satisfies $f(x) = g(||x||)$ for some function g. Then it is well-known that the magnitude of the gradient of f is a function of the function f.

In this regard, for a function $f : \mathbb{R}^n \to \mathbb{R}$, we consider the following condition

$$||\nabla f(p)|| = \phi(f(p)), \quad p \in \mathbb{R}^n, \tag{C}$$

where $\phi : \mathbb{R} \to [0, +\infty)$ is a real valued function. Then, the radial functions on $\mathbb{R}^n$ satisfy condition (C).

Therefore, it is natural to ask a question:

"What kinds of functions on $\mathbb{R}^n$ satisfy condition (C)*?"*

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In this article, we study the functions defined on $\mathbb{R}^n$ which satisfy condition (C). As a result, first, we establish a local characterization of functions satisfying condition (C).

Proposition 1. *For a function f defined on an open set $V \subset \mathbb{R}^n$ without critical points, the following are equivalent*:

1) *The function f satisfies condition* (C)*.*
2) *Every integral curve of ∇f is a straight line.*
3) *For a level hypersurface M, f is constant on each parallel hypersurface of M.*

Second, using Proposition 1, we prove the following characterization theorem of functions satisfying condition (C) globally on $\mathbb{R}^n$.

Theorem 2. *Suppose that a function $f : \mathbb{R}^n \to \mathbb{R}$ with isolated critical points satisfies condition* (C)*. Then f is a function of either a distance function $r = ||p - o||$ from a fixed point o or a linear function. That is, the level sets are either concentric hyperspheres or parallel hyperplanes.*

As applications of Theorem 2, we get the following characterizations.

Corollary 3. *Suppose that a function f defined on $\mathbb{R}^n$ with isolated critical points satisfies condition* (C)*. Then we have the following.*

1) *If f has no critical points, then f is a function of some linear function.*
2) *If f has at least one critical point, then f is a radial function from a fixed point o.*

In particular, we have:

Corollary 4. *For a function f defined on $\mathbb{R}^n$, the following are equivalent*:

1) *The magnitude $||\nabla f||$ of the gradient of f is constant on $\mathbb{R}^n$.*
2) *The function f is a linear function.*

It follows from Corollary 4 that a conservative vector field with constant magnitude defined on the whole space $\mathbb{R}^n$ is a constant vector field.

For a conservative vector field F on $\mathbb{R}^n$, condition (C) is equivalent to the following:

"The magnitude of a conservative vector field F is a function of its scalar potential function."

For a fixed point $o \in \mathbb{R}^n$, a vector field F defined on $\mathbb{R}^n \setminus \{o\}$ is called a *central vector field* if it is invariant under orthogonal transformations around o. The point o is called the center of the vector field F. The gradient vector field of a radial function from a point $o \in \mathbb{R}^n$ is an example of such vector fields.

Since orthogonal transformations around a fixed point o are actually rotations and reflections, the invariance conditions show that vectors of a central vector field are always directed towards, or away from, its center o. Hence, it is straightforward to show that every central vector field with center $o \in \mathbb{R}^n$ is the gradient vector field of a radial function from o.

Now, we may restate Theorem 2 as follows.

Theorem 5. *Suppose that the magnitude of a conservative vector field F on $\mathbb{R}^n$ with isolated singularities is a function of its scalar potential function. Then we have the following.*

1) *If F has no singularities, then F has constant direction.*
2) *If F has at least one singularity, then F has exactly one zero $o \in \mathbb{R}^n$ and F is a central vector field with center o.*

Theorem 2 or Theorem 5 might be well-known, but we could not find the references for them (cf. [7]).

Throughout this article, all objects are smooth (that is, C^2), unless otherwise mentioned. For notations and terminologies, see [2], [3], [5] or [9].

2. Proofs

Suppose that $||\nabla f|| = \phi(f)$ for a real valued function ϕ. Let R_f denote the set of regular values of f. For $k \in R_f$, let's denote by U the unit normal vector to the level hypersurface $f^{-1}(k)$ in the direction of ∇f. Hence, it follows from condition (C) that

$$\nabla f = \phi(k) U \quad \text{on} \quad f^{-1}(k). \tag{1}$$

First, we prove the following lemma.

