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A CLASS OF ARITHMETIC FUNCTIONS ON $\mathbf{PSL_2(\mathbb{Z})}$, II

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Abstract. Atanassov introduced the irrational factor function and the strong restrictive factor function, which he defined as

$$I(n) = \prod_{p^\alpha || n} p^{1/\alpha} \qquad \text{and} \qquad R(n) = \prod_{p^\alpha || n} p^{\alpha - 1}$$

in [2] and [3]. Various properties of these functions have been investigated by Alkan, Ledoan, Panaitopol, and the authors. In the prequel, we expanded these functions to a class of elements of $\mathrm{PSL}_2(\mathbb{Z})$, and studied some of the properties of these maps. In the present paper we generalize the previous work by introducing real moments and considering a larger class of maps. This allows us to explore new properties of these arithmetic functions.

1. Introduction

In the present paper, we continue the work done in [11], namely, a study of a class of arithmetic functions that generalize the so-called irrational factor function $I(n)$ and the strong restrictive factor function $R(n)$, which are defined in [2] and [3] by

$$I(n) = \prod_{p^\alpha || n} p^{1/\alpha} \quad \text{and} \quad R(n) = \prod_{p^\alpha || n} p^{\alpha - 1}.$$

Panaitopol, Alkan, Ledoan, and the authors develop a number of results on these arithmetic functions ([1], [8], [9], and [10]). In the prequel, the authors establish results on the average values of the functions

$$f_A(n) = \prod_{p^\alpha || n} p^{\frac{a\alpha + b}{c\alpha + d}}$$

for a class of matrices $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{PSL}_2(\mathbb{Z})$. Here, $\mathrm{PSL}_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\pm I$, where I is the identity matrix.

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We complement the results of the previous paper by considering real-valued λ-moments of these arithmetic functions $f_A(n)$, defined over a larger class of matrices in $\mathrm{PSL}_2(\mathbb{Z})$. This method offers greater flexibility, since the λ-weighted moments of the Dirichlet series associated with $f_A(n)$ may have meromorphic continuation to a region in which the original Dirichlet series has an essential singularity. As we shall see, this expands previous results to a more general framework, and leads to new results within this more general setting.

Consider the subset $\mathcal{A}$ of $\mathrm{SL}_2(\mathbb{Z})$ given by

$$\mathcal{A} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : c \geq 0, d > 0 \right\}.$$

For each matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $\mathrm{PSL}_2(\mathbb{Z})$ consider the fractional linear transformation

$$Az = \frac{az+b}{cz+d}.$$

For each $\lambda > 0$ we examine the λ-moment

$$(f_A(n))^\lambda = \prod_{p^\alpha || n} p^{\lambda A \alpha}.$$

A key tool in our study of the λ-moment is the Dirichlet series

$$F_{A,\lambda}(s) = \sum_{n=1}^{\infty} \frac{(f_A(n))^\lambda}{n^s}.$$

We say that the pair (A, λ) is *good* if there exists a half-plane where $F_{A,\lambda}$ has meromorphic continuation with at least one pole. Consider the space $\mathbb{G}$ in $\mathcal{A} \times \mathbb{R}_+$ of pairs (A, λ) that are good. Information about $\mathbb{G}$ leads to information about λ-moments

$$S_{A,\lambda}(x) = \sum_{n \leq x} (f_A(n))^\lambda$$

and more precise estimates about the weighted λ-moments such as

$$M_{A,\lambda}(x) = \sum_{n \leq x} \left(1 - \frac{n}{x}\right) (f_A(n))^\lambda.$$

Asymptotic formulas for such moments are given in the following section.

