Bull. Korean Math. Soc. **51** (2014), No. 2, pp. 519–530
http://dx.doi.org/10.4134/BKMS.2014.51.2.519

SOME CHARACTERIZATIONS OF COHEN-MACAULAY MODULES IN DIMENSION $> s$

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Abstract. Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. For an integer $s > -1$, we say that M is *Cohen-Macaulay in dimension* $> s$ if every system of parameters of M is an M-sequence in dimension $> s$ introduced by Brodmann-Nhan [1]. In this paper, we give some characterizations for Cohen-Macaulay modules in dimension $> s$ in terms of the Noetherian dimension of the local cohomology modules $H^i_{\mathfrak{m}}(M)$, the polynomial type of M introduced by Cuong [5] and the multiplicity $e(\underline{x}; M)$ of M with respect to a system of parameters $\underline{x}$.

1. Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module with $\dim M = d$.

It is well known that the Cohen-Macaulay modules play an important role in the theory of Noetherian rings and finitely generated modules. Recall that M is called *Cohen-Macaulay* if every system of parameters (s.o.p. for short) of M is an M-sequence. The structure of Cohen-Macaulay modules are well-known in the multiplicity, local cohomology, $\mathfrak{m}$-adic completion, localization, etc (see [3]). There are some extensions of the concepts of M-sequence and Cohen-Macaulay modules, among which are the notions of M-sequence in dimension $> s$ introduced by Brodmann-Nhan [1] and Cohen-Macaulay modules in dimension $> s$ defined by Zamani [21].

Definition. Let $s \geqslant -1$ be an integer. A sequence $(x_1, \ldots, x_r)$ of elements in $\mathfrak{m}$ is said to be an *M-sequence in dimension* $> s$ if $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_R(M/(x_1, \ldots, x_{i-1})M)$ satisfying $\dim(R/\mathfrak{p}) > s$ for all $i = 1, \ldots, r$. We say that M is a *Cohen-Macaulay module in dimension* $> s$ if every s.o.p. of M is an M-sequence in dimension $> s$.

Received December 20, 2012; Revised October 10, 2013.
2010 *Mathematics Subject Classification.* 13D45, 13E05.
Key words and phrases. Cohen-Macaulay modules in dimension $> s$, M-sequence in dimension $> s$, multiplicity, Noetherian dimension, local cohomology modules.
This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.01-2011.20.

It is clear that M-sequences in dimension $> s$ for $s = -1, 0, 1$ are exactly M-sequences, f-sequences with respect to M in sense of Cuong-Schenzel-Trung [9], and generalized regular sequences with respect to M in sense of Nhan [17], respectively. Therefore Cohen-Macaulay modules in dimension $> s$ for $s = -1, 0, 1$ are, respectively, Cohen-Macaulay modules, f-modules defined in [9] and generalized f-modules introduced in Nhan-Morales [18]. Moreover, for each ideal I of R, all maximal f-sequences with respect to M in I have the same length and the length of a maximal f-sequence of M in I is exactly the least integer r such that the local cohomology module $H^r_I(M)$ is not Artinian (cf. [14]). Also, all maximal generalized regular sequences of M in an ideal I have the same length and this common length is the least integer i such that $\mathrm{Supp}(H^i_I(M))$ is a finite set (see [17]).

Zamani [21] gave some properties of Cohen-Macaulay modules in dimension $> s$ concerning the $\mathfrak{m}$-adic completion, the localization, the catenarity, the equidimension up to primary components of dimension $\leqslant s$ of the support of M. He also presented some results concerning the finiteness of associated primes of local cohomology modules as extensions of previous results by Hellus [11] and Nhan-Morales [18].

The purpose of this paper is to give some characterizations for Cohen-Macaulay modules in dimension $> s$ in terms of the multiplicity $e(\underline{x}; M)$ of M, the Noetherian dimension $\text{N-dim}_R\, H^i_{\mathfrak{m}}(M)$ of local cohomology modules $H^i_{\mathfrak{m}}(M)$, and the polynomial type $p(M)$ of M introduced by Cuong [5]. Note that $H^i_{\mathfrak{m}}(M)$ is an Artinian R-module and the Noetherian dimension for Artinian modules was introduced in [19] and [13]. It is clear that if $s \geq d$, then M is always Cohen-Macaulay in dimension $> s$. Moreover, the structure of Cohen-Macaulay modules in dimension > -1 (i.e., Cohen-Macaulay modules) can be described in terms of the theories of multiplicity and local cohomology. Therefore we only consider the case $0 \leqslant s < d$.

The main result of this paper is the following theorem.

Main Theorem. *Suppose that* $0 \leqslant s < d$.

(i) *The following statements are equivalent*:

(a) $\text{N-dim}_R(H^i_{\mathfrak{m}}(M)) \leqslant s$ *for all* $i < d$.

(b) $p(M) \leqslant s$.

(c) *There exist a s.o.p.* $\underline{x} = (x_1, \dots, x_d)$ *of* M *and* $k_1, \dots, k_s \in \{1, \dots, d\}$ *such that*

$$I(y_1, \dots, y_d; M) = I(x_1, \dots, x_d; M),$$

where $y_j = x_j^2$ *if* $j \notin \{k_1, \dots, k_s\}$ *and* $y_j = x_j$ *if* $j \in \{k_1, \dots, k_s\}$.

(d) *There exist a s.o.p.* $\underline{x} = (x_1, \dots, x_d)$ *of* M *and a constant* $C_{\underline{x}}$ *(not depending on* n*) such that for all integer* $n > 0$,

$$I(x_1^n, \dots, x_d^n; M) \leqslant n^s C_{\underline{x}}.$$

(ii) *If one of the conditions* (a), (b), (c), (d) *is satisfied, then M is Cohen-Macaulay in dimension $> s$.*

(iii) *Assume that R is universally catenary and all whose formal fibers are Cohen-Macaulay. Then M is Cohen-Macaulay in dimension $> s$ if and only if one of the conditions* (a), (b), (c), (d) *is satisfied.*

The proof of Main Theorem will be given in Section 3. In the next section, we recall some definitions and earlier results which will be used later.

