

CHARACTERIZATIONS AND THE MOORE-PENROSE INVERSE OF HYPERGENERALIZED K-PROJECTORS

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ABSTRACT. We characterize hypergeneralized k -projectors (i.e., $A^k = A^\dagger$). Also, some representation for the Moore-Penrose inverse of a linear combination of hypergeneralized k -projectors is found and the invertibility for some linear combinations of commuting hypergeneralized k -projectors is considered.

1. Introduction

Let $\mathbb{C}^{n \times m}$ denote the set of all $n \times m$ complex matrices. The symbols $\mathcal{R}(A)$ and $r(A)$ will denote the range (column space) and the rank of a matrix A , respectively. For a matrix $A \in \mathbb{C}^{n \times n}$, $tr(A)$ and $\sigma(A)$ will denote the trace and the spectrum of a matrix A , respectively. Also, we will use the following notation: for $k \in \mathbb{N}$ and $k > 1$, the set of complex roots of 1 shall be denoted by σ_k and if we set $\omega_k = e^{2\pi i/k}$, then $\sigma_k = \{\omega_k^0, \omega_k^1, \dots, \omega_k^{k-1}\}$. The Moore-Penrose inverse of A is the unique matrix A^\dagger satisfying the equations

$$(1) AA^\dagger A = A, \quad (2) A^\dagger AA^\dagger = A^\dagger, \quad (3) AA^\dagger = (AA^\dagger)^*, \quad (4) A^\dagger A = (A^\dagger A)^*.$$

I_n will denote the identity matrix of order n and 0_r will denote the null-matrix of order r . Also, P_S denotes the orthogonal projector onto subspace S . We use the notations $C_n^P, C_n^{OP}, C_n^{EP}, C_n^{GP}$ and C_n^{HGP} for the subsets of $\mathbb{C}^{n \times n}$ consisting of projectors (idempotent matrices), orthogonal projectors (Hermitian idempotent matrices), EP (range-Hermitian) matrices, generalized and hypergeneralized projectors, respectively, i.e.,

$$\begin{aligned} C_n^P &= \{A \in \mathbb{C}^{n \times n} : A^2 = A\}, \\ C_n^{OP} &= \{A \in \mathbb{C}^{n \times n} : A^2 = A = A^*\}, \\ C_n^{EP} &= \{A \in \mathbb{C}^{n \times n} : \mathcal{R}(A) = \mathcal{R}(A^*)\} = \{A \in \mathbb{C}^{n \times n} : AA^\dagger = A^\dagger A\}, \\ C_n^{GP} &= \{A \in \mathbb{C}^{n \times n} : A^2 = A^*\}, \\ C_n^{HGP} &= \{A \in \mathbb{C}^{n \times n} : A^2 = A^\dagger\}. \end{aligned}$$

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Further, we will use the abbreviations “g-p” and “hg-p” for generalized projector and hypergeneralized projector, respectively.

A matrix $B \in \mathbb{C}^{n \times n}$ is said to be similar to a matrix $A \in \mathbb{C}^{n \times n}$ if there exists a nonsingular matrix $P \in \mathbb{C}^{n \times n}$ such that $B = P^{-1}AP$. If a matrix $A \in \mathbb{C}^{n \times n}$ is similar to a diagonal matrix, then A is said to be diagonalizable.

The concepts of g-p and hg-p were introduced by Groß and Trenkler [11] who presented very interesting properties of the classes of g-p and hg-p. A characterization of nonnegative matrices such that $A = A^\dagger$ is derived by Berman [7].

In [5], the authors introduced the following concept: A square matrix A is said to be a k -generalized projector (g-kp) if $A^k = A^*$. This class of matrices obviously generalizes to the class of g-p. In [9], the g-kp have been generalized on the set of all bounded linear operators on Hilbert space. They defined the hypergeneralized k -projectors (hg-kp): Let \mathcal{H} be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the all bounded linear operators on \mathcal{H} . For $A \in \mathcal{B}(\mathcal{H})$, A is said to be a hypergeneralized k -projector if there exists a natural number $k > 1$ such that $A^k = A^\dagger$. Also, they proved the following inclusion: The set of all g-kp is the subset of all hg-kp. Hence, the class of g-kp may be generalized by considering the class of hg-kp. This leads our interest to the subset of the class of square matrices A with the property $A^k = A^\dagger$ for $k \in \mathbb{N}$ and $k > 1$, called as hypergeneralized k -projectors. Specially, if $k = 2$, we get the class of h-p (see [1], [2], [3], [11], [13], [14]).