Theorem 1.1. *Suppose λ is a positive real number. Given a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathcal{A}$, necessary and sufficient conditions for the pair (A, λ) to be in $\mathbb{G}$ are*

- *if $A = R_k = \left(\begin{smallmatrix} 1 & -k \\ 0 & 1 \end{smallmatrix}\right)$, $k = 1, 2, 3, \ldots$ and $0 < \lambda < 1/k$,*
- *if $A \neq R_k$, $b \geq -1$ and $\lambda \in (0, \infty)$,*
- *if $A \neq R_k$, $b < -1$ and $0 < \lambda < -\frac{dc}{bc+1}$.*

For any $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathcal{A}$ define $h(A) = \max\{|a|, |b|, |c|, |d|\}$. For any positive integer Q, define

$$g_Q(\lambda) = \frac{\#\{A \in \mathcal{A} : h(A) \leq Q \text{ and } (A, \lambda) \in \mathbb{G}\}}{\#\{A \in \mathcal{A} : h(A) \leq Q\}}.$$

Figure 1 shows the behavior of $g_Q(\lambda)$ for values of Q up to 75. We will prove

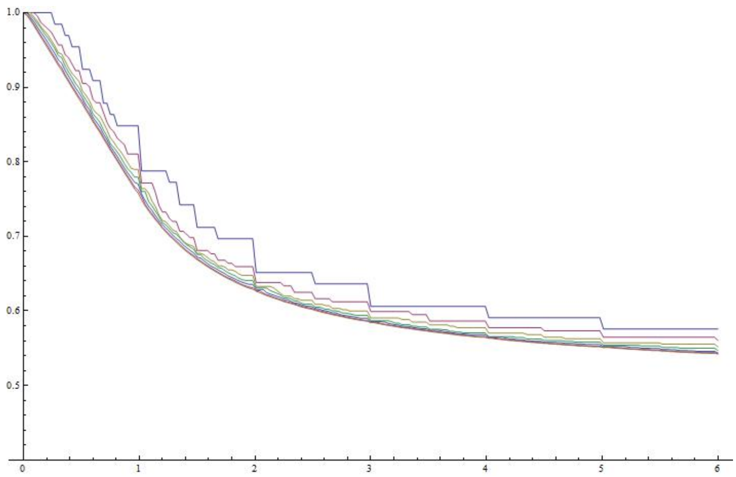

FIGURE 1. Plot of the function $g_Q(\lambda)$

the following:

Theorem 1.2. *The functions g_Q converge uniformly on compact subintervals of $(0, \infty)$ to g as $Q \to \infty$, where*

$$g(\lambda) = \begin{cases} 1 - \dfrac{\lambda}{4} & \text{if } 0 \leq \lambda < 1, \\ \dfrac{1}{2} + \dfrac{1}{4\lambda} & \text{if } \lambda \geq 1. \end{cases}$$

2. Proof of Theorem 1.1

Lemma 2.1. *Suppose $F(s)$ is a Dirichlet series with Euler product*

$$F(s) = \prod_{p \text{ prime}} \left(1 + \frac{p^{c_1}}{p^s} + \frac{p^{c_2}}{p^{2s}} + \frac{p^{c_3}}{p^{3s}} + \cdots\right),$$

where $c_1, c_2, \ldots$ are real numbers independent of p. Assume there exists a finite set of natural numbers $\mathcal{N} = \{n_1, n_2, \ldots, n_l\}$ such that for all $1 \leq j, k \leq l$ we have $\frac{1}{n_j}\left(1 + c_{n_j}\right) = \frac{1}{n_k}\left(1 + c_{n_k}\right)$ and such that

$$\frac{1}{n_j}\left(1 + c_{n_j}\right) > \sup_{n \notin \mathcal{N}} \left\{\frac{1}{n}\left(1 + c_n\right)\right\}.$$

Then $F(s)$ satisfies

$$F(s) = G(s) \prod_{j=1}^{l} \zeta(n_j s - c_{n_j})$$

where $G(s)$ is analytic in the half plane

$$\Re(s) > \sup_{\substack{m \in \mathcal{N} \\ n \notin \mathcal{N}}} \left\{ \frac{1}{n}\left(1 + c_n\right), \frac{1}{m}\left(1/2 + c_m\right) \right\}$$

and is bounded in any closed half plane contained in this region.