2. Preliminaries

From the definition of M-sequence in dimension $> s$ and Cohen-Macaulay modules in dimension $> s$, we have following immediate properties.

Lemma 2.1. *Let $(x_1, \ldots, x_r)$ be a sequence of elements in $\mathfrak{m}$.*

(i) *$(x_1, \ldots, x_r)$ is an M-sequence in dimension $> s$ if and only if for all $i = 1, \ldots, r$ we have $\dim\big((x_1, \ldots, x_{i-1})M :_M x_i/(x_1, \ldots, x_{i-1})M\big) \leqslant s$.*

(ii) *$(x_1, \ldots, x_r)$ is an M-sequence in dimension $> s$ if and only if $x_1/1, \ldots, x_r/1$ is a poor $M_{\mathfrak{p}}$-sequence for all $\mathfrak{p} \in \operatorname{Spec} R$ such that $\dim R/\mathfrak{p} > s$.*

Let $\underline{x} = (x_1, \ldots, x_t) \subseteq \mathfrak{m}$ be a multiplicative system of M, i.e., it satisfies the condition $\ell(M/(x_1, \ldots, x_t)M) < \infty$. Denote by $e(\underline{x}; M)$ the *multiplicity* of M with respect to $\underline{x}$. Then $e(\underline{x}; M) \geq 0$ and $e(\underline{x}; M) = 0$ if and only if $\underline{x}$ is a s.o.p. of M, i.e., $t = d$. For some other basic properties of multiplicity that will be used in the sequel, we refer to the book by H. Matsumura [16].

Recall that if $\ell_R(H^i_{\mathfrak{m}}(M)) < \infty$ for all $i < d$, then M is called *generalized Cohen-Macaulay* (see [9]). Now we recall some characterizations of generalized Cohen-Macaulay modules introduced by [9] and [20]. From now on, for a s.o.p. $\underline{x} = (x_1, \ldots, x_d)$ of M, we set

$$I(\underline{x}; M) = \ell_R(M/(x_1, \ldots, x_d)M) - e(\underline{x}; M).$$

Lemma 2.2. *The following statements are equivalent*:

(i) *M is generalized Cohen-Macaulay.*

(ii) *There exists a constant $I(M)$ such that $I(\underline{x}; M) \leqslant I(M)$ for all s.o.p. $\underline{x}$ of M.*

(iii) *There exist a s.o.p. $\underline{x}$ of M and a constant $C_{\underline{x}}$ such that $I(x_1^n, \ldots, x_d^n; M) \leqslant C_{\underline{x}}$ for all integers n.*

(iv) *There exists a s.o.p. $\underline{x}$ of M such that $I(x_1^2, \ldots, x_d^2; M) = I(\underline{x}; M)$.*

A s.o.p. $\underline{x}$ of M satisfies Lemma 2.2, (iv) is called a *standard s.o.p.* of M. Note that if a s.o.p. $\underline{x}$ of M is standard, then $I(x_1^{n_1}, \ldots, x_d^{n_d}; M) = I(\underline{x}; M)$ for all $n_1, \ldots, n_d \geqslant 1$ (see [20, Theorem 2.1]).

Recall that the *Noetherian dimension* $\text{N-dim}_R A$ of an Artinian R-module A is defined inductively as follows (cf. Kirby [13], Roberts [19]). If $A = 0$, we put $\text{N-dim}\, A = -1$. For an integer $d \geq 0$, we put $\text{N-dim}_R A = d$ if $\text{N-dim}_R A < d$ is false, and for every ascending sequence $A_0 \subseteq A_1 \subseteq \cdots$ of submodules of A, there exists n_0 such that $\text{N-dim}_R(A_n/A_{n+1}) < d$ for all $n > n_0$.

Lemma 2.3 ([7])**.** (i) *Let A be Artinian R-module. Then A has a natural structure $\widehat{R}$-module and*

$$\text{N-dim}_R A = \text{N-dim}_{\widehat{R}} A = \dim_{\widehat{R}}(\widehat{R}/\operatorname{Ann}_{\widehat{R}} A) \leqslant \dim(R/\operatorname{Ann}_R A).$$

(ii) $\text{N-dim}\, A = 0$ *if and only if* $\dim_R A = 0$. *In this case, the length of A is finite and the ring $R/\operatorname{Ann}_R A$ is Artinian.*

(iii) *Let I be an ideal of R and M a non zero f.g. R-module. Then*

$$\text{N-dim}(H^i_{\mathfrak{m}}(M)) \leqslant i$$

and in particular, $\text{N-dim}(H^d_{\mathfrak{m}}(M)) = d$.

The theory of secondary representation introduced by I. G. Macdonald [15] is in some sense dual to the more known theory of primary decomposition. It has shown in [15] that every Artinian R-module A has a secondary representation $A = A_1 + \cdots + A_n$ of $\mathfrak{p}_i$-secondary submodules A_i. The set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is independent of the minimal secondary representation of A and it is denoted by $\operatorname{Att}_R A$.

Lemma 2.4. (i) $A \neq 0$ *if and only if* $\operatorname{Att}_R A \neq \emptyset$. *In this case, the minimal elements in* $\operatorname{Att}_R A$ *are exactly the minimal prime ideals containing* $\operatorname{Ann}_R A$.

(ii) $\text{N-dim}\, A \leqslant \dim\big(R/\operatorname{Ann}_R A\big) = \max\{\dim R/\mathfrak{p} : \ \mathfrak{p} \in \operatorname{Att}_R A\}$.

Now we recall the notion of polynomial type introduced by Cuong [5]. Let $\underline{x} = (x_1, \ldots, x_d)$ be a s.o.p. of M and $n_1, \ldots, n_d$ be integers. Consider

$$I(x_1^{n_1}, \ldots, x_d^{n_d}; M) = \ell(M/(x_1^{n_1}, \ldots, x_d^{n_d})M - n_1 \cdots n_d e(\underline{x}; M)$$

as a function in $n_1, \ldots, n_d$. Then this function always takes non-negative values and bounded above by polynomials, but it is not a polynomial for $n_1, \ldots, n_d$ large enough. However, the least degree of all polynomials in $n_1, \ldots, n_d$ bounding above the function $I(x_1^{n_1}, \ldots, x_d^{n_d}; M)$ is independent of the choice of $\underline{x}$. This least degree is called *the polynomial type* of M and denoted by $p(M)$ (see [5]). If we stipulate that the degree of polynomial zero is $-\infty$, then M is Cohen-Macaulay if and only if $p(M) = -\infty$. Moreover, M is generalized Cohen-Macaulay if and only if $p(M) \leqslant 0$ (see [9]).