In this paper, we characterize this class of matrices and, as simple corollaries, we deduce the characterizations of hg-p presented in [2] and [3]. Also, we give the form for the Moore-Penrose inverse and study the nonsingularity of a linear combination $c_1A + c_2B$, where A and B are commuting hg-kp, as well as the nonsingularity of a linear combination $c_1A + c_2B + c_3C$, where A, B and C are commuting hg-kp such that $BC = 0$. Also, as corollaries for commuting g-p and hg-p, we give results presented in [14].

2. Characterizations of hypergeneralized k -projectors

In this section, we give some characterizations of hg-kp. First, we give necessary and sufficient conditions that A is a hg-kp.

Theorem 2.1. *Let $A \in \mathbb{C}^{n \times n}$ and $k \in \mathbb{N}$, $k > 1$. Then the following statements are equivalent:*

- (i) A is a hg-kp (i.e., $A^k = A^\dagger$);
- (ii) A is a EP matrix, $\sigma(A) \subseteq \sigma_{k+1} \cup \{0\}$ and A is diagonalizable;
- (iii) A is a EP matrix and $A^{k+2} = A$.

Proof. Let us prove that (i) is equivalent to (iii).

(i) \Rightarrow (iii) Matrix A is EP because $AA^\dagger = AA^k = A^kA = A^\dagger A$. Also, matrix A is $(k+2)$ -potent because $A^{k+2} = AA^kA = AA^\dagger A = A$.

(iii) \Rightarrow (i) Since A is an EP matrix, there exist a unitary matrix $U \in \mathbb{C}^{n \times n}$ and a nonsingular matrix $K \in \mathbb{C}^{r \times r}$ such that

$$(1) \quad A = U(K \oplus 0)U^*$$

(see [8]). Also,

$$(2) \quad A^\dagger = U(K^{-1} \oplus 0)U^*.$$

From $A^{k+2} = A$, we have $K^k = K^{-1}$, which implies that A is a hg-kp. Hence, (i) holds.

(ii) \Leftrightarrow (iii) This follows from the well known fact that $A^{k+2} = A$ if and only if A is diagonalizable and the spectrum of A is contained in $\sigma_{k+1} \cup \{0\}$ (see [6, Theorem 2.1]). \square

From Theorem 2.1, it follows that A is a hg-kp if and only if

$$(3) \quad A = U(K \oplus 0)U^*,$$

where $U \in \mathbb{C}^{n \times n}$ is a unitary matrix and $K \in \mathbb{C}^{r \times r}$ is a nonsingular matrix such that $K^{k+1} = I_r$.

If A is a hg-kp, then $A^{k+1} = AA^\dagger$, i.e., A^{k+1} is the orthogonal projector onto $\mathcal{R}(A)$. Also, the converse implication is valid.

Theorem 2.2. *Let $A \in \mathbb{C}^{n \times n}$. Then A is a hg-kp if and only if A^{k+1} is the orthogonal projector onto $\mathcal{R}(A)$.*

Proof. (\Leftarrow) By Corollary 6 in [12], every matrix $A \in \mathbb{C}^{n \times n}$ of rank r has the form

$$(4) \quad A = U \begin{bmatrix} DK & DL \\ 0 & 0 \end{bmatrix} U^*,$$

where $U \in \mathbb{C}^{n \times n}$ is unitary, $D = \text{diag}(\lambda_1 I_{r_1}, \dots, \lambda_t I_{r_t})$ is the diagonal matrix of nonzero singular values of A , $\lambda_1 > \lambda_2 > \dots > \lambda_t > 0$, $r_1 + r_2 + \dots + r_t = r$ and $K \in \mathbb{C}^{r \times r}$, $L \in \mathbb{C}^{r \times (n-r)}$ satisfy

$$KK^* + LL^* = I_r.$$

From (4), it follows that

$$(5) \quad A^{k+1} = U \begin{bmatrix} (DK)^{k+1} & (DK)^k DL \\ 0 & 0 \end{bmatrix} U^*$$

and

$$A^\dagger = U \begin{bmatrix} K^* D^{-1} & 0 \\ L^* D^{-1} & 0 \end{bmatrix} U^*.$$

