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BOUNDEDNESS FOR FRACTIONAL HARDY-TYPE OPERATOR ON HERZ-MORREY SPACES WITH VARIABLE EXPONENT

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Abstract. In this paper, the fractional Hardy-type operator of variable order $\beta(x)$ is shown to be bounded from the Herz-Morrey spaces $M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)$ with variable exponent $q_1(x)$ into the weighted space $M\dot{K}^{\alpha,\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n,\omega)$, where $\omega=(1+|x|)^{-\gamma(x)}$ with some $\gamma(x)>0$ and $1/q_1(x)-1/q_2(x)=\beta(x)/n$ when $q_1(x)$ is not necessarily constant at infinity. It is assumed that the exponent $q_1(x)$ satisfies the logarithmic continuity condition both locally and at infinity that $1<q_1(\infty)\le q_1(x)\le (q_1)_+<\infty$ $(x\in\mathbb{R}^n)$.

1. Introduction

Let f be a locally integrable function on $\mathbb{R}^n$. The n-dimensional Hardy operator is defined by

$$\mathscr{H}(f)(x):=\frac{1}{|x|^n}\int_{|t|<|x|}f(t)\mathrm{d}t,\quad x\in\mathbb{R}^n\setminus\{0\}.$$

In 1995, Christ and Grafakos [2] obtained the result for the boundedness of $\mathscr{H}$ on $L^p(\mathbb{R}^n)$ $(1<p<\infty)$ spaces, and they also found the exact operator norms of $\mathscr{H}$ on this space. In 2007, Fu et al. [3] gave the central BMO estimates for commutators of n-dimensional fractional and Hardy operators. And recently, author [4, 5, 6, 7, 8, 9] also considers the boundedness for Hardy operator and its commutator in (variable exponent) Herz-Morrey spaces.

The theory of variable exponent Lebesgue spaces is started by Orlicz (see [10], 1931) and Nakano (see [11, 12], 1950 and 1951). In particular, the definition of Musielak-Orlicz spaces is clearly stated in [11]. However, the variable exponent function space, due to the failure of translation invariance and related properties, is very difficult to analyze.

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Nowadays there is an evident increase of investigations related to both the theory of the spaces $L^{q(\cdot)}(\Omega)$ themselves and the operator theory in these spaces. This is caused by possible applications to models with non-standard local growth (in elasticity theory, fluid mechanics, differential equations, see for example [13], [14] and references therein) and is based on recent breakthrough result on boundedness of the Hardy-Littlewood maximal operator in these spaces. By virtue of the fine works [15, 16, 17, 18, 20, 21, 22, 25, 26, 27, 28], some important conditions on variable exponent, for example, the log-Hölder conditions and the Muckenhoupt type condition, have been obtained.

Now, we define the n-dimensional fractional Hardy-type operators of variable order $\beta(x)$ as follows.

Definition 1.1. Let f be a locally integrable function on $\mathbb{R}^n$, $0 \leq \beta(x) < n$. The n-dimensional fractional Hardy-type operators of variable order $\beta(x)$ are defined by

$$\mathscr{H}_{\beta(\cdot)}(f)(x) := \frac{1}{|x|^{n-\beta(x)}} \int_{|t|<|x|} f(t)\mathrm{d}t, \tag{1.0a}$$

$$\mathscr{H}^*_{\beta(\cdot)}(f)(x) := \int_{|t|\geq|x|} \frac{f(t)}{|t|^{n-\beta(x)}}\mathrm{d}t, \tag{1.0b}$$

where $x \in \mathbb{R}^n \setminus \{0\}$.

Obviously, when $\beta(x) = 0$, $\mathscr{H}_{\beta(\cdot)}$ is just $\mathscr{H}$, and denote by $\mathscr{H}^* := \mathscr{H}^*_{\beta(\cdot)} = \mathscr{H}^*_0$. And when $\beta(x)$ is constant, $\mathscr{H}_{\beta(\cdot)}$ and $\mathscr{H}^*_{\beta(\cdot)}$ will become $\mathscr{H}_\beta$ and $\mathscr{H}^*_\beta$ [3], respectively.