When $p(M) > 0$, we can compute $p(M)$ in terms of the Noetherian dimension of Artinian local cohomology modules $H^i_{\mathfrak{m}}(M)$.

Lemma 2.5 ([6, Lemma 3.1])**.** *Let $p(M) > 0$. Then we have*

(i) $p(M) = \max\limits_{i<d} \text{N-dim}(H^i_{\mathfrak{m}}(M))$.

(ii) *If $x \in \mathfrak{m}$ such that $x \notin \mathfrak{p}$ for all* $\mathfrak{p} \in \bigcup\limits_{i=1}^{d} \operatorname{Att}(H^i_{\mathfrak{m}}(M)) \setminus \{\mathfrak{m}\}$, *then*

$$p(M/xM) = p(M) - 1.$$

3. Proof of Main Theorem

Proof of Main Theorem (i). (a)⇔(b) follows from Lemma 2.5(i).

(a)⇒(c). Let $d = 1$. Then $s = 0$ and M is generalized Cohen-Macaulay. By Lemma 2.2(iv), there exists a standard s.o.p. x_1 of M, i.e., $I(x_1^2; M) = I(x_1; M)$. Therefore (c) is true.

Let $d > 1$. We prove the result by induction on s, where $0 \leqslant s < d$. Let $s = 0$. Then $\text{N-dim}_R\, H^i_{\mathfrak{m}}(M) \leqslant 0$ for all $i < d$. By Lemma 2.3(ii), $\ell_R(H^i_{\mathfrak{m}}(M)) < \infty$ for all $i < d$, i.e., M is generalized Cohen-Macaulay. Therefore there exists by Lemma 2.2(iv) a s.o.p. $(x_1, \ldots, x_d)$ of M such that $I(x_1^2, \ldots, x_d^2; M) = I(x_1, \ldots, x_d; M)$. It means that condition (c) is true for $s = 0$. Let $1 \leqslant s < d$ and assume that the result is true for the case $s - 1$. If $p(M) \leqslant 0$, then M is generalized Cohen-Macaulay. Therefore there exists a standard s.o.p. $\underline{x} = (x_1, \ldots, x_d)$ of M. Therefore by [20, Theorem 2.1] we have

$$I(\underline{x}; M) \leqslant I(y_1, \ldots, y_d; M) \leqslant I(x_1^2, \ldots, x_d^2; M) = I(\underline{x}; M),$$

where $y_j = x_j^2$ if $j \notin \{k_1, \ldots, k_s\}$ and $y_j = x_j$ if $j \in \{k_1, \ldots, k_s\}$ for all $j = 1, \ldots, d$. Hence $I(\underline{x}; M) = I(y_1, \ldots, y_d; M)$ and the result is true in this case. Let $p(M) > 0$. Let $x_1 \in \mathfrak{m}$ such that $x_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in (\bigcup_{i=1}^{d} \operatorname{Att}(H^i_{\mathfrak{m}}(M))) \setminus \{\mathfrak{m}\}$. Note that $p(M) \leqslant s$ by Lemma 2.5(i), we get by Lemma 2.5(ii) that $p(M/x_1M) = p(M) - 1 \leqslant s - 1$. Hence $\text{N-dim}(H^i_{\mathfrak{m}}(M/x_1M) \leqslant s - 1$ for all $i < d - 1$ by Lemma 2.5(i). Applying the induction hypothesis for M/x_1M, there exist a s.o.p. $(x_2, \ldots, x_d)$ of M and integers $k_2, \ldots, k_s \in \{2, \ldots, d\}$ such that

$$I(y_2, \ldots, y_d; M) = I(x_2, \ldots, x_d; M),$$

where $y_j = x_j^2$ if $j \notin \{k_2, \ldots, k_s\}$ and $y_j = x_j$ if $j \in \{k_2, \ldots, k_s\}$, for all $j = 2, \ldots, d$. Without loss of generality we can assume that $k_2 = 2, \ldots, k_s = s$, i.e.,

$$I(x_2, \ldots, x_s, x_{s+1}^2, \ldots, x_d^2; M/x_1M) = I(x_2, \ldots, x_d; M/x_1M). \tag{1}$$

By the choice of x_1, we have $\dim(0 :_M x_1) \leqslant 0$. Since $d > 1$, we have

$$e(x_2, \ldots, x_s, x_{s+1}^2, \ldots, x_d^2; 0 :_M x_1) = 0 = e(x_2, \ldots, x_s, x_{s+1}, \ldots, x_d; 0 :_M x_1).$$

Therefore, we have

$$\begin{aligned}
&I(x_2, \ldots, x_s, x_{s+1}^2, \ldots, x_d^2; M/x_1M)\\
&= \ell_R(M/(x_1, \ldots, x_s, x_{s+1}^2, \ldots, x_d^2)M) - e(x_1, \ldots, x_s, x_{s+1}^2, \ldots, x_d^2; M)\\
&\quad + e(x_2, \ldots, x_s, x_{s+1}^2, \ldots, x_d^2; 0 :_M x_1)\\
&= I(x_1, \ldots, x_s, x_{s+1}^2, \ldots, x_d^2; M), \text{ and}
\end{aligned}$$

$$\begin{aligned}
&I(x_2, \ldots, x_d; M/x_1M)\\
&= \ell_R(M/(x_1, x_2, \ldots, x_d)M) - e(x_1, x_2, \ldots, x_d; M) + e(x_2, \ldots, x_d; 0 :_M x_1)\\
&= I(x_1, \ldots, x_d; M).
\end{aligned}$$

So, it follows from (1) that

$$I(x_1,\dots,x_s,x_{s+1}^2,\dots,x_d^2;M)=I(x_1,\dots,x_d;M),$$

and (c) is proved.