Hence,

$$AA^\dagger = P_{\mathcal{R}(A)} = U(I_r \oplus 0)U^*.$$

Now, $P_{\mathcal{R}(A)} = A^{k+1}$ if and only if $(DK)^{k+1} = I_r$ and $L = 0$. Thus, A has the form (3), which is equivalent to the fact that A is a hg-kp. \square

Corollary 2.3. *Let $A \in \mathbb{C}^{n \times n}$ be a hg-kp. Then $\text{r}(A) = \text{tr}(A^{k+1})$.*

Proof. From Theorem 2.2 and (3) we get $r(A) = r(A^{k+1}) = tr(A^{k+1})$. \square

The converse result is invalid, as can be seen by taking

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

in which case $r(A) = tr(A^k)$ and $A^\dagger = A^{-1} \neq A^k$ for $k \in \mathbb{N}$. Hence, A is not a hg-kp.

As corollaries, we get Theorem 1 and Corollary 1 in [3].

Corollary 2.4 ([3]). *Let $A \in \mathbb{C}^{n \times n}$. Then $A \in C_n^{HGP}$ if and only if A^3 is the orthogonal projector onto $\mathcal{R}(A)$.*

Corollary 2.5 ([3]). *Let $A \in C_n^{HGP}$. Then $r(A) = tr(A^3)$.*

By definition of the Moore-Penrose inverse, the group inverse and the Drazin inverse, it is easy to see that if A is a hg-kp, then

$$A^\dagger = A^\# = A^d = A^k = A^{m(k+1)+k}, \quad m \in \mathbb{N}.$$

Generally A is a hg-kp if and only if its Moore-Penrose inverse A^\dagger is:

Theorem 2.6. *Let $A \in \mathbb{C}^{n \times n}$. The following are equivalent:*

- (i) A is a hg-kp;
- (ii) A^* is a hg-kp;
- (iii) A^\dagger is a hg-kp.

Proof. Let A and A^\dagger be given by (1) and (2), respectively.

(i) \Rightarrow (ii) This follows from $K^k = K^{-1} \Leftrightarrow (K^*)^k = (K^*)^{-1}$.

(ii) \Rightarrow (i) Since $(A^*)^* = A$, the proof follows directly by (i) \Rightarrow (ii) replacing A by A^* .

(i) \Rightarrow (iii) This follows from $K^k = K^{-1} \Leftrightarrow K = (K^{-1})^k$.

(iii) \Rightarrow (i) Since $(A^\dagger)^\dagger = A$, the proof follows directly by (i) \Rightarrow (iii) replacing A by A^\dagger . \square

As a corollary we get the part of Theorem 5 in [2].

Corollary 2.7 ([2]). *Let $A \in \mathbb{C}^{n \times n}$. Then*

$$A \in C_n^{HGP} \Leftrightarrow A^\dagger \in C_n^{HGP}.$$

The following theorem singles out a sufficient condition for the equivalence of A being a hg-kp and A being an EP matrix.

Theorem 2.8. *Let $A \in \mathbb{C}^{n \times n}$. Assume there exists $B \in \mathbb{C}^{n \times n}$ such that B is a hg-kp and $A^2 = AB$ or $A^2 = BA$. Then A is a hg-kp if and only if $A \in C_n^{EP}$.*

Proof. (\Rightarrow) This follows from Theorem 2.1.

(\Leftarrow) Since B is a hg-kp, it is clear that $A^2 = AB$ leads to

$$A^2 = ABB^\dagger B = AB^{k+2} = A^{k+3}.$$

Now, by using that $AA^\dagger = A^\dagger A$ and $A^\dagger AA^\dagger = A^\dagger$ and multiplying $A^2 = A^{k+3}$ three times by A^\dagger , we have that $A^\dagger = A^k$. The proof with the condition $A^2 = BA$ follows similarly. \square

As the following corollary, we get Theorem 8 in [3].

Corollary 2.9 ([3]). *Let $A, B \in \mathbb{C}^{n \times n}$ be such that $B \in C_n^{HGP}$ and $A^2 = AB$ or $A^2 = BA$. Then $A \in C_n^{HGP}$ if and only if $A \in C_n^{EP}$.*

3. The Moore-Penrose inverse and the invertibility of a linear combination of commuting hypergeneralized k-projectors

It is well known that any g-kp is a hg-kp. So, following results also hold for g-kp.

The following lemma is furthermore very useful in this section.