Proof. We restrict ourselves to the case $l = 1$; the remaining cases are similar. We factor

$$\prod_p \left(1 + \frac{p^{c_1}}{p^s} + \frac{p^{c_2}}{p^{2s}} + \frac{p^{c_3}}{p^{3s}} + \cdots \right)$$

$$= \prod_p \left(1 + \frac{p^{c_{n_1}}}{p^{n_1 s}} \right) \left(1 + \left(1 + \frac{p^{c_{n_1}}}{p^{n_1 s}} \right)^{-1} \sum_{\substack{n \geq 1 \\ n \neq n_1}} \frac{p^{c_n}}{p^{ns}} \right)$$

and note that

$$\prod_p \left(1 + \frac{p^{c_{n_1}}}{p^{n_1 s}} \right) = \frac{\zeta(n_1 s - c_{n_1})}{\zeta(2 n_1 s - 2 c_{n_1})}.$$

Write $s = \sigma + it$. The function $1/\zeta(2n_1 s - 2c_{n_1})$ is analytic for $\sigma > \frac{1}{n_1}\left(\frac{1}{2} + c_{n_1}\right)$. Also, for any fixed $\epsilon > 1/2$ and $\sigma \geq \frac{1}{n_1}\left(\epsilon + c_{n_1}\right)$ we have that

$$\left(1 + \frac{p^{c_{n_1}}}{p^{n_1 s}} \right)^{-1} \ll_\epsilon 1,$$

and for $\sigma > \epsilon + \sup_{n \neq n_1} \left\{ \frac{1}{n}\left(1 + c_n\right) \right\}$ we have

$$\sum_{\substack{n \geq 1 \\ n \neq n_1}} \frac{p^{c_n}}{p^{ns}} \ll \sup_{n \neq n_1} \frac{p^{c_n}}{p^{n\sigma}} \ll_\epsilon \frac{1}{p^{1+\epsilon}}.$$

So

$$\sum_p \left| \left(1 + \frac{p^{c_{n_1}}}{p^{n_1 s}} \right)^{-1} \sum_{\substack{n \geq 1 \\ n \neq n_1}} \frac{p^{c_n}}{p^{ns}} \right|$$

converges in any half plane of the form

$$\sigma \geq \sigma_0 > \epsilon + \sup_{n \neq n_1} \left\{ \frac{c_{n_1}}{n_1}, \frac{1}{n}\left(1 + c_n\right) \right\}.$$

It follows that the product

$$\prod_p \left(1 + \left(1 + \frac{p^{c_{n_1}}}{p^{n_1 s}}\right)^{-1} \sum_{\substack{n\geq 1\\ n\neq n_1}} \frac{p^{c_n}}{p^{ns}} \right)$$

is uniformly bounded on the half-plane $\Re s > \sigma_0$ (see §14.2, p.15 of [12]). Hence

$$G(s) = \frac{1}{\zeta(2n_1 s - 2c_{n_1})} \prod_p \left(1 + \left(1 + \frac{p^{c_{n_1}}}{p^{n_1 s}}\right)^{-1} \sum_{\substack{n\geq 1\\ n\neq n_1}} \frac{p^{c_n}}{p^{ns}} \right)$$

is uniformly bounded on $\Re s \geq \max\left\{\sigma_0, \frac{1}{n_1}\left(\frac{1}{2} + c_{n_1}\right)\right\}$. Since $G(s)$ is analytic in this half-plane, the Dirichlet series $F(s)$ has a meromorphic continuation to this region, where it satisfies $F(s) = \zeta(n_1 s - c_{n_1})G(s)$. □

We now give the proof of Theorem 1.1. Given a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $\mathcal{A}$, we have that if $\alpha \geq 1$, then $\frac{a\alpha+b}{c\alpha+d} < (|a|+|b|)\alpha$. So $f_{A,\lambda}(n) \leq n^{|a|+|b|}$ and hence the Dirichlet series $F_{A,\lambda}(s)$ will be analytic in the region $\Re s > 1+|a|+|b|$. In this region, F has an Euler product

$$F_{A,\lambda}(s) = \prod_p \left(1 + \frac{p^{\lambda A1}}{p^s} + \frac{p^{\lambda A2}}{p^{2s}} + \frac{p^{\lambda A3}}{p^{3s}} + \cdots \right).$$

Let

$$\theta^{(1)} = \sup_{n\geq 1}\{\theta_n(\lambda)\}, \tag{1}$$

where $\theta_n = \theta_n(\lambda) = \frac{1}{n}(1 + \lambda An)$. If the supremum in (1) is attained, then by employing Lemma 2.1 one can show that $(A, \lambda) \in \mathbb{G}$. Next, we identify this supremum by considering the function

$$\theta(x, \lambda) = \frac{1}{x}\left(1 + \lambda\frac{ax+b}{cx+d}\right),$$

where x is positive and real-valued.