(c)⇒(d). Let $d=1$. Then $s=0$ and M is generalized Cohen-Macaulay. So, there exists a standard s.o.p. x_1 of M and by [20, Theorem 2.1], we have $I(x_1;M)=I(x_1^2;M)=I(x_1^n;M)$ for all $n\in\mathbb{N}$. Set $C_{\underline{x}}=I(x_1;M)$. Then $I(x_1^n;M)=C_{\underline{x}}=n^0C_{\underline{x}}$ for all $n\geqslant 1$. Hence (d) is true.

Let $d>1$. We prove the result by induction on s, where $0\leqslant s<d$. Let $s=0$. From the hypothesis (c), there exists a s.o.p. $\underline{x}=(x_1,\dots,x_d)$ of M such that

$$I(x_1^2,\dots,x_d^2;M)=I(x_1,\dots,x_d;M).$$

It means that M is generalized Cohen-Macaulay and $\underline{x}$ is a standard s.o.p. of M. Set $C_{\underline{x}}=I(x_1,\dots,x_d;M)$. Then

$$I(x_1^n,\dots,x_d^n;M)=n^0C_{\underline{x}}$$

for all $n\geqslant 1$ and (d) is true for the case $s=0$. Let $s>0$ and assume that the result is true for $s-1$. Let $\underline{x}=(x_1,\dots,x_d)$ be a s.o.p. of M which satisfies (c). Without loss of generality we can assume that $k_1=d-s+1,\dots,k_s=d$, i.e.,

$$I(x_1^2,\dots,\dots,x_{d-s}^2,x_{d-s+1},\dots,x_d;M)=I(x_1,\dots,x_d;M). \tag{2}$$

We have by the property of multiplicity that

$$\begin{aligned}&I(x_1^2,\dots,x_{d-s}^2,x_{d-s+1},\dots,x_d;M)\\&=I(x_1^2,\dots,x_{d-s}^2,x_{d-s+1},\dots,x_{d-1};M/x_dM)+2^{d-s}e(x_1,\dots,x_{d-1};0:_M x_d)\end{aligned}$$

and

$$I(x_1,\dots,x_d;M)=I(x_1,\dots,x_{d-1};M/x_dM)+e(x_1,\dots,x_{d-1};0:_M x_d).$$

Note that $I(x_1^2,\dots,x_{d-s}^2,x_{d-s+1},\dots,x_{d-1}\geqslant I(x_1,\dots,x_{d-1};M/x_dM)$. Since $s<d$, we have

$$2^{d-s}e(x_1,\dots,x_{d-1};0:_M x_d)\geqslant e(x_1,\dots,x_{d-1};0:_M x_d).$$

Therefore it follows from (2) that $e(x_1,\dots,x_{d-1};0:_M x_d)=0$ and

$$\begin{aligned}&I(x_1,\dots,x_{d-s},x_{d-s+1},\dots,x_{d-1};M/x_dM)\\&=I(x_1^2,\dots,x_{d-s}^2,x_{d-s+1},\dots,x_{d-1};M/x_dM).\end{aligned}$$

Hence, $\dim(0:_M x_d)\leqslant d-2$ and hence $e(x_1^n,\dots,x_{d-1}^n;0:_M x_d)=0$ for all $n>0$. Therefore, using the induction hypothesis for M/x_dM, there exists a constant $C_{\underline{x}}$ such that

$$\begin{aligned}I(x_1^n,\dots,x_d^n;M)&\leqslant nI(x_1^n,\dots,x_{d-1}^n,x_d;M)\\&=n\big(I(x_1^n,\dots,x_{d-1}^n;M/x_dM)+e(x_1^n,\dots,x_{d-1}^n;0:_M x_d)\big)\\&\leqslant nn^{s-1}C_{\underline{x}}=n^sC_{\underline{x}}\end{aligned}$$

for all integers $n>0$. Thus (d) is proved.

(d)⇒(b). Since $I(x_1^n, \dots, x_d^n; M) \leqslant n^s I(\underline{x}; M)$ for all integers n, by the definition of the polynomial type $p(M)$ we have $p(M) \leqslant s$. □

Proof of Main Theorem (ii). Suppose that (a) is true. Set $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$. Since there is an isomorphism $H^i_{\widehat{\mathfrak{m}}}(\widehat{M}) \cong H^i_{\mathfrak{m}}(M)$ of $\widehat{R}$-modules, we have by Lemma 2.3(i) and assumption (a) that $\text{N-dim}_{\widehat{R}}(H^i_{\widehat{\mathfrak{m}}}(\widehat{M})) \leqslant s$ for all $i < d$. We first claim that $\widehat{M}$ is a Cohen-Macaulay module in dimension $> s$. We prove this by induction on d. Let $d = 1$. Then $s = 0$ and $\widehat{M}$ is generalized Cohen-Macaulay. By [9], each s.o.p. of $\widehat{M}$ is an $\widehat{M}$-sequence in dimension > 0. Let $d > 1$ and assume that the claim is true for $d-1$. Let $\underline{x} = (x_1, \dots, x_d)$ be a s.o.p. of $\widehat{M}$. Let $\widehat{\mathfrak{p}} \in \operatorname{Ass}_{\widehat{R}} \widehat{M}$ such that $\dim(\widehat{R}/\widehat{\mathfrak{p}}) := k > s$. If $k = d$, then $x_1 \notin \widehat{\mathfrak{p}}$ as x_1 is a parameter element of $\widehat{M}$. So, we assume that $k < d$. Note that $\widehat{\mathfrak{p}} \in \operatorname{Att}_{\widehat{R}}(H^k_{\widehat{\mathfrak{m}}}(\widehat{M}))$ by [2, Corollary 11.3.3]. Hence $\widehat{\mathfrak{p}} \supseteq \operatorname{Ann}_{\widehat{R}}(H^k_{\widehat{\mathfrak{m}}}(\widehat{M}))$ by Lemma 2.4. So we get by Lemma 2.3(i) that

$$\text{N-dim}_{\widehat{R}}(H^k_{\widehat{\mathfrak{m}}}(\widehat{M})) = \dim(\widehat{R}/\operatorname{Ann}_{\widehat{R}}(H^k_{\widehat{\mathfrak{m}}}(\widehat{M})) \geqslant \dim(\widehat{R}/\widehat{\mathfrak{p}}) = k > s.$$