Lemma 3.1. *Let $X, Y \in \mathbb{C}^{n \times n}$ and $c_1, c_2 \in \mathbb{C}$. If $X^{k+1} = Y^{k+1} = I_n$ and $XY = YX$, then*

$$(6) \quad (c_1 X + c_2 Y) \sum_{i=0}^k (-1)^i c_1^{k-i} c_2^i X^{k-i} Y^i = (c_1^{k+1} + (-1)^k c_2^{k+1}) I_n.$$

Proof. The result follows from

$$\begin{aligned} (c_1 X + c_2 Y) \sum_{i=0}^k (-1)^i c_1^{k-i} c_2^i X^{k-i} Y^i &= c_1^{k+1} X^{k+1} + (-1)^k c_2^{k+1} Y^{k+1} \\ &= (c_1^{k+1} + (-1)^k c_2^{k+1}) I_n. \end{aligned} \quad \square$$

In the following theorem, we present the form for the Moore-Penrose inverse and we give some necessary and sufficient conditions for the invertibility of the linear combination $c_1 A + c_2 B$, where A and B are two commuting hg-kp.

Theorem 3.1. *Let $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{n \times n}$ be commuting hg-kp and let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such that $c_1^{k+1} + (-1)^k c_2^{k+1} \neq 0$. Then*

$$\begin{aligned} (c_1 A + c_2 B)^\dagger &= \frac{1}{c_1^{k+1} + (-1)^k c_2^{k+1}} \left(\sum_{i=0}^k (-1)^i c_1^{k-i} c_2^i A^{k-i} B^i \right) A^{k+1} B^{k+1} \\ &\quad + \frac{1}{c_1} A^k (I_n - B^{k+1}) + \frac{1}{c_2} B^k (I_n - A^{k+1}). \end{aligned} \quad (7)$$

Furthermore, $c_1 A + c_2 B$ is nonsingular if and only if $n = r(A) + r(B) - r(AB)$ and in this case $(c_1 A + c_2 B)^{-1}$ is given by (7).

Proof. By Theorem 2.1 and Corollary 3.9 from [4], we can suppose that A and B have the form

$$A = U(A_1 \oplus A_2 \oplus 0_t \oplus 0)U^*, \quad B = U(B_1 \oplus 0_s \oplus B_2 \oplus 0)U^*,$$

where $U \in \mathbb{C}^{n \times n}$ is unitary, $A_1, B_1 \in \mathbb{C}^{r \times r}$, $A_2 \in \mathbb{C}^{s \times s}$, $B_2 \in \mathbb{C}^{t \times t}$ are nonsingular, $A_1 B_1 = B_1 A_1$, $A_1^{k+1} = B_1^{k+1} = I_r$, $A_2^{k+1} = I_s$ and $B_2^{k+1} = I_t$. Also, notice that

$$(8) \quad A^{k+1} = U(I_r \oplus I_s \oplus 0_t \oplus 0)U^*, \quad B^{k+1} = U(I_r \oplus 0_s \oplus I_t \oplus 0)U^*.$$

Now, we have

$$(9) \quad c_1 A + c_2 B = U \left((c_1 A_1 + c_2 B_1) \oplus c_1 A_2 \oplus c_2 B_2 \oplus 0_{n-(r+s+t)} \right) U^*.$$

By Lemma 3.1, we get that $c_1 A_1 + c_2 B_1$ is nonsingular and that

$$(c_1 A_1 + c_2 B_1)^{-1} = \frac{1}{c_1^{k+1} + (-1)^k c_2^{k+1}} \sum_{i=0}^k (-1)^i c_1^{k-i} c_2^i A_1^{k-i} B_1^i.$$

Also, $c_1 A_2$ and $c_2 B_2$ are nonsingular and $(c_1 A_2)^{-1} = \frac{1}{c_1} A_2^k$ and $(c_2 B_2)^{-1} = \frac{1}{c_2} B_2^k$. Now, we have

$$\begin{aligned} & (c_1 A + c_2 B)^\dagger \\ &= U \left(\left(\frac{1}{c_1^{k+1} + (-1)^k c_2^{k+1}} \sum_{i=0}^k (-1)^i c_1^{k-i} c_2^i A_1^{k-i} B_1^i \right) \oplus \frac{1}{c_1} A_2^k \oplus \frac{1}{c_2} B_2^k \oplus 0 \right) U^*. \end{aligned}$$

Finally, using (8), we get (7).