If $c = 0$, then

$$\begin{aligned} \theta(x, \lambda) &= \frac{1}{x}\left(1 + \lambda\frac{ax+b}{d}\right) \\ &= \frac{\lambda a}{d} + \frac{1}{x}\left(1 + \frac{\lambda b}{d}\right). \end{aligned}$$

If $c \neq 0$, then

$$\frac{ax+b}{cx+d} = \frac{a}{c} - \frac{\det A}{c(cx+d)},$$

and upon writing

$$\frac{1}{x(cx+d)} = \frac{1/d}{x} - \frac{c/d}{cx+d}$$

we find that

$$\theta(x,\lambda) = \frac{1}{x}\left(1 + \frac{\lambda b}{d}\right) + \frac{1}{cx+d}\frac{\lambda \det A}{d}.$$

Since $cx+d > 0$ for all positive x, the expression $\theta(x,\lambda)$ will be a decreasing function of x if the coefficients of $\frac{1}{x}$ and $\frac{1}{cx+d}$ are positive. If $\det A = 1$ with $a \geq 1$ and $b \geq 0$, then $\theta(x,\lambda)$ is a decreasing function of x for any $\lambda > 0$, so $\sup_{n\geq 2}\{\theta_n\} < \theta_1$, and $(A,\lambda) \in \mathbb{G}$ for any λ in $(0,\infty)$. If $b \leq 0$, then $1 + \lambda\frac{b}{d}$ is positive provided that $\lambda < -d/b$.

The partial derivative

$$\frac{\partial}{\partial x}\theta(x,\lambda) = -\frac{\left(1+\lambda\frac{b}{d}\right)(cx+d)^2 + \frac{c\lambda}{d}x^2}{x^2(cx+d)^2}$$

is negative for large enough x provided that

$$0 < c^2\left(1 + \lambda\frac{b}{d}\right) + \frac{c\lambda}{d}. \tag{2}$$

If $b = -1$ and $c = 1$, then (3) gives that $(A,\lambda) \in \mathbb{G}$ for any λ in $(0,\infty)$ for such matrices A. More generally, if $b \neq -1$, then $bc \neq -1$, and so (2) is equivalent to

$$\lambda < -\frac{dc}{bc+1}. \tag{3}$$

We see that $\theta(x,\lambda)$ has a maximum provided that $\lambda > 0$ is in this range, and hence so does $\theta_n(\lambda)$. This completes the proof of Theorem 1.1.

If we take x_0 to be the value of x for which $\theta(x,\lambda)$ is maximal, then $\theta^{(1)}$ is equal to θ_{n_1}, where $n_1 = \lfloor x_0 \rfloor$ or $n_1 = \lceil x_0 \rceil$. We remark that if λ is such that the above maximum is attained at both $\lfloor x_0 \rfloor$ and $\lceil x_0 \rceil$, where x_0 is not an integer, then $F_{A,\lambda}(s)$ has a double pole at $s = \theta^{(1)}$. Furthermore, we note that for a given matrix A, the set of such exceptional λ is at most countable.

One can obtain an asymptotic formula for $M_{A,\lambda}(x)$ using the techniques of the prequel to this paper, which can be summarized as follows:

Write $F_{A,\lambda}(s)$ in the form given by Lemma 2.1 and use a variant of Perron's formula, namely

$$\sum_{n\leq x}\left(1 - \frac{n}{x}\right) f_{A,\lambda}(n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_{A,\lambda}(s)\frac{x^s}{s(s+1)}ds,$$

where $\sigma_0 < c < \sigma_0 + \delta$ for some $\delta > 0$.

Apply the zero-free region for $\zeta(s)$ due to Korobov [7] and Vinogradov [14] (see also Chapters 2 and 5 of [15])

$$\sigma \geq 1 - c_0(\log t)^{-2/3}(\log\log t)^{-1/3}$$

for $t \geq t_0$, in which

$$\frac{1}{|\zeta(s)|} \ll (\log t)^{2/3}(\log\log t)^{1/3}$$

(Recent improvements on explicit forms of this result can be found in [4] and [5]).