On the other hand, $\text{N-dim}_{\widehat{R}}(H^k_{\widehat{\mathfrak{m}}}(\widehat{M})) \leqslant s$ by the above fact. This is impossible. Therefore x_1 is $\widehat{M}$-regular in dimension $> s$. Thus $\dim(0 :_{\widehat{M}} x_1) \leqslant s$ by Lemma 2.1(i). Hence $H^i_{\widehat{\mathfrak{m}}}(0 :_{\widehat{M}} x_1) = 0$ for all $> s$. From the exact sequence

$$0 \longrightarrow 0 :_{\widehat{M}} x_1 \longrightarrow \widehat{M} \longrightarrow \widehat{M}/(0 :_{\widehat{M}} x_1) \longrightarrow 0$$

we have an isomorphism $H^i_{\widehat{\mathfrak{m}}}(\widehat{M}) \cong H^i_{\widehat{\mathfrak{m}}}(\widehat{M}/(0 :_{\widehat{M}} x_1)))$ for all $i > s$. Therefore from the exact sequence

$$0 \longrightarrow \widehat{M}/(0 :_{\widehat{M}} x_1) \xrightarrow{x_1} \widehat{M} \longrightarrow \widehat{M}/x_1\widehat{M} \longrightarrow 0,$$

we get the exact sequence $H^i_{\widehat{\mathfrak{m}}}(\widehat{M}) \longrightarrow H^i_{\widehat{\mathfrak{m}}}(\widehat{M}/x_1\widehat{M}) \longrightarrow H^{i+1}_{\widehat{\mathfrak{m}}}(\widehat{M})$ for all $i \geqslant s$. Since $\text{N-dim}_{\widehat{R}}(H^i_{\widehat{\mathfrak{m}}}(\widehat{M})) \leqslant s$ for all $i < d$, $\text{N-dim}_{\widehat{R}}(H^i_{\widehat{\mathfrak{m}}}(\widehat{M}/x_1\widehat{M})) \leqslant s$ for all $i < d-1$. So, by the induction hypothesis applying to $\widehat{M}/x_1\widehat{M}$, we have $(x_2, \dots, x_d)$ is an $\widehat{M}/x_1\widehat{M}$-sequence in dimension $> s$. Therefore, $(x_1, \dots, x_d)$ is an $\widehat{M}$-sequence in dimension $> s$, i.e., $\widehat{M}$ is Cohen-Macaulay in dimension $> s$. Thus, M is Cohen-Macaulay in dimension $> s$ by [21, Proposition 2.6]. □

Proof of Main Theorem (iii). By Lemma 2.3(i) we need to show that

$$\dim_{\widehat{R}}(\widehat{R}/\operatorname{Ann}_{\widehat{R}}(H^i_{\widehat{\mathfrak{m}}}(\widehat{M}))) \leqslant s$$

for all $i < d$. Let $i < d$ and $\widehat{\mathfrak{p}} \in \operatorname{Att}_{\widehat{R}}(H^i_{\widehat{\mathfrak{m}}}(\widehat{M}))$. Then $\dim(\widehat{R}/\widehat{\mathfrak{p}}) := k \leqslant i < d$ by [2, 11.3.5] and $\widehat{\mathfrak{p}} \in \operatorname{Ass}_{\widehat{R}} \widehat{M}$ by [2, 11.3.3]. Since R is universally catenary and all whose formal fibers are Cohen-Macaulay, we have by [12, Corollary 1.2] that R is a quotient ring of a Cohen-Macaulay ring. So we have by [21, Proposition 2.6] that $\widehat{M}$ is Cohen-Macaulay in dimension $> s$. Suppose $k > s$. Since $k < d$, there exists a s.o.p. $(x_1, \dots, x_d)$ of $\widehat{M}$ such that $x_1 \in \widehat{\mathfrak{p}}$. Therefore $(x_1, \dots, x_d)$ is not a $\widehat{M}$-sequence in dimension $> s$. This is impossible. Hence $k \leqslant s$. Thus

$$\dim_{\widehat{R}}(\widehat{R}/\operatorname{Ann}_{\widehat{R}}(H^i_{\mathfrak{m}}(\widehat{M}))) = \max_{\widehat{\mathfrak{p}} \in \operatorname{Att}_{\widehat{R}} H^i_{\mathfrak{m}}(\widehat{M})} \dim(\widehat{R}/\widehat{\mathfrak{p}}) \leqslant s.$$

□

As a consequence of Main Theorem, we have the following characterization for the Cohen-Macaulayness in dimension $> s$ in term of the dimension of the non-Cohen-Macaulay locus.

Denote by $\mathrm{NC}(M)$ the non-Cohen-Macaulay locus of M, i.e.,

$$\mathrm{NC}(M) = \{\mathfrak{p} \in \operatorname{Spec} R \mid M_{\mathfrak{p}} \text{ is not Cohen-Macaulay}\}.$$

If R is universally catenary and all its formal fibers are Cohen-Macaulay, then $\mathrm{NC}(M)$ is closed in $\operatorname{Spec} R$ under the Zariski topology (cf. [8]). Therefore $\dim(\mathrm{NC}(M))$ is well defined. Set $\mathfrak{a}(M) = \mathfrak{a}_0(M) \cdots \mathfrak{a}_{d-1}(M)$, where $\mathfrak{a}_i(M) = \operatorname{Ann}_R(H^i_{\mathfrak{m}}(M))$ for all $i \leqslant d-1$.

Corollary 3.1. *If R is universally catenary and all its formal fibers are Cohen-Macaulay, then the following statements are equivalent*:

(i) *M is Cohen-Macaulay in dimension $> s$.*

(ii) $\dim(R/\mathfrak{a}(M)) \leqslant s$.