Also, from (9) we can conclude that $c_1 A + c_2 B$ is nonsingular if and only if $n = r + t + s$. Since $r(A) = r + t$, $r(B) = r + s$ and $r(AB) = r$, we get that $c_1 A + c_2 B$ is nonsingular if and only if $n = r(A) + r(B) - r(AB)$. \square

In the following corollaries, we study the problem of when linear combinations $c_1 I_n + c_2 \prod_{i=1}^m A_i^{k_i}$ and $c_1 I_n + c_2 A$ are nonsingular, where $\{A_i\}_{i=1}^m$ is a finite commuting family of hg-kp and A is a hg-kp, respectively. First, we state the following lemma:

Lemma 3.2. *Let $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{n \times n}$ be commuting hg-kp. Then AB is a hg-kp.*

Proof. Since A and B are two commuting hg-kp, then $BB^\dagger A^* AB = A^* AB$ and $ABB^* A^\dagger A = ABB^*$, so $(AB)^\dagger = A^\dagger B^\dagger$ (see [10]). Now,

$$(AB)^\dagger = A^\dagger B^\dagger = A^k B^k = (AB)^k.$$

Hence, AB is a hg-kp. \square

Corollary 3.2. *Let all of $A_i \in \mathbb{C}^{n \times n}$, $i = \overline{1, m}$ be commuting hg-kp, $m, k_1, \dots, k_m \in \mathbb{N}$, $c_1, c_2 \in \mathbb{C}$, $c_1 \neq 0$ and $c_1^{k+1} + (-1)^k c_2^{k+1} \neq 0$. Then $c_1 I_n + c_2 \prod_{i=1}^m A_i^{k_i}$ is nonsingular.*

Proof. From Lemma 3.2 we get: if $\{A_i\}_{i=1}^m$ is a finite commuting family of hg-kp, then $\prod_{i=1}^m A_i^{k_i}$ is also a hg-kp, where $m, k_1, \dots, k_m \in \mathbb{N}$. Now, the proof follows from Theorem 3.1. \square

Corollary 3.3. *Let $A \in \mathbb{C}^{n \times n}$ be a hg-kp, $c_1, c_2 \in \mathbb{C}$, $c_1 \neq 0$ and $c_1^{k+1} + (-1)^k c_2^{k+1} \neq 0$. Then $c_1 I_n + c_2 A$ is nonsingular and*

$$(c_1 I_n + c_2 A)^{-1} = \frac{1}{c_1^{k+1} + (-1)^k c_2^{k+1}} \left(\sum_{i=0}^k (-1)^i c_1^{k-i} c_2^i A^i \right) A^{k+1} + \frac{1}{c_1} (I_n - A^{k+1}).$$

As corollaries we get Theorem 2.1, Proposition 2.3 and Theorem 2.2 in [14], respectively.

Corollary 3.4 ([14]). *Let $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{n \times n}$ be commuting g-p or hg-p and let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such that $c_1^3 + c_2^3 \neq 0$. Then*

$$(10) \quad \begin{aligned} (c_1 A + c_2 B)^\dagger &= \frac{1}{c_1^3 + c_2^3} \left(c_1^2 A^2 B^3 - c_1 c_2 A B + c_2^2 A^3 B^2 \right) \\ &\quad + \frac{1}{c_1} A^2 (I_n - B^3) + \frac{1}{c_2} B^2 (I_n - A^3). \end{aligned}$$

Furthermore, $c_1 A + c_2 B$ is nonsingular if and only if $n = r(A) + r(B) - r(AB)$ and in this case $(c_1 A + c_2 B)^{-1}$ is given by (10).

Corollary 3.5 ([14]). *Let all of $A_i \in \mathbb{C}^{n \times n}$, $i = \overline{1, m}$ be commuting g-p or hg-p, $m, k_1, \dots, k_m \in \mathbb{N}$, $c_1, c_2 \in \mathbb{C}$, $c_1 \neq 0$ and $c_1^3 + c_2^3 \neq 0$. Then $c_1 I_n + c_2 \prod_{i=1}^m A_i^{k_i}$ is nonsingular.*

Corollary 3.6 ([14]). *Let $A \in \mathbb{C}^{n \times n}$ be a g-p or hg-p, $c_1, c_2 \in \mathbb{C}$, $c_1 \neq 0$ and $c_1^3 + c_2^3 \neq 0$. Then $c_1 I_n + c_2 A$ is nonsingular and*

$$(c_1 I_n + c_2 A)^{-1} = \frac{1}{c_1^3 + c_2^3} \left(c_1^2 A^3 - c_1 c_2 A + c_2^2 A^2 \right) + \frac{1}{c_1} (I_n - A^3).$$

In the following corollary, we present the form for the Moore-Penrose inverse and we give some necessary and sufficient conditions for the invertibility of linear combination $c_1 A + c_2 B$, where A and B are two commuting hg-kp such that $AB = 0$.