Fix U and T to be chosen later, with $0 < U < T < x^2$, and let $\nu = \frac{1}{n_1}\left(\frac{1}{2} + \lambda A_{n_1}\right)$ and

$$\eta = \nu - c_0(\log U)^{-2/3}(\log\log U)^{-1/3}.$$

Deform the path of integration into the union of the line segments

$$\begin{cases} \gamma_1, \gamma_9 : s = c + it & \text{if } |t| \geq T \\ \gamma_2, \gamma_8 : s = \sigma \pm iT & \text{if } \nu \leq \sigma \leq c \\ \gamma_3, \gamma_7 : s = \nu + it & \text{if } U \leq |t| \leq T \\ \gamma_4, \gamma_6 : s = \sigma \pm iU & \text{if } \eta \leq \sigma \leq \nu \\ \gamma_5 : s = \eta + it & \text{if } |t| \leq U. \end{cases}$$

The integrand is analytic on and within this modified contour, and by the residue theorem,

$$M_{A,\lambda}(x) = K_1 x^{\theta^{(1)}} + \sum_{k=1}^{9} J_k,$$

the main contribution being due to the residue of the simple pole at the point $s = \theta^{(1)}$.

In order to estimate the integral along the modified contour one makes use of the bounds

$$|\zeta(\sigma + it)| = \begin{cases} O(t^{(1-\sigma)/2}, & \text{if } 0 \leq \sigma \leq 1 \text{ and } |t| \geq 1 \\ O(\log t), & \text{if } 1 \leq \sigma \leq 2 \\ O(1), & \text{if } \sigma \geq 2 \end{cases}$$

(see [13], §3.11 and §5.1).

Upon estimating $|J_i|$, $i = 1, 2, \ldots, 9$ and selecting U and T so as to optimize the error terms, we get that if $(A, \lambda) \in \mathbb{G}$ and the pole of $F_{A,\lambda}(s)$ at $\theta^{(1)}$ is simple, then

$$M_{A,\lambda}(x) = K_1 x^{\theta^{(1)}} + R_{A,\lambda}(x),$$

where

$$R_{A,\lambda}(x) \ll_{A,\lambda} \max\left\{x^{\theta^{(2)}}, x^{\frac{1}{n_1}\left(\frac{1}{2}+\lambda A_{n_1}\right)} \exp\{-c(\log x)^{3/5}(\log\log x)^{-1/5}\}\right\}.$$

Notice that in the case where $x^{\theta^{(2)}}$ is of a larger order than

$$x^{\frac{1}{n_1}\left(\frac{1}{2}+\lambda A_{n_1}\right)} \exp\{-c(\log x)^{3/5}(\log\log x)^{-1/5}\},$$

one obtains a secondary term in the asymptotic formula for $M_{A,\lambda}(x)$ of the form $K_2 x^{\theta^{(2)}}$.

We remark that given a matrix A there may possibly be a finite or countable set of λ for which $F_{A,\lambda}(s)$ has a double pole at $s = \theta^{(1)}$. In these rare cases $M_{A,\lambda}(x)$ has a different order of magnitude. More precisely,

$$M_{A,\lambda}(x) \sim K' x^{\theta^{(1)}} \log x$$

as $x \to \infty$, where K' is a positive constant that depends only on A and λ.

For asymptotic formulas for the sums $S_{A,\lambda}(x)$ we use the following form of Perron's formula (see [12], Sections 9.42 and 9.44; and [13], Section 3.12):

$$\sum_{n\le x} a(n) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} A(s)\frac{x^s}{s}ds + R(x,c,T),$$

where $A(s)$ is the Dirichlet series associated with $a(n)$ and

$$|R(x,c,T)| \le \frac{x^c}{T}\sum_{n=1}^{\infty}\frac{|a(n)|}{n^c|\log x/n|}.$$

It is more natural to consider this sum $S_{A,\lambda}(x)$ instead of $M_{A,\lambda}(x)$, but we note that the variant of Perron's formula used for $M_{A,\lambda}(x)$ has an extra factor of $\frac{1}{s+1}$ in the integrand, which makes estimations easier.