(iii) $\dim \mathrm{NC}(M) \leqslant s$ *and* $\dim(R/\mathfrak{p}) = d$ *for all* $\mathfrak{p} \in (\min(\operatorname{Supp}_R M))_{>s}$.

Proof. (i)⇔(ii). By [4, Theorem 1.2] we have $p(M) = \dim(R/\mathfrak{a}(M))$. Therefore the assertion follows from Main Theorem.

(i)⇒(iii) follows from [21, Proposition 2.4 (i)⇒(iv)].

(iii)⇒(i). Let $\mathfrak{p} \in (\operatorname{Supp}_R M)_{>s}$. Since $\dim \mathrm{NC}(M) \leqslant s$ by hypothesis (iii), $M_{\mathfrak{p}}$ is Cohen-Macaulay. Let $\mathfrak{q} \in \min(\operatorname{Supp}_R M)_{>s}$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. Then $\dim(R/\mathfrak{q}) = d$ by (iii). Since R is universally catenary, it is catenary. Therefore

$$d \geq \dim(R/\mathfrak{p}) + \dim M_{\mathfrak{p}} \geq \dim(R/\mathfrak{p}) + \operatorname{ht}(\mathfrak{p}/\mathfrak{q}) = \dim(R/\mathfrak{q}) = d.$$

Hence M is Cohen-Macaulay in dimension $> s$ by [21, Proposition 2.4(iv)⇒(i)].

□

Corollary 3.2. *Suppose that R is universally catenary and all its formal fibers are Cohen-Macaulay. Then the following statements are true*:

(i) *$M = \oplus_{i=1}^n M_i$ is Cohen-Macaulay in dimension $> s$ if and only if for every i, M_i is of dimension at most s or is of dimension d and Cohen-Macaulay in dimension $> s$.*

(ii) *Let $x_1, \dots, x_{d-s}$ be a part of s.o.p of M. Then M is Cohen-Macaulay in dimension $> s$ if and only if so is $(x_1, \dots, x_{d-s})M$.*

Proof. (i) It follows from the assumption and from Main Theorem that M is Cohen-Macaulay in dimension $> s$ if and only if $\text{N-dim}(H^j_{\mathfrak{m}}(M)) \leqslant s$ for all $j < d$. Therefore M is Cohen-Macaulay in dimension $> s$ if and only if either $\dim M_i = d$ and $\text{N-dim}(H^j_{\mathfrak{m}}(M_i)) \leqslant s$ for all $j < d$ or $\dim M_i \leqslant s$ for all $i = 1, \dots, n$. Now the assertion follows from Main Theorem.

(ii) Set $N = (x_1, \dots, x_{d-s})M$. From the exact sequence $0 \to N \to M \to M/N \to 0$ and the fact that $(x_1, \dots, x_{d-s})$ is a part of s.o.p of M, we can deduce that $\dim M/N = s$, and hence $H^i_{\mathfrak{m}}(M/N) = 0$ for all $i > s$. Therefore, the long exact sequence

$$\cdots \to H^i_{\mathfrak{m}}(M/N) \to H^{i+1}_{\mathfrak{m}}(N) \to H^{i+1}_{\mathfrak{m}}(M) \to H^{i+1}_{\mathfrak{m}}(M/N) \to \cdots$$

gives us $H^i_{\mathfrak{m}}(N) \cong H^i_{\mathfrak{m}}(M)$ for all $i > s$. Now the assertion follows from Main Theorem. □

Now we consider the Cohen-Macaulayness in dimension $> s$ of the polynomial rings and the formal power series rings.

Proposition 3.3. *Let $S = R[[x_1, \dots, x_t]]$ be the ring of all formal power series in t variables $x_1, \dots, x_t$ with coefficients in R. Then $p(S) = p(R) + t$.*

Proof. By induction, we only need to prove the case $n = 1$. It is clear that $\mathfrak{n} = (\mathfrak{m}, x_1, \dots, x_t)$ is the unique maximal ideal of S and $\dim S = \dim R + t$. Set $x_1 = x$ and let $(a_1, \dots, a_d)$ be a s.o.p. of R. Then we have the canonical epimorphism of local rings $\varphi : S \longrightarrow R$ given by $\varphi(\sum c_i x^i) = c_0$. Hence, we can consider each R-module as a S-module by mean of φ. It is clear that $\operatorname{Ker} \varphi = xS$. Therefore, there is an isomorphism of S-modules

$$S/(a_1, \dots, a_d, x)S \cong R/(a_1, \dots, a_d)R.$$

It follows that $S/(a_1, \dots, a_d, x)S$ is of finite length, i.e., $(a_1, \dots, a_d, x)$ is a s.o.p. of S. Let $n_1, \dots, n_d, n$ be a tuple of $(d + 1)$ positive integers. Since x is S-regular, so is x^n and hence $(0 :_S x^n) = 0$. Thus, we have

$$\begin{aligned} e(a_1^{n_1}, \dots, a_d^{n_d}, x^n; S) &= e(a_1^{n_1}, \dots, a_d^{n_d}; S/x^n S) - e(a_1^{n_1}, \dots, a_d^{n_d}; 0 :_S x^n) \\ &= e(a_1^{n_1}, \dots, a_d^{n_d}; S/x^n S). \end{aligned}$$

It is clear that $\psi : S \longrightarrow R^n$ defined by $\psi(\sum c_i x^i) = (c_0, \dots, c_{n-1})$ is a surjection with $\operatorname{Ker} \psi = x^n S$. Hence $S/x^n S \cong R^n$. Thus

$$e(a_1^{n_1}, \dots, a_d^{n_d}; S/x^n S) = e(a_1^{n_1}, \dots, a_d^{n_d}; R^n) = n e(a_1^{n_1}, \dots, a_d^{n_d}; R).$$

On the other hand, by the above isomorphism $S/x^n S \cong R^n$, we have

$$\begin{aligned} \ell_S\Big(S/(a_1^{n_1}, \dots, a_d^{n_d}, x^n)S\Big) &= \ell_S\Big(S/x^n S/(a_1^{n_1}, \dots, a_d^{n_d})S/x^n S\Big) \\ &= n\ell_R\Big(R/(a_1^{n_1}, \dots, a_d^{n_d})R\Big). \end{aligned}$$