Corollary 3.7. *Let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. If A and B are commuting hg-kp such that $AB = 0$, then*

$$(11) \quad (c_1 A + c_2 B)^\dagger = \frac{1}{c_1} A^k + \frac{1}{c_2} B^k.$$

Furthermore, $c_1 A + c_2 B$ is nonsingular if and only if $n = r(A) + r(B)$ and in this case $(c_1 A + c_2 B)^{-1}$ is given by (11).

If we specialize to $k = 2$ in the previous corollary we obtain Corollary 2.4 in [14], which deals with g-p or hg-p.

In the following theorem, we give some necessary and sufficient conditions for the invertibility of linear combination $c_1 A + c_2 B + c_3 C$, where A, B and C are commuting hg-kp such that $BC = 0$.

Theorem 3.8. *Let $c_1, c_2, c_3 \in \mathbb{C} \setminus \{0\}$, $c_1^{k+1} + (-1)^k c_2^{k+1} \neq 0$ and $c_1^{k+1} + (-1)^k c_3^{k+1} \neq 0$. If $A, B, C \in \mathbb{C}^{n \times n}$ are commuting hg-kp such that $BC = 0$, then the following conditions are equivalent:*

- (i) $c_1 A + c_2 B + c_3 C$ is nonsingular,
- (ii) $B^{k+1} + C^{k+1} + A(I_n - B^{k+1} - C^{k+1})$ is nonsingular,
- (iii) $\text{r}(A(I_n - B^{k+1} - C^{k+1})) = n - (\text{r}(B) + \text{r}(C))$.

Proof. By Theorem 2.1 and Corollary 3.9 from [4], we can suppose that B and C have the form

$$B = U(B_1 \oplus 0_t \oplus 0)U^*, \quad C = U(0_r \oplus C_1 \oplus 0)U^*,$$

where $U \in \mathbb{C}^{n \times n}$ is unitary, $B_1 \in \mathbb{C}^{r \times r}$, $C_1 \in \mathbb{C}^{t \times t}$ are nonsingular, $B_1^{k+1} = I_r$ and $C_1^{k+1} = I_t$. Since A, B, C are commuting hg-kp, then A has the form $A = U(A_1 \oplus A_2 \oplus A_3)U^*$, where A_1, A_2, A_3 are hg-kp and A_1 commutes with B_1 and A_2 commutes with C_1 . Now, we get

$$c_1 A + c_2 B + c_3 C = U\left((c_1 A_1 + c_2 B_1) \oplus (c_1 A_2 + c_3 C_1) \oplus c_1 A_3\right)U^*.$$

Notice that $c_1 A_1 + c_2 B_1$ and $c_1 A_2 + c_3 C_1$ are nonsingular. Indeed, since $(c_1 A_1)^{k+1} + (-1)^k (c_2 B_1)^{k+1} = c_1^{k+1} A_1^{k+1} + (-1)^k c_2^{k+1} I_r$, then $(c_1 A_1)^{k+1} + (-1)^k (c_2 B_1)^{k+1}$ is nonsingular because it is the linear combination of an orthogonal projector and the identity matrix for all constants $c_1, c_2 \in \mathbb{C}$ such that $c_2 \neq 0$ and $c_1^{k+1} + (-1)^k c_2^{k+1} \neq 0$. From nonsingularity of $(c_1 A_1)^{k+1} + (-1)^k (c_2 B_1)^{k+1}$ and Lemma 3.1, it follows that $c_1 A_1 + c_2 B_1$ is nonsingular. Similarly, we can conclude that $c_1 A_2 + c_3 C_1$ is nonsingular for all constants $c_1, c_3 \in \mathbb{C}$ such that $c_3 \neq 0$ and $c_1^{k+1} + (-1)^k c_3^{k+1} \neq 0$. Now, it follows that $c_1 A + c_2 B + c_3 C$ is nonsingular if and only if A_3 is nonsingular for all constants $c_1, c_2, c_3 \in \mathbb{C} \setminus \{0\}$ such that $c_1^{k+1} + (-1)^k c_2^{k+1} \neq 0$ and $c_1^{k+1} + (-1)^k c_3^{k+1} \neq 0$ and we can prove that (i) \Leftrightarrow (ii) and (i) \Leftrightarrow (iii). Indeed, A_3 is nonsingular if and only if $B^{k+1} + C^{k+1} + A(I_n - B^{k+1} - C^{k+1})$ is nonsingular, so (i) \Leftrightarrow (ii). On the other hand, A_3 is nonsingular if and only if $\text{r}(A_3) = n - (r + t)$, i.e., $\text{r}(A(I_n - B^{k+1} - C^{k+1})) = n - (\text{r}(B) + \text{r}(C))$. So (i) \Leftrightarrow (iii). \square