Lemma 6. *For a function f defined on an open set $V \subset \mathbb{R}^n$ without critical points, the following are equivalent*:

1) *$||\nabla f|| = \phi(f)$ for a function ϕ.*
2) *Every integral curve $y(t)$ of ∇f is a straight line.*

Proof. For a point $p \in f^{-1}(k)$ with $k \in R_f$, we denote by $\kappa_1(p), \ldots, \kappa_{n-1}(p)$ the principal curvatures of the level hypersurface $f^{-1}(k)$ associated with the corresponding principal directions $e_1(p), \ldots, e_{n-1}(p)$ with respect to U at p.

For each $i = 1, 2, \ldots, n-1$, we let $x_i(s)$ denote a unit speed curve of $f^{-1}(k)$ starting from p with $x_i'(0) = e_i(p), i = 1, 2, \ldots, n-1$. Then (1) implies for each $i = 1, 2, \ldots, n-1$

$$\nabla f(x_i(s)) = \phi(k) U(x_i(s)). \tag{2}$$

By differentiating (2) with respect to s, we get at $s = 0$

$$H^f(p) e_i(p) = -\phi(k) \kappa_i(p) e_i(p), \tag{3}$$

where H^f denotes the Hessian matrix of f. Since $H^f(p)$ is symmetric, from (3) we see that for any point $p \in f^{-1}(k)$

$$H^f(p) \nabla f(p) = h(p) \nabla f(p), \tag{4}$$

where h is a function. It follows from (4) that for all $x \in \mathbb{R}^n$, f satisfies

$$H^f(x) \nabla f(x) = h(x) \nabla f(x). \tag{5}$$

Let's denote by $y(t)$ the integral curve of ∇f with $y(0) = p \in f^{-1}(k)$. Then we have

$$y''(t) = H^f(y(t))\nabla f(y(t)) = h(t)\nabla f(y(t)) = h(t)y'(t), \tag{6}$$

where $h(t) = h(y(t))$ and the second equality follows from (5). This shows that $y(t)$ is a parametrization of a straight line.

Conversely, suppose that every integral curve $y(t)$ of ∇f is a straight line. Then every integral curve $y(t)$ satisfies $y''(t) = h(t)y'(t)$ for some function h. Hence (6) shows that f satisfies (5) for all $x \in \mathbb{R}^n$.

Now, for a fixed unit speed curve $x(s)$ on the level hypersurface $f^{-1}(k)$, we differentiate $||\nabla f(x(s))||^2$ as follows.

$$\begin{aligned}\frac{d}{ds}||\nabla f(x(s))||^2 &= 2\left\langle \frac{d}{ds}\nabla f(x(s)), \nabla f(x(s))\right\rangle \\ &= 2\left\langle H^f(x(s))x'(s), \nabla f(x(s))\right\rangle \\ &= 2\left\langle x'(s), H^f(x(s))\nabla f(x(s))\right\rangle \\ &= 2\left\langle x'(s), h(x(s))\nabla f(x(s))\right\rangle \\ &= 0,\end{aligned} \tag{7}$$

where the 3rd and 4th equalities follow from the symmetry of H^f and (5), respectively. Thus, $||\nabla f||^2$ is constant on each level hypersurface of f. This completes the proof. $\square$

For a hypersurface M of $\mathbb{R}^n$ with a unit normal vector field U, parallel hypersurfaces M_t, $t \in \mathbb{R}$ of M are defined by

$$M_t = \{p + tU(p) \mid p \in M\}.$$

Next, we show that the level hypersurfaces of f are parallel.

Lemma 7. *Suppose that a function f defined on an open set $V \subset \mathbb{R}^n$ without critical points satisfies condition* (C). *Then, for a level hypersurface M of f, f is constant on each parallel hypersurface of M.*

Proof. Let $x(s)$ denote a fixed unit speed curve of a level hypersurface $f^{-1}(k)$ of f. We consider the integral curve $y_s(t)$ of ∇f with $y_s(0) = x(s)$. Then, from Lemma 6 we have

$$y_s(t) = x(s) + a(t)\nabla f(x(s)), \tag{8}$$

where $a(t)$ is a function with $a(0)= 0$. Since $\nabla f(x(s)) = y_s'(0) = a'(0)\nabla f(x(s))$, we get $a'(0) = 1$.