Therefore we get

$$\begin{aligned} &I(a_1^{n_1}, \dots, a_d^{n_d}, x^n; S) \\ &= \ell_S\Big(S/(a_1^{n_1}, \dots, a_d^{n_d}, x^n)S\Big) - e(a_1^{n_1}, \dots, a_d^{n_d}, x^n; S) \\ &= \ell_S\Big(S/x^n S/(a_1^{n_1}, \dots, a_d^{n_d})S/x^n S\Big) - e(a_1^{n_1}, \dots, a_d^{n_d}; S/x^n S) \\ &= n\ell_R\Big(R/(a_1^{n_1}, \dots, a_d^{n_d})R\Big) - ne(a_1^{n_1}, \dots, a_d^{n_d}; R) \end{aligned}$$

$$= nI(a_1^{n_1}, \ldots, a_d^{n_d}; R).$$

Thus, by the definition of polynomial type, we get $p(S) = p(R) + 1$. $\square$

Let $S = R[[x_1, \ldots, x_t]]$ be the ring of formal power series and $S' = R[x_1, \ldots, x_t]$ the polynomial ring in t variables over R. It is well known that R is Cohen-Macaulay (i.e., Cohen-Macaulay in dimension > -1) if and only if so is S and S'. Below we consider for the case $s \geq 0$.

Corollary 3.4. *Let $s \geq 0$ be an integer. Assume that R is universally catenary and all its formal fibers are Cohen-Macaulay. Let $\mathfrak{n} = (\mathfrak{m}, x_1, \ldots, x_t)S'$ be the unique homogeneous maximal ideal of S'. The following statements are equivalent*:

(i) *R is a Cohen-Macaulay ring in dimension $> s$.*
(ii) *S is Cohen-Macaulay in dimension $> s + t$.*
(iii) *$S'_\mathfrak{n}$ is a Cohen-Macaulay ring in dimension $> s + t$.*

Proof. (i)⇒(ii). Since R is a Cohen-Macaulay ring in dimension $> s$ and R is universally catenary and all whose formal fibers are Cohen-Macaulay, $p(R) \leqslant s$ by Main Theorem. Hence $p(S) = p(R)+t \leqslant s+t$ by Proposition 3.3. Therefore S is Cohen-Macaulay in dimension $> s + t$ by Main Theorem.

(ii)⇒(i). Since R is universally catenary and all whose formal fibers are Cohen-Macaulay, we have by [12, Corollary 1.2] that R is a quotient ring A/I of the Cohen-Macaulay ring A. From the isomorphism

$$R[[x]] \cong \frac{A}{I}[[x]] \cong A[[x]]/I[[x]],$$

where $I[[x]]$ is an ideal of $A[[x]]$ with coefficients in I, it follows that $S = R[[x]]$ is a quotient ring of the Cohen-Macaulay ring $A[[x]]$. Hence S is universally catenary and all whose formal fibres are Cohen-Macaulay. Therefore, we have by Main Theorem, (iii) that $p(S) < s + t$ and hence $p(R) < s$ by Proposition 3.3. Thus the assertion follows from Main Theorem, (ii). Similarly, we can prove the case (i)⇔(iii). $\square$

Similar to the cases of f-module and generalized f-module, the assumption of R being a universally catenary and all whose formal fibers being Cohen-Macaulay in Main Theorem is not redundant. The following example illustrates this fact.

Example 3.5. There exists a Noetherian local domain $(S, \mathfrak{n})$ such that:

(i) $\dim S = 4, \operatorname{depth} S = 3$ and S is Cohen-Macaulay in dimension > 2.
(ii) N-$\dim(H^3_\mathfrak{n}(S)) = 3$, $\dim(S/\operatorname{Ann}_S(H^3_\mathfrak{n}(S)) = 4$ and $\dim S/\mathfrak{a}(S) = 4$.
(iii) $p(S) = 3, \dim(\widehat{S}/\mathfrak{a}(\widehat{S})) = 3$ and $\widehat{S}$ is not Cohen-Macaulay in dimension > 2, where $\widehat{S}$ is the $\mathfrak{n}$-adic completion of S.

Proof. Let $(R, \mathfrak{m})$ be a Noetherian local domain of dimension 2 constructed by D. Ferrand and M. Raynaud [10] for which the $\mathfrak{m}$-adic completion $\widehat{R}$ of R has an associated prime $\mathfrak{q}$ of dimension 1. We have by [7, Example 4.1] that

$$\dim_{\widehat{R}}(H^1_{\mathfrak{m}}(R)) = \text{N-dim}(H^1_{\mathfrak{m}}(R)) = 1 < \dim_R(H^1_{\mathfrak{m}}(R)) = 2.$$

(i) Let $S = R[[x, y]]$ be the ring of all formal power series in two variables x, y with coefficients in R. Then $\dim S = 4$ and $\operatorname{depth} S = 3$. Since S is a Noetherian local domain of dimension 4, it is clear that S is a Cohen-Macaulay ring in dimension > 2.

(ii) It is clear that $\mathfrak{n} = (\mathfrak{m}, x, y)S$ is the unique maximal ideal of S and $\widehat{S} = \widehat{R}[[x, y]]$ is the $\mathfrak{n}$-adic completion of S. As $\widehat{\mathfrak{p}} \in \operatorname{Ass} \widehat{R}$, there exists $a \in \widehat{R}$ such that $\widehat{\mathfrak{p}} = \operatorname{Ann}_{\widehat{R}} a$. Set

$$\widehat{\mathfrak{p}}[[x, y]] = \Big\{ \sum_{i=0}^{\infty} a_i x^i + b_i y^i \in S \mid a_i, b_i \in \widehat{\mathfrak{p}}, \forall i \Big\}.$$