The Riesz-type potential operator of variable order $\beta(x)$ is defined by

$$I_{\beta(\cdot)}(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta(x)}}\mathrm{d}y, \quad 0 < \beta(x) < n. \tag{1.1}$$

The boundedness of the operator $I_{\beta(\cdot)}$ from the space $L^{p(\cdot)}(\mathbb{R}^n)$ with the variable exponent $p(x)$ into the space $L^{q(\cdot)}(\mathbb{R}^n)$ with the limiting Sobolev exponent

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\beta(x)}{n}$$

was an open problem for a long time. It was solved in the case of bounded domains. First, in [29], in the case of bounded domains Ω, there was proved a conditional result: the Sobolev theorem is valid for the potential operator $I_{\beta(\cdot)}$ within the framework of the spaces $L^{p(\cdot)}(\Omega)$ with the variable exponent $p(x)$ satisfying the logarithmic Dini condition, if the maximal operator is bounded in the space $L^{p(\cdot)}(\Omega)$. In 2004, Diening [20] proved the boundedness of the maximal operator.

In 2004, Diening [19] proved Sobolev's theorem for the potential I_β on the whole space $\mathbb{R}^n$ assuming that $p(x)$ is constant at infinity ($p(x)$ is always constant outside some large ball) and satisfies the same logarithmic condition as in [29]. Another progress for unbounded domains is the result of Cruz-Uribe

et al. [18] on the boundedness of the maximal operator in unbounded domains for exponents $p(x)$ satisfying the logarithmic smoothness condition both locally and at infinity.

In [24], Kokilashvili and Samko prove Sobolev-type theorem for the potential $I_{\beta(\cdot)}$ from the space $L^{p(\cdot)}(\mathbb{R}^n)$ into the weighted space $L^{q(\cdot)}_{\omega}(\mathbb{R}^n)$ with the power weight ω fixed to infinity, under the logarithmic condition for $p(x)$ satisfied locally and at infinity, not supposing that $p(x)$ is constant at infinity but assuming that $p(x)$ takes its minimal value at infinity.

Motivated by the above results, we are to investigate mapping properties of the fractional Hardy-type operators $\mathscr{H}_{\beta(\cdot)}$ and $\mathscr{H}^*_{\beta(\cdot)}$ within the framework of the Herz-Morrey spaces with variable exponent.

Throughout this paper, we will denote by $|S|$ the Lebesgue measure and by χ_S the characteristic function for a measurable set $S \subset \mathbb{R}^n$; $B(x,r)$ is the ball centered at x and of radius r; $B_0 = B(0,1)$. C denotes a constant that is independent of the main parameters involved but whose value may differ from line to line. For any index $1 < q(x) < \infty$, we denote by $q'(x)$ its conjugate index, namely, $q'(x) = \frac{q(x)}{q(x)-1}$. For $A \sim D$, we mean that there is a constant $C > 0$ such that$C^{-1}D \leq A \leq CD$.

2. Preliminaries

In this section, we give the definition of Lebesgue and Herz-Morrey spaces with variable exponent, and give basic properties and useful lemmas.

2.1. Function spaces with variable exponent

Let Ω be a measurable set in $\mathbb{R}^n$ with $|\Omega| > 0$. We first define Lebesgue spaces with variable exponent.

Definition 2.1. Let $q(\cdot) : \Omega \to [1, \infty)$ be a measurable function.