In order to estimate $S_{A,\lambda}(x)$, we apply the above version of Perron's formula with $a(n) = f_{A,\lambda}(n)$ and $A(s) = F_{A,\lambda}(s)$. Upon shifting the path of integration and replacing it with a rectangular path with vertices $c - iT$, $c + iT$, $\nu + iT$, and $\nu - iT$, one can apply Cauchy's theorem as before to obtain

$$\sum_{n\le x} (f_A(n))^\lambda = K'' x^{\theta^{(1)}} + \sum_{k=1}^{3} J_k,$$

where K'' is a positive constant depending only on A and λ. Upon estimating the remaining terms J_k and $R(x,c,T)$ and selecting T so as to optimize the resulting error terms, one finds that

$$\sum_{n\le x} (f_A(n))^\lambda = K'' x^{\theta^{(1)}} + R'_{A,\lambda}(x),$$

where

$$R'_{A,\lambda}(x) \ll_{A,\lambda} \max\left\{x^{\theta^{(2)}}, x^{\frac{1}{n_1}\left(\frac{1}{2}+\lambda A_{n_1}\right)}(\log x)^r\right\}$$

and r is a positive constant that depends only on A and λ.

We also remark that even if (A,λ) is not in $\mathbb{G}$, one can still use Perron's formula to get nontrivial upper and lower bounds for $M_{A,\lambda}(x)$ of the form

$$x^C \ll_{A,\lambda} M_{A,\lambda}(x) \ll_{A,\lambda,\epsilon} x^{C+\epsilon}$$

for some positive constant C depending on A and λ.

3. Proof of Theorem 1.2

We begin with the following simple result.

Lemma 3.1. *For any* $0 < \alpha < \beta$ *and any* $\epsilon > 0$ *we have*

$$\#\{\alpha q \leq m \leq \beta q : (m,q) = 1\} = (\beta - \alpha)\phi(q) + O_\epsilon(q^\epsilon).$$

Proof. We have

$$\begin{aligned}\#\{\alpha q \leq m \leq \beta q : (m,q) = 1\} &= \sum_{\alpha q \leq m \leq \beta q} \sum_{\substack{d|m \\ d|q}} 1 \\ &= \sum_{d|q} \mu(d) \sum_{\substack{\alpha q \leq m \leq \beta q \\ d|m}} 1 \\ &= \sum_{d|q} \mu(d) \left(\left\lfloor \beta \frac{q}{d} \right\rfloor - \left\lfloor \alpha \frac{q}{d} \right\rfloor \right).\end{aligned}$$

Since $|\sum_{d|q} \mu(d)| \leq \sum_{d|q} 1 \ll_\epsilon q^\epsilon$ we have

$$\begin{aligned}\#\{\alpha q \leq m \leq \beta q : (m,q) = 1\} &= (\beta - \alpha) q \sum_{d|q} \frac{\mu(d)}{d} + O_\epsilon(q^\epsilon) \\ &= (\beta - \alpha)\phi(q) + O_\epsilon(q^\epsilon).\end{aligned}$$ □

Note that if $c \geq 0$ and $b < 0$, then the relation $ad - bc = 1$ implies that $a \leq 0$. Letting $b \to -b$ and $a \to -a$, the given conditions can be replaced by $bc - ad = 1$, $a, b, c, d > 0$, and

$$0 < \lambda < \frac{dc}{bc - 1}. \tag{4}$$

Let

$$N_{\mathcal{A},a}(\lambda) = \#\left\{A \in \mathcal{A} : h(A) = c, (\lambda, A) \in \mathbb{G}\right\},$$

and define $N_{\mathcal{A},b}(\lambda)$, $N_{\mathcal{A},c}(\lambda)$, and $N_{\mathcal{A},d}(\lambda)$ similarly for the cases where $h(A) = b$, $h(A) = c$, and $h(A) = d$, so that

$$\begin{aligned}&\#\left\{A \in \mathcal{A} : h(A) \leq Q, (\lambda, A) \in \mathbb{G}\right\} \\ &= N_{\mathcal{A},a}(\lambda) + N_{\mathcal{A},b}(\lambda) + N_{\mathcal{A},c}(\lambda) + N_{\mathcal{A},d}(\lambda) + O(Q).\end{aligned}$$