By differentiating $f(y_s(t))$ with respect to s, it follows from (5) and (8) that

$$\begin{aligned} \frac{d}{ds}f(y_s(t)) &= \left\langle \nabla f(y_s(t)), \frac{d}{ds}y_s(t) \right\rangle \\ &= \left\langle \nabla f(y_s(t)), x'(s) + a(t)H^f(x(s))x'(s) \right\rangle \\ &= a(t)a'(t) \left\langle \nabla f(x(s)), H^f(x(s))x'(s) \right\rangle \\ &= a(t)a'(t) \left\langle H^f(x(s))\nabla f(x(s)), x'(s) \right\rangle \\ &= 0. \end{aligned} \tag{9}$$

Hence $f(y_s(t))$ is a function of t only. This shows that f is constant on each parallel hypersurface of M. □

If we let $k(t) = f(y_s(t))$, then we have from condition (C)

$$k'(t) = \langle \nabla f(y_s(t)), \nabla f(y_s(t)) \rangle = \phi(k(t))^2, \quad k(0) = k. \tag{10}$$

On the other hand, from (8) we get

$$k'(t) = \langle y_s'(t), y_s'(t) \rangle = \phi(k)^2 a'(t)^2. \tag{11}$$

It follows from (10) and (11) that

$$\phi(k(t)) = \phi(k)a'(t). \tag{12}$$

Thus $a(t)$ is determined by (12) with $a(0) = 0$, which is independent of $x(s)$.

Remark 8. If we let $z_s(t)$ denote the integral curve of $U = \nabla f/||\nabla f||$ with $z_s(0) = x(s)$, then we have $z_s(t) = x(s) + tU(x(s))$ and $\frac{d}{ds}f(z_s(t)) = 0$. For $k(t) = f(z_s(t))$, we get $k'(t) = \phi(k(t))$.

Conversely, suppose that f is constant on each parallel hypersurface M_t. That is, $f(p + tU(p)) = k(t), p \in M$, where $k(t)$ is a function of t. Then we have

$$\nabla f(p + tU(p)) = k'(t)U(p). \tag{13}$$

Hence f satisfies condition (C) with $\phi = \pm k' \circ k^{-1}$ if $k'(t) \neq 0$.

Thus, we have the following local characterization of functions satisfying condition (C).

Proposition 1. *For a function f defined on an open set $V \subset \mathbb{R}^n$ without critical points, the following are equivalent*:

1) *f satisfies condition* (C).
2) *Every integral curve of ∇f is a straight line.*
3) *For a level hypersurface M, f is constant on each parallel hypersurface of M.*

Now, we prove the main theorem as follows.

Theorem 2. *Suppose that a function $f : \mathbb{R}^n \to \mathbb{R}$ with isolated critical points satisfies condition* (C)*. Then f is a function of either a distance function $r = ||p - o||$ from a fixed point o or a linear function. That is, the level sets are either concentric hyperspheres or parallel hyperplanes.*

Proof. Suppose that a function $f : \mathbb{R}^n \to \mathbb{R}$ with isolated critical points satisfies condition (C) globally on $\mathbb{R}^n$. Without loss of generality, we may assume that $0 \in R_f$. Then, the above discussion shows that for the unit normal $U = \nabla f/||\nabla f||$ to the level hypersurface $M_0 = f^{-1}(0)$, the function f and ∇f is given by

$$f(p + tU(p)) = k(t) \quad \text{and} \quad \nabla f(p + tU(p)) = k'(t)U(p), \tag{14}$$

where $k(t)$ is a function with $k(0) = 0$ and $k'(0) \neq 0$.