(I)) The Lebesgue spaces with variable exponent $L^{q(\cdot)}(\Omega)$ is defined by

$$L^{q(\cdot)}(\Omega) = \{f \text{ is measurable function} : F_q(f/\eta) < \infty \text{ for some constant } \eta > 0\},$$

where $F_q(f) := \int_\Omega |f(x)|^{q(x)} \mathrm{d}x$. The Lebesgue space $L^{q(\cdot)}(\Omega)$ is a Banach function space with respect to the norm

$$\|f\|_{L^{q(\cdot)}(\Omega)} = \inf\Big\{\eta > 0 : F_q(f/\eta) = \int_\Omega \Big(\frac{|f(x)|}{\eta}\Big)^{q(x)} \mathrm{d}x \leq 1\Big\}.$$

(II)) The space $L^{q(\cdot)}_{\mathrm{loc}}(\Omega)$ is defined by

$$L^{q(\cdot)}_{\mathrm{loc}}(\Omega) = \{f \text{ is measurable} : f \in L^{q(\cdot)}(\Omega_0) \text{ for all compact subsets } \Omega_0 \subset \Omega\}.$$

III) The weighted Lebesgue space $L^{q(\cdot)}_{\omega}(\Omega)$ is defined by as the set of all measurable functions for which

$$\|f\|_{L^{q(\cdot)}_{\omega}(\Omega)} = \|\omega f\|_{L^{q(\cdot)}(\Omega)} < \infty.$$

Next we define some classes of variable exponent functions. Given a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{r>0} r^{-n} \int_{B(x,r)} |f(y)|\mathrm{d}y,$$

where $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$.

Definition 2.2. Given a measurable function $q(\cdot)$ defined on $\mathbb{R}^n$, we write

$$q_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} q(x), \quad q_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} q(x).$$

(I) $q'_- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} q'(x) = \frac{q_+}{q_+-1}$, $q'_+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} q'(x) = \frac{q_-}{q_--1}$.

(II) Denote by $\mathscr{P}(\mathbb{R}^n)$ the set of all measurable functions $q(\cdot) : \mathbb{R}^n \to (1,\infty)$ such that

$$1 < q_- \le q(x) \le q_+ < \infty, \quad x \in \mathbb{R}^n.$$

(III) The set $\mathscr{B}(\mathbb{R}^n)$ consists of all measurable functions $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ satisfying that the Hardy-Littlewood maximal operator M is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$.

(IV) The set $\mathscr{C}^{\log}_0(\mathbb{R}^n)$ consists of all locally log-Hölder continuous functions $q(\cdot) : \mathbb{R}^n \to (0,\infty)$ satisfies the condition

$$|q(x)-q(y)| \le \frac{-C}{\ln(|x-y|)}, \quad |x-y| \le 1/2,\ x,y \in \mathbb{R}^n. \tag{2.1}$$

(V) The set $\mathscr{C}^{\log}_\infty(\mathbb{R}^n)$ consists of all log-Hölder continuous at infinity functions $q(\cdot) : \mathbb{R}^n \to (0,\infty)$ satisfies the condition

$$|q(x)-q(\infty)| \le \frac{C_\infty}{\ln(e+|x|)}, \quad x \in \mathbb{R}^n, \tag{2.2}$$

where $q(\infty) = \lim_{|x|\to\infty} q(x)$.

(VI) Denote by $\mathscr{C}^{\log}(\mathbb{R}^n) := \mathscr{C}^{\log}_0(\mathbb{R}^n) \cap \mathscr{C}^{\log}_\infty(\mathbb{R}^n)$ the set of all globally log-Hölder continuous functions $q(\cdot) : \mathbb{R}^n \to (0,\infty)$.

Remark 1. The $\mathscr{C}^{\log}_\infty(\mathbb{R}^n)$ condition is equivalent to the uniform continuity condition

$$|q(x)-q(y)| \le \frac{C}{\ln(e+|x|)}, \quad |y| \ge |x|,\ x,y \in \mathbb{R}^n. \tag{2.3}$$

The $\mathscr{C}^{\log}_\infty(\mathbb{R}^n)$ condition was originally defined in this form in [18].

Next we define the Herz-Morrey spaces with variable exponent. Let $B_k = B(0,2^k) = \{x \in \mathbb{R}^n : |x| \le 2^k\}, A_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$.