If $h(A) = c$, then $bc - ad = 1$ implies $ad \equiv -1 \pmod{c}$, hence $a \equiv -\bar{d} \pmod{c}$. Since also $1 \leq a < c$, we have $a = c - \bar{d}$, where $\bar{d}$ is the unique inverse of d mod c satisfying $1 \leq \bar{d} \leq c$. So

$$b = \frac{1 + ad}{c} = \frac{1 + cd - d\bar{d}}{c}.$$

Inserting this into equation (4), we get

$$\bar{d} > c\frac{\lambda - 1}{\lambda}.$$

We note that there are only $O(1)$ matrices A with $h(A) = 1$. Hence

$$\begin{aligned} N_{\mathcal{A},c}(\lambda) &= \sum_{2\le q\le Q} \#\left\{1 \le d \le q : (d,q) = 1, \bar{d} > q\frac{\lambda-1}{\lambda}\right\} + O(1) \\ &= \sum_{2\le q\le Q} \#\left\{1 \le m \le q : (m,q) = 1, m > q\frac{\lambda-1}{\lambda}\right\} + O(1) \\ &= \sum_{2\le q\le Q} \#\left\{\max\left\{1, q\frac{\lambda-1}{\lambda}\right\} \le m \le q : (m,q) = 1\right\} + O(1) \\ &= \begin{cases} \displaystyle\sum_{2\le q\le Q} \phi(q) + O_\epsilon(Q^{1+\epsilon}) & \text{if } 0 \le \lambda < 1, \\ \displaystyle\frac{1}{\lambda}\sum_{2\le q\le Q} \phi(q) + O_\epsilon(Q^{1+\epsilon}) & \text{if } \lambda \ge 1. \end{cases} \end{aligned}$$

Using the well-known estimate

$$\sum_{n\le X} \phi(n) = \frac{1}{2\zeta(2)}X^2 + O(X \log X)$$

(see for example [15] or Chapter 18 of [6]), we see that

$$N_{\mathcal{A},c}(\lambda) = \begin{cases} \dfrac{1}{2\zeta(2)}Q^2 + O_\epsilon(Q^{1+\epsilon}) & \text{if } 0 \le \lambda < 1, \\ \dfrac{1}{2\lambda\zeta(2)}Q^2 + O_\epsilon(Q^{1+\epsilon}) & \text{if } \lambda \ge 1. \end{cases}$$

If $h(A) = b$, then $bc - ad = 1$ implies $ad \equiv -1 \pmod b$, hence $a \equiv -\bar{d} \pmod b$. Since also $1 \le a < b$, we have $a = b - \bar{d}$, where $\bar{d}$ is the unique inverse of d mod b satisfying $1 \le \bar{d} \le b$. So

$$c = \frac{1 + ad}{b} = \frac{1 + bd - d\bar{d}}{b}.$$

Inserting this into equation (4), we get

$$d > b\lambda + \frac{1}{\bar{d} - 1}.$$

This can only hold if $\lambda < 1$, hence for $\lambda \ge 1$, we have

$$\#\{A \in \mathcal{A} : h(A) = b, (\lambda, A) \in \mathbb{G}\} = 0.$$

When $0 < \lambda < 1$, we note that $\bar{d} = 1$ only when $d = 1$ since $1 \le d < b$. We get that $d > b\lambda$ for all but a bounded number of integers b. Also, we note that again $h(A) = 1$ for only $O(1)$ of these matrices. Hence for $0 < \lambda < 1$,

$$\begin{aligned} N_{\mathcal{A},b}(\lambda) &= \sum_{2 \le q \le Q} \# \left\{ 1 < d \le q : (d,q) = 1, d > b\lambda + \frac{1}{\bar{d} - 1} \right\} + O(1) \\ &= \sum_{2 \le q \le Q} \left(\# \left\{ 1 < d \le q : (d,q) = 1, d > b\lambda + \frac{1}{\bar{d} - 1} \right\} + O(1) \right) + O(1) \\ &= \sum_{2 \le q \le Q} \# \{ \lambda q \le m \le q : (m,q) = 1 \} + O(Q) \\ &= (1 - \lambda) \sum_{2 \le q \le Q} \phi(q) + O_\epsilon(Q^{1+\epsilon}) \\ &= \frac{1-\lambda}{2\zeta(2)} Q^2 + O_\epsilon(Q^{1+\epsilon}). \end{aligned}$$