Note that Theorem 2.10 in [14] which deals with g-p or hg-p is the corollary of the previous theorem.

In the following corollary, we study the problem of when a linear combination $c_1 I_n + c_2 A + c_3 B$ of commuting hg-kp is nonsingular and we give the form of the inverse in this case.

Corollary 3.9. *Let $c_1, c_2, c_3 \in \mathbb{C}$, $c_1 \neq 0$, $c_1^{k+1} + (-1)^k c_2^{k+1} \neq 0$ and $c_1^{k+1} + (-1)^k c_3^{k+1} \neq 0$. If $A, B \in \mathbb{C}^{n \times n}$ are commuting hg-kp such that $AB = 0$, then $c_1 I_n + c_2 A + c_3 B$ is nonsingular and*

$$(c_1 I_n + c_2 A + c_3 B)^{-1} = \frac{1}{c_1^{k+1} + (-1)^k c_2^{k+1}} \left(\sum_{i=0}^k (-1)^i c_1^{k-i} c_2^i A^i \right) A^{k+1}$$

$$(12) \quad \begin{aligned} & + \frac{1}{c_1^{k+1} + (-1)^k c_3^{k+1}} \left(\sum_{i=0}^k (-1)^i c_1^{k-i} c_3^i B^i \right) B^{k+1} \\ & + \frac{1}{c_1} (I_n - A^{k+1} - B^{k+1}). \end{aligned}$$

Proof. The invertibility of $c_1 I_n + c_2 A + c_3 B$ follows from Theorem 3.8. Also, from the proof of Theorem 3.8 follows that $c_1 I_n + c_2 A + c_3 B$ has the form

$$c_1 I_n + c_2 A + c_3 B = U \left((c_1 I_r + c_2 A_1) \oplus (c_1 I_t + c_3 B_1) \oplus c_1 I_{n-r-t} \right) U^*,$$

where $U \in \mathbb{C}^{n \times n}$ is unitary, $A_1 \in \mathbb{C}^{r \times r}$, $B_1 \in \mathbb{C}^{t \times t}$ are nonsingular, $A_1 B_1 = B_1 A_1$, $A_1^{k+1} = I_r$ and $B_1^{k+1} = I_t$. By Corollary 3.3, we get

$$(c_1 I_r + c_2 A_1)^{-1} = \frac{1}{c_1^{k+1} + (-1)^k c_2^{k+1}} \left(\sum_{i=0}^k (-1)^i c_1^{k-i} c_2^i A_1^i \right) A_1^{k+1}$$

and

$$(c_1 I_t + c_3 B_1)^{-1} = \frac{1}{c_1^{k+1} + (-1)^k c_3^{k+1}} \left(\sum_{i=0}^k (-1)^i c_1^{k-i} c_3^i B_1^i \right) B_1^{k+1}.$$

Now,

$$\begin{aligned} (c_1 I_n + c_2 A + c_3 B)^{-1} &= U \left(\frac{1}{c_1^{k+1} + (-1)^k c_2^{k+1}} \left(\sum_{i=0}^k (-1)^i c_1^{k-i} c_2^i A_1^i \right) A_1^{k+1} \right. \\ &\quad \oplus \frac{1}{c_1^{k+1} + (-1)^k c_3^{k+1}} \left(\sum_{i=0}^k (-1)^i c_1^{k-i} c_3^i B_1^i \right) B_1^{k+1} \\ &\quad \left. \oplus \frac{1}{c_1} I_{n-r-t} \right) U^*, \end{aligned}$$

which is equivalent to (12). \square

If we specialize to $k = 2$ in the previous corollary we obtain Theorem 2.11 in [14], which deals with g-p or hg-p.

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