Definition 2.3. Suppose that $\alpha \in \mathbb{R},\ 0 \le \lambda < \infty,\ 0 < p < \infty,\ q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. The Herz-Morrey space with variable exponent $M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ is defined by

$$M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n) = \Big\{ f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}) : \|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} < \infty \Big\},$$

where

$$\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)} = \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{p}}.$$

Compare the variable Herz-Morrey space $M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ with the variable Herz space $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, where

$$\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \Big\{f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}) : \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} < \infty\Big\},$$

Obviously, $M\dot{K}^{\alpha,0}_{p,q(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

In 2012, Almeida and Drihem [1] discuss the boundedness of a wide class of sublinear operators on Herz spaces $K^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n)$ and $\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n)$ with variable exponent $\alpha(\cdot)$ and $q(\cdot)$. In this paper, the author only considers Herz-Morrey space $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$ with variable exponent $q(\cdot)$ but fixed $\alpha \in \mathbb{R}$ and $p \in (0,\infty)$. However, for the case of the exponent $\alpha(\cdot)$ is variable as well, which can be found in the furthermore work for the author.

2.2. Auxiliary propositions and lemmas

In this part we state some auxiliary propositions and lemmas which will be needed for proving our main theorems. And we only describe partial results we need.

Proposition 2.1. *Let $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$.*

(I) *If $q(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n)$, then we have $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$.*
(II) *$q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ if and only if $q'(\cdot) \in \mathscr{B}(\mathbb{R}^n)$.*

The first part in Proposition 2.1 is independently due to Cruz-Uribe et al. [18] and to Nekvinda [27], respectively. The second of Proposition 2.1 belongs to Diening [21] (see Theorem 8.1 or Theorem 1.2 in [17]).

Remark 2. Since

$$|q'(x) - q'(y)| \le \frac{|q(x) - q(y)|}{(q_- - 1)^2},$$

it follows at once that if $q(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n)$, then so does $q'(\cdot)$—i.e., if the condition hold, then M is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$ and $L^{q'(\cdot)}(\mathbb{R}^n)$. Furthermore, Diening has proved general results on Musielak-Orlicz spaces.

The order $\beta(x)$ of the fractional Hardy-type operators in Definition 1.1 is not assumed to be continuous. We assume that it is a measurable function on

$\mathbb{R}^n$ satisfying the following assumptions

$$\begin{cases} \beta_0 := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} \beta(x) > 0, \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x)\beta(x) < n, \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(\infty)\beta(x) < n. \end{cases} \tag{2.4}$$

In order to prove our main results, we need the Sobolev type theorem for the space $\mathbb{R}^n$ which was proved in ref. [24] for the exponents $p(x)$ not necessarily constant in a neigbourhood of infinity, but with some extra power weight fixed to infinity and under the assumption that $p(x)$ takes its minimal value at infinity.

Proposition 2.2. *Suppose that* $p(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n) \cap \mathscr{P}(\mathbb{R}^n)$. *Let*

$$1 < p(\infty) \le p(x) \le p_+ < \infty, \tag{2.5}$$

and $\beta(x)$ *meet condition* (2.4). *Then the following weighted Sobolev-type estimate is valid for the operator* $I_{\beta(\cdot)}$:

$$\Big\|(1+|x|)^{-\gamma(x)} I_{\beta(\cdot)}(f)\Big\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\beta(x)}{n}$$

is the Sobolev exponent and

$$\gamma(x) = C_\infty \beta(x)\Big(1 - \frac{\beta(x)}{n}\Big) \le \frac{n}{4} C_\infty, \tag{2.6}$$

C_∞ *being the Dini-Lipschitz constant from* (2.2) *which* $q(\cdot)$ *is replaced by* $p(\cdot)$.

Remark 3. (i) If $\beta(x)$ satisfies the condition of type (2.2): $|\beta(x) - \beta(\infty)| \le \frac{C_\infty}{\ln(e+|x|)}$ $(x \in \mathbb{R}^n)$, then the weight $(1+|x|)^{-\gamma(x)}$ is equivalent to the weight $(1+|x|)^{-\gamma(\infty)}$.

(ii) One can also treat operator (1.1) with $\beta(x)$ replaced by $\beta(y)$. In the case of potentials over bounded domains Ω such potentials differ unessentially, if the function $\beta(x)$ satisfies the smoothness logarithmic condition as (2.1), since

$$C_1|x-y|^{n-\beta(y)} \le |x-y|^{n-\beta(x)} \le C_2|x-y|^{n-\beta(y)}$$

in this case (see [29], p. 277).