We consider the flow $y : M_0 \times \mathbb{R} \to \mathbb{R}^n$ given by

$$y(p, t) = p + tU(p). \tag{15}$$

For a point $p \in M_0$, we denote by $\kappa_1(p), \ldots, \kappa_{n-1}(p)$ the principal curvatures of M associated with the corresponding principal directions $e_1(p), \ldots, e_{n-1}(p)$ with respect to U at p. For each $i = 1, 2, \ldots, n-1$, we let $x_i(s)$ denote a unit speed curve of M_0 starting from p with $x_i'(0) = e_i(p), i = 1, 2, \ldots, n-1$. Then, we have for each $i = 1, 2, \ldots, n-1$

$$\frac{d}{ds} y(x_i(s), t)|_{s=0} = (1 - t\kappa_i(p))e_i(p). \tag{16}$$

Since $k'(0) \neq 0$, we see that $k'(t) \neq 0$ on an open interval I_0 containing 0. Hence for each $t \in I_0$, the level set $M_t = f^{-1}(k(t))$ is a nonsingular hypersurface. Consider an open interval I containing 0. Then it follows from (15) and (16) that the level sets $M_t, t \in I$ are all nonsingular hypersurfaces if and only if each $t \in I$ satisfies the following

$$1 - t\kappa_i(p) > 0, i = 1, 2, \ldots, n-1 \quad \text{and} \quad p \in M_0. \tag{17}$$

Let's denote by I^* the maximal open interval containing 0 such that each $t \in I^*$ satisfies (17).

First, suppose that the maximal interval I^* has an end point t_0. Then we have $k'(t_0) = 0$, otherwise t_0 is contained in I^*. Hence every point of the level set M_{t_0} is a critical point of the function f. Since such points are isolated, we see that the level set M_{t_0} is a fixed point $o \in \mathbb{R}^n$.

This shows that M_0 is a hypersphere of radius $|t_0|$ centered at o. Hence we have $\kappa_i(p) = 1/t_0$ for all $i = 1, 2, \ldots, n-1$ and $p \in M_0$. It follows from (17) that $I^* = (t_0, \infty)$ or $I^* = (-\infty, t_0)$ according to the sign of t_0. Since each level set M_t of f is a parallel hypersurface of M_0, it is also a hypersphere centered at o. Therefore f is a function of the distance function $r = ||p - o||$ from the point o. Thus, f is a radial function from the point o.

Finally, suppose that the maximal interval I^* is the real line $\mathbb{R}$. Then, it follows from (17) and the definition of I^* that

$$\kappa_i(p) = 0, i = 1, 2, \ldots, n-1 \quad \text{and} \quad p \in M_0. \tag{18}$$

Therefore M_0 is a hyperplane, and hence every level set of f is also a hyperplane. Thus f is a function of a linear function. This completes the proof of Theorem 2. □

From Theorem 2, immediately we get Corollaries 3, 4 and Theorem 5.

Remark. Suppose that a function $f : \mathbb{R}^n \to \mathbb{R}$ with isolated critical points satisfies condition (C) for some $\phi : \mathbb{R} \to [0, +\infty)$.

If we additionally assume that f has at most one critical value (say, $f(p) = a$, and hence we have $\phi(a) = 0$), then setting $h(r) := \int_{r_0}^{r} \phi(s)^{-1} ds$ ($[r_0, r]$ being a segment outside $\phi^{-1}(0) = \{a\}$), we obtain $||\nabla(h \circ f)(x)|| = 1$ on a domain in $\mathbb{R}^n$. This is the classical eikonal equation (cf. [4] and [8]). But, the solutions of eikonal equations do not imply our theorems in this article.

Instead, in case f has no critical points, $h \circ f$ is a function globally defined on $\mathbb{R}^n$ with $||\nabla(h \circ f)(x)|| = 1$. Hence $h \circ f$ is a linear function (Remark 2.3 of [1] and [6]). This gives a proof of 1) in Corollary 3.

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Dong-Soo Kim
Department of Mathematics
Chonnam National University
Kwangju 500-757, Korea
E-mail address: dosokim@chonnam.ac.kr

Young Ho Kim
Department of Mathematics
Kyungpook National University
Taegu 702-701, Korea
E-mail address: yhkim@knu.ac.kr