If $h(A) = a$, then $bc - ad = 1$ implies $bc \equiv 1 \pmod a$, hence $b \equiv \bar{c} \pmod a$. Since also $1 \le b < a$, we have $b = \bar{c}$, where $\bar{c}$ is the unique inverse of c mod a satisfying $1 \le \bar{c} \le a$. So

$$d = \frac{bc - 1}{a} = \frac{c\bar{c} - 1}{a}.$$

Inserting this into equation (4), we get that $a\lambda < c$. This can only hold if $\lambda < 1$, hence for $\lambda \ge 1$, we have $\# \{ A \in \mathcal{A} : h(A) = a, (\lambda, A) \in \mathbb{G} \} = 0$. Furthermore, if $bc = 1$, then $ad = 2$, so there are $O(1)$ such matrices. When $0 < \lambda < 1$,

$$\begin{aligned} N_{\mathcal{A},a}(\lambda) &= \sum_{2 \le q \le Q} \# \{ 1 \le c \le q : (c,q) = 1, c > a\lambda \} + O(1) \\ &= \sum_{2 \le q \le Q} \# \{ \lambda q \le m \le q : (m,q) = 1 \} + O(1) \\ &= (1 - \lambda) \sum_{2 \le q \le Q} \phi(q) + O_\epsilon(Q^{1+\epsilon}) \\ &= \frac{1-\lambda}{2\zeta(2)} Q^2 + O_\epsilon(Q^{1+\epsilon}). \end{aligned}$$

If $h(A) = d$, then $bc - ad = 1$ implies $bc \equiv 1 \pmod d$, hence $b \equiv \bar{c} \pmod d$. Since also $1 \le b < c$, we have $b = \bar{c}$, where $\bar{c}$ is the unique inverse of c mod d satisfying $1 \le \bar{c} \le d$. So

$$d = \frac{bc - 1}{d} = \frac{c\bar{c} - 1}{d}.$$

Inserting this into equation (4), we get

$$\bar{c} < \frac{d}{\lambda} + \frac{1}{c}.$$

So $\bar{c} < \frac{d}{\lambda}$ for all but a bounded number of integers c. Since again there are $O(1)$ matrices with $bc = 1$, we get

$$\begin{aligned}
N_{\mathcal{A},d}(\lambda) &= \sum_{2\le q\le Q} \# \left\{ 1 \le c \le q : (c,q) = 1, \bar{c} < \frac{d}{\lambda} + \frac{1}{c} \right\} + O(1) \\
&= \sum_{2\le q\le Q} \left(\# \left\{ 1 \le c \le q : (c,q) = 1, \bar{c} < \frac{d}{\lambda} \right\} + O(1) \right) + O(1) \\
&= \sum_{2\le q\le Q} \# \left\{ 1 \le m \le \min\{q, \frac{q}{\lambda}\} : (m,q) = 1 \right\} + O(Q) \\
&= \begin{cases} \displaystyle\sum_{2\le q\le Q} \phi(q) + O_\epsilon(Q^{1+\epsilon}) & \text{if } 0 \le \lambda < 1, \\ \displaystyle\frac{1}{\lambda} \sum_{2\le q\le Q} \phi(q) + O_\epsilon(Q^{1+\epsilon}) & \text{if } \lambda \ge 1. \end{cases} \\
&= \begin{cases} \displaystyle\frac{1}{2\zeta(2)} Q^2 + O_\epsilon(Q^{1+\epsilon}) & \text{if } 0 \le \lambda < 1, \\ \displaystyle\frac{1}{2\lambda\zeta(2)} Q^2 + O_\epsilon(Q^{1+\epsilon}) & \text{if } \lambda \ge 1. \end{cases}
\end{aligned}$$

Combining the above four cases, and the fact that if $\det A = -1$, then $(A, \lambda) \in \mathbb{G}$ for all $\lambda > 0$, one obtains after a short calculation the desired result.

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