(iii) Under the assumptions of Proposition 2.2, similar conclusion is also valid for the fractional maximal operator

$$M_{\beta(\cdot)}(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{n-\beta(x)}} \int_{B(x,r)} |f(y)|\mathrm{d}y.$$

(iv) When $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, the assumption that $p(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n)$ is equivalent to assuming $1/p(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n)$, since

$$\Big|\frac{p(x)-p(y)}{(p_+)^2}\Big| \le \Big|\frac{1}{p(x)} - \frac{1}{p(y)}\Big| = \Big|\frac{p(x)-p(y)}{p(x)p(y)}\Big| \le \Big|\frac{p(x)-p(y)}{(p_-)^2}\Big|.$$

And further, $1/p(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n)$ implies that $1/q(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n)$ as well.

The next lemma known as the generalized Hölder's inequality on Lebesgue spaces with variable exponent, and the proof can be found in [15].

Lemma 2.1 (Generalized Hölder's inequality)**.** *Suppose that $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, then for any $f \in L^{q(\cdot)}(\mathbb{R}^n)$ and any $g \in L^{q'(\cdot)}(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\mathrm{d}x \le C_q \|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{q'(\cdot)}(\mathbb{R}^n)},$$

where $C_q = 1 + 1/q_- - 1/q_+$.

The following lemma can be found in [23].

Lemma 2.2. *Let $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$.*

(I) *Then there exist positive constants $\delta \in (0,1)$ and $C > 0$ such that*

$$\frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \le C\Big(\frac{|S|}{|B|}\Big)^{\delta}$$

for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$.

(II) *Then there exists a positive constant $C > 0$ such that*

$$C^{-1} \le \frac{1}{|B|}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \le C$$

for all balls B in $\mathbb{R}^n$.

Remark 4. (i) If $q_1(\cdot),\ q_2(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n) \cap \mathscr{P}(\mathbb{R}^n)$, then we see that $q_1'(\cdot),\ q_2(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Hence we can take positive constants $0 < \delta_1 < 1/(q_1')_+,\ 0 < \delta_2 < 1/(q_2)_+$ such that

$$\frac{\|\chi_S\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}} \le C\Big(\frac{|S|}{|B|}\Big)^{\delta_1}, \quad \frac{\|\chi_S\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}} \le C\Big(\frac{|S|}{|B|}\Big)^{\delta_2} \tag{2.7}$$

hold for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$ (see [7, 23]).

(ii) On the other hand, Kopaliani [25] has proved the conclusion: If the exponent $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ equals to a constant outside some large ball, then $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ if and only if $q(\cdot)$ satisfies the Muckenhoupt type condition

$$\sup_{Q:\text{cube}} \frac{1}{|Q|}\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{q'(\cdot)}(\mathbb{R}^n)} < \infty.$$

3. Main results and their proofs

Our main result can be stated as follows.

Theorem 3.1. *Suppose that $q_1(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n) \cap \mathscr{P}(\mathbb{R}^n)$ satisfies condition* (2.5)*, and $\beta(x)$ meet condition* (2.4) *which $p(\cdot)$ is replaced by $q_1(\cdot)$. Define the variable exponent $q_2(\cdot)$ by*

$$\frac{1}{q_2(x)} = \frac{1}{q_1(x)} - \frac{\beta(x)}{n}.$$

Let $0 < p_1 \le p_2 < \infty$, $\lambda \ge 0$, $\alpha < \lambda + n\delta_1$, where $\delta_1 \in (0, 1/(q_1')_+)$ is the constant appearing in (2.7)*. Then*

$$\Big\|(1+|x|)^{-\gamma(x)}\mathscr{H}_{\beta(\cdot)}(f)\Big\|_{M\dot{K}^{\alpha,\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)},$$

where $\gamma(x)$ is defined as in (2.6)*, and C_∞ is the Dini-Lipschitz constant from* (2.2) *which $q_1(\cdot)$ instead of $q(\cdot)$.*