Then $\widehat{\mathfrak{p}}[[x, y]]$ is a prime ideal of $\widehat{S}$ and

$$\operatorname{Ann}_{\widehat{S}} a = \Big\{ \sum_{i=0}^{\infty} a_i x^i + b_i y^i \in \widehat{S} \mid \sum_{i=0}^{\infty} (aa_i)x^i + (ab_i)y^i = 0 \Big\} = \widehat{\mathfrak{p}}[[x, y]].$$

Therefore, $\widehat{\mathfrak{p}}[[x, y]] \in \operatorname{Ass} \widehat{S}$ and

$$\dim\big(\frac{\widehat{S}}{\widehat{\mathfrak{p}}[[x, y]]}\big) = \dim\big(\frac{\widehat{R}}{\widehat{\mathfrak{p}}}\big)[[x, y]] = \dim(\widehat{R}/\widehat{\mathfrak{p}}) + 2 = 3.$$

By [2, Corollary 11.3.3], it implies that $\widehat{\mathfrak{p}}[[x, y]] \in \operatorname{Att}_{\widehat{S}}(H^3_{\widehat{\mathfrak{n}}}(\widehat{S})) \cong \operatorname{Att}_{\widehat{S}}(H^3_{\mathfrak{n}}(S))$. Hence $\widehat{\mathfrak{p}}[[x, y]] \supseteq \operatorname{Ann}_{\widehat{S}}(H^3_{\mathfrak{n}}(S))$. As $\widehat{\mathfrak{p}}[[x, y]] \in \operatorname{Ass} \widehat{S} \cap \operatorname{Att}_{\widehat{S}}(H^3_{\mathfrak{n}}(S))$, we have

$$\widehat{\mathfrak{p}}[[x, y]] \cap S \in \operatorname{Ass}(S) \cap \operatorname{Att}_{\widehat{S}}(H^3_{\mathfrak{n}}(S)) = 0$$

since S is a domain. Therefore, we get

$$\operatorname{Ann}_S(H^3_{\mathfrak{n}}(S)) = \operatorname{Ann}_{\widehat{S}}(H^3_{\mathfrak{n}}(S)) \cap S \subseteq \widehat{\mathfrak{p}}[[x, y]] \cap S = 0.$$

Thus, $\dim_S S/\operatorname{Ann}_S(H^3_{\mathfrak{n}}(S)) = \dim S/\mathfrak{a}(S) = \dim S = 4$.

(iii) We get by Proposition 3.3 that $p(S) = 3$. Hence $\dim \widehat{S}/\mathfrak{a}(\widehat{S}) = p(\widehat{S}) = p(S) = 3$ by [5]. Therefore $\widehat{S}$ is not a Cohen-Macaulay ring in dimension > 2 by Main Theorem. $\square$

Acknowledgment. The author would like to thank Prof. Le Thanh Nhan for her useful suggestions.

References

[1] M. Brodmann and L. T. Nhan, *A finiteness result for associated primes of certain* Ext-*modules*, Comm. Algebra **36** (2008), no. 4, 1527–1536.

[2] M. Brodmann and R. Y. Sharp, *Local Cohomology: An Algebraic Introduction with Geometric Applications*, Cambridge University Press, 1998.

[3] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, 1993.

[4] N. T. Cuong, *On the dimension of the non-Cohen-Macaulay locus of local rings admitting dualizing Complexes*, Math. Proc. Cambridge. Philos. Soc. **109** (1991), no. 3, 479–488.

[5] ———, *On the least degree of polynomials bounding above the differences between lengths and multiplicities of certain system of parameters in local rings*, Nagoya Math. J. **125** (1992), 105–114.

[6] N. T. Cuong, M. Morales, and L. T. Nhan, *On the length of generalized fractions*, J. Algebra **265** (2003), no. 1, 100–113.

[7] N. T. Cuong and L. T. Nhan, *On Noetherian dimension of Artinian modules*, Vietnam J. Math. **30** (2002), no. 2, 121–130.

[8] N. T. Cuong, L. T. Nhan, and N. T. K. Nga, *On pseudo supports and non-Cohen-Macaulay locus of finitely generated modules*, J. Algebra **323** (2010), no. 10, 3029–3038.

[9] N. T. Cuong, P. Schenzel, and N. V. Trung, *Verallgemeinerte Cohen-Macaulay-Moduln*, Math. Nachr. **85** (1978), 57–73.

[10] D. Ferrand and M. Raynaud, *Fibres formelles d'un anneau local Noetherian*, Ann. Sci. Éc. Norm. Sup. (4) **3** (1970), 295–311.

[11] M. Hellus, *On the set of associated primes of a local cohomology modules*, J. Algebra **237** (2001), no. 1, 406–419.

[12] T. Kawasaki, *On arithmetic Macaulayfication of Noetherian rings*, Trans. Amer. Math. Soc. **354** (2002), no. 1, 123–149.

[13] D. Kirby, *Dimension and length for Artinian modules*, Quart. J. Math. Oxford Ser. (2) **41** (1990), no. 164, 419–429.

[14] R. Lu and Z. Tang, *The f-depth of an ideal on a module*, Proc. Amer. Math. Soc. **130** (2002), no. 7, 1905–1912.

[15] I. G. Macdonald, *Secondary representation of modules over a commutative ring*, Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971), pp. 23–43. Academic Press, London, 1973.

[16] H. Matsumura, *Commutative Ring Theory*, Cambridge, Cambridge University Press, 1986.

[17] L. T. Nhan, *On generalized regular sequences and the finiteness for associated primes of local cohomology modules*, Comm. Algebra **33** (2005), no. 3, 793–806.

[18] L. T. Nhan and M. Morales, *Generalized f-modules and the associated prime of local cohomology modules*, Comm. Algebra **34** (2006), no. 3, 863–878.

[19] R. N. Roberts, *Krull dimension for Artinian modules over quasi-local commutative rings*, Quart. J. Math. Oxford Ser. (2) **26** (1975), no. 103, 269–273.

[20] N. V. Trung, *Toward a theory of generalized Cohen-Macaulay modules*, Nagoya Math J. **102** (1986), 1–49.

[21] N. Zamani, *Cohen-Macaulay modules in dimension $> s$ and results on local cohomology*, Comm. Algebra **37** (2009), no. 4, 1297–1307.

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