Proof. For any $f \in M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$, if we denote $f_j := f \cdot \chi_j = f \cdot \chi_{A_j}$ for each $j \in \mathbb{Z}$, then we can write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\cdot\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

By (1.0a) and Lemma 2.1, we have

$$\begin{aligned}
|\mathscr{H}_{\beta(\cdot)}(f)(x)\cdot\chi_k(x)| &\le \frac{1}{|x|^{n-\beta(x)}}\int_{|t|<|x|}|f(t)|\mathrm{d}t\cdot\chi_k(x)\\
&\le \frac{1}{|x|^{n-\beta(x)}}\int_{B_k}|f(t)|\mathrm{d}t\cdot\chi_k(x)\\
&\le C2^{-kn}|x|^{\beta(x)}\Big(\sum_{j=-\infty}^{k}\int_{A_j}|f(t)|\mathrm{d}t\Big)\cdot\chi_k(x) \qquad (3.1)\\
&\le C2^{-kn}\sum_{j=-\infty}^{k}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\cdot|x|^{\beta(x)}\chi_k(x).
\end{aligned}$$

For Proposition 2.2, we note that

$$\begin{aligned}
I_{\beta(\cdot)}(\chi_{B_k})(x) &\ge I_{\beta(\cdot)}(\chi_{B_k})(x)\cdot\chi_{B_k}(x) = \int_{B_k}\frac{1}{|x-y|^{n-\beta(x)}}\mathrm{d}y\cdot\chi_{B_k}(x) \qquad (3.2)\\
&\ge C|x|^{\beta(x)}\cdot\chi_{B_k}(x) \ge C|x|^{\beta(x)}\cdot\chi_k(x).
\end{aligned}$$

Using Proposition 2.2, Lemma 2.2, (2.7), (3.1) and (3.2), we have

(3.3)
$$
\begin{aligned}
&\Big\|(1+|x|)^{-\gamma(x)}\mathscr{H}_{\beta(\cdot)}(f)\cdot\chi_k(\cdot)\Big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{-kn}\sum_{j=-\infty}^{k}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big\|(1+|x|)^{-\gamma(x)}|\cdot|^{\beta(x)}\cdot\chi_k(\cdot)\Big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{-kn}\sum_{j=-\infty}^{k}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Big\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(\chi_{B_k})\Big\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{-kn}\sum_{j=-\infty}^{k}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{-kn}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\sum_{j=-\infty}^{k}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\\
&\le C\sum_{j=-\infty}^{k}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}\\
&\le C\sum_{j=-\infty}^{k}2^{(j-k)n\delta_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Because of $0<p_1/p_2\le 1$, then we apply inequality

$$
\Big(\sum_{i=-\infty}^{\infty}|a_i|\Big)^{p_1/p_2}\le\sum_{i=-\infty}^{\infty}|a_i|^{p_1/p_2}, \tag{3.4}
$$

and obtain

$$
\begin{aligned}
&\Big\|(1+|x|)^{-\gamma(x)}\mathscr{H}_{\beta(\cdot)}(f)\Big\|^{p_1}_{M\dot{K}^{\alpha,\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n)}\\
&=\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda p_1}\Big(\sum_{k=-\infty}^{k_0}2^{k\alpha p_2}\Big\|(1+|x|)^{-\gamma(x)}\mathscr{H}_{\beta(\cdot)}(f)\cdot\chi_k(\cdot)\Big\|^{p_2}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\Big)^{p_1/p_2}\\
&\le\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda p_1}\Big(\sum_{k=-\infty}^{k_0}2^{k\alpha p_1}\Big\|(1+|x|)^{-\gamma(x)}\mathscr{H}_{\beta(\cdot)}(f)\cdot\chi_k(\cdot)\Big\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\Big).
\end{aligned}
$$

On the other hand, note the following fact

$$
\begin{aligned}
\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}&=2^{-j\alpha}\Big(2^{j\alpha p_1}\|f_j\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{1/p_1}\\
&\le 2^{-j\alpha}\Big(\sum_{i=-\infty}^{j}2^{i\alpha p_1}\|f_i\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{1/p_1}
\end{aligned}
\tag{3.5}
$$

$$= 2^{j(\lambda-\alpha)}\Big(2^{-j\lambda}\Big(\sum_{i=-\infty}^{j} 2^{i\alpha p_1}\|f_i\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{1/p_1}\Big)$$
$$\le C2^{j(\lambda-\alpha)}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}.$$

Thus, combining (3.3) and (3.5), and using $\alpha < \lambda + n\delta_1$, it follows that

$$\begin{aligned}
&\Big\|(1+|x|)^{-\gamma(x)}\mathscr{H}_{\beta(\cdot)}(f)\Big\|^{p_1}_{M\dot{K}^{\alpha,\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n)}\\
&\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda p_1}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p_1}\Big(\sum_{j=-\infty}^{k} 2^{(j-k)n\delta_1}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}\Big)\\
&\le C\|f\|^{p_1}_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda p_1}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p_1}\Big(\sum_{j=-\infty}^{k} 2^{(j-k)n\delta_1}2^{j(\lambda-\alpha)}\Big)^{p_1}\Big)\\
&\le C\|f\|^{p_1}_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda p_1}\Big(\sum_{k=-\infty}^{k_0} 2^{k\lambda p_1}\Big(\sum_{j=-\infty}^{k} 2^{(j-k)(n\delta_1+\lambda-\alpha)}\Big)^{p_1}\Big)\\
&\le C\|f\|^{p_1}_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda p_1}\Big(\sum_{k=-\infty}^{k_0} 2^{k\lambda p_1}\Big)\\
&\le C\|f\|^{p_1}_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Consequently, the proof of Theorem 3.1 is completed. □

Theorem 3.2. *Suppose that $q_1(\cdot) \in \mathscr{C}^{\log}(\mathbb{R}^n)\cap\mathscr{P}(\mathbb{R}^n)$ satisfies condition (2.5), and $\beta(x)$ meet condition (2.4) which $q_1(\cdot)$ instead of $p(\cdot)$. Define the variable exponent $q_2(\cdot)$ by*

$$\frac{1}{q_2(x)} = \frac{1}{q_1(x)} - \frac{\beta(x)}{n}.$$

Let $0 < p_1 \le p_2 < \infty$, $\lambda \ge 0$, $\alpha > \lambda - n\delta_2$, where $\delta_2 \in (0, 1/(q_2)_+)$ is the constant appearing in (2.7). Then

$$\Big\|(1+|x|)^{-\gamma(x)}\mathscr{H}^*_{\beta(\cdot)}(f)\Big\|_{M\dot{K}^{\alpha,\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)},$$

where $\gamma(x)$ is defined as in (2.6), and the Dini-Lipschitz constant from condition (2.2) which $q(\cdot)$ is replaced by $q_1(\cdot)$.

Proof. For simplicity, for any $f \in M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\mathbb{R}^n)$, we write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\cdot\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

By (1.0b) and Lemma 2.1, we have

$$
\begin{aligned}
&\left|(1+|x|)^{-\gamma(x)}\mathscr{H}^*_{\beta(\cdot)}(f)(x)\cdot\chi_k(x)\right| \\
&\le \int_{|t|\ge|x|}\frac{|f(t)|}{|t|^{n-\beta(x)}}\mathrm{d}t\cdot(1+|x|)^{-\gamma(x)}\chi_k(x) \\
&\le C\int_{\mathbb{R}^n\setminus B_k}|f(t)||x|^{\beta(x)-n}\mathrm{d}t\cdot(1+|x|)^{-\gamma(x)}\chi_k(x) \\
&\le C\sum_{j=k+1}^{\infty}\int_{A_j}|f(t)||x|^{\beta(x)-n}(1+|x|)^{-\gamma(x)}\mathrm{d}t\cdot\chi_k(x) \\
&\le C\sum_{j=k+1}^{\infty}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\left\|(1+|x|)^{-\gamma(x)}|\cdot|^{\beta(x)-n}\chi_j(\cdot)\right\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\cdot\chi_k(x).
\end{aligned}
\tag{3.6}
$$

Similar to (3.2), we give

$$
\begin{aligned}
I_{\beta(\cdot)}(\chi_{B_j})(x) &\ge I_{\beta(\cdot)}(\chi_{B_j})(x)\cdot\chi_{B_j}(x) \\
&= \int_{B_j}\frac{1}{|x-y|^{n-\beta(x)}}\mathrm{d}y\cdot\chi_{B_j}(x) \\
&\ge C|x|^{\beta(x)}\cdot\chi_{B_j}(x)\ge C|x|^{\beta(x)}\cdot\chi_j(x).
\end{aligned}
\tag{3.7}
$$

Since $q_1(\cdot)\in\mathscr{C}^{\log}(\mathbb{R}^n)\cap\mathscr{P}(\mathbb{R}^n)$ and $\beta(x)$ satisfy condition (2.4) and (2.5) which $q_1(\cdot)$ instead of $p(\cdot)$. So, applying Proposition 2.2, Lemma 2.2, (2.7), (3.6) and (3.7), we obtain

$$
\begin{aligned}
&\left\|(1+|x|)^{-\gamma(x)}\mathscr{H}^*_{\beta(\cdot)}(f)\cdot\chi_k(\cdot)\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
&\le C\sum_{j=k+1}^{\infty}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\left\|(1+|x|)^{-\gamma(x)}|\cdot|^{\beta(x)-n}\chi_j(\cdot)\right\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \\
&\le C\sum_{j=k+1}^{\infty}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\cdot 2^{-jn}\left\|(1+|x|)^{-\gamma(x)}I_{\beta(\cdot)}(\chi_{B_j})\right\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \\
&\le C\sum_{j=k+1}^{\infty}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\cdot 2^{-jn}\|\chi_{B_j}\|_{L^{q_2'(\cdot)}(\mathbb{R}^n)} \\
&\le C\sum_{j=k+1}^{\infty}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}} \\
&\le C\sum_{j=k+1}^{\infty}2^{(k-j)n\delta_2}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
\tag{3.8}
$$

Because of $0 < p_1/p_2 \le 1$, therefore, applying (3.4), and combining (3.5) and (3.8), and using $\alpha > \lambda - n\delta_2$, it follows that

$$\begin{aligned}
&\Big\|(1+|x|)^{-\gamma(x)}\mathscr{H}^*_{\beta(\cdot)}(f)\Big\|^{p_1}_{M\dot{K}^{\alpha,\lambda}_{p_2,q_2(\cdot)}(\mathbb{R}^n)}\\
&\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda p_1}\Bigg(\sum_{k=-\infty}^{k_0} 2^{k\alpha p_1}\Big(\sum_{j=k+1}^{\infty} 2^{(k-j)n\delta_2}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\Big)^{p_1}\Bigg)\\
&\le C\|f\|^{p_1}_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda p_1}\Bigg(\sum_{k=-\infty}^{k_0} 2^{k\alpha p_1}\Big(\sum_{j=k+1}^{\infty} 2^{(k-j)n\delta_2}2^{j(\lambda-\alpha)}\Big)^{p_1}\Bigg)\\
&\le C\|f\|^{p_1}_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda p_1}\Bigg(\sum_{k=-\infty}^{k_0} 2^{k\lambda p_1}\Big(\sum_{j=k+1}^{\infty} 2^{(k-j)(n\delta_2+\alpha-\lambda)}\Big)^{p_1}\Bigg)\\
&\le C\|f\|^{p_1}_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda p_1}\Bigg(\sum_{k=-\infty}^{k_0} 2^{k\lambda p_1}\Bigg) \le C\|f\|^{p_1}_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Consequently, the proof of Theorem 3.2 is completed. □

In particular, when $\gamma(x) = 0$ and $\beta(\cdot)$ is constant exponent, the main results above are proved by Zhang and Wu in [9].

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