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ON SUPERLINEAR $p(x)$-LAPLACIAN-LIKE PROBLEM WITHOUT AMBROSETTI AND RABINOWITZ CONDITION

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Abstract. This paper deals with the superlinear elliptic problem without Ambrosetti and Rabinowitz type growth condition of the form:

$$\begin{cases} -\mathrm{div}\left((1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}})|\nabla u|^{p(x)-2}\nabla u\right)=\lambda f(x,u), \text{ a.e. in } \Omega, \\ u=0, \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathrm{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$ is a parameter. The purpose of this paper is to obtain the existence results of nontrivial solutions for every parameter λ. Firstly, by using the mountain pass theorem a nontrivial solution is constructed for almost every parameter $\lambda > 0$. Then we consider the continuation of the solutions. Our results are a generalization of that of Manuela Rodrigues.

1. Introduction

During the last fifteen years, variational problems and partial differential equations with various types of nonstandard growth conditions have become increasingly popular. This is partly due to their frequent appearance in applications such as the modeling of electrorheological fluids [1, 12] and image processing [2], but these problems are very interesting from a purely mathematical point of view as well.

In this paper, we consider the following nonlinear eigenvalue problems for $p(x)$-Laplacian-like operators originated from a capillary phenomena of the following form:

$$(P)\quad \begin{cases} -\mathrm{div}\left((1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}})|\nabla u|^{p(x)-2}\nabla u\right)=\lambda f(x,u), \text{ a.e. in } \Omega, \\ u=0, \text{ on } \partial\Omega, \end{cases}$$

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where $\Omega \subset \mathrm{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$ is a parameter.

M. Manuela Rodrigues [11] established the existence of nontrivial solution of problem (P), by assuming the following conditions:

(f_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheodory condition and

$$|f(x,t)| \leq c_1 + c_2|t|^{\beta(x)-1}, \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $\beta \in C_+(\overline{\Omega})$ and $1 < \beta(x) < p^*(x)$ for $x \in \overline{\Omega}$, $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$, $p^*(x) = \infty$ if $p(x) \geq N$.

(f_2) $\exists M > 0$, $\theta > p^+$ such that

$$0 < \theta F(x,t) \leq tf(x,t), \quad \forall |t| \geq M,\ x \in \Omega,$$

where $F(x,t) = \int_0^t f(x,s)ds$.

(f_3) $f(x,t) = o(|t|^{p^+-1}), t \to 0$ for $x \in \Omega$ uniformly and $\beta^- > p^+$.

It is well known, condition (f_2) is quite important not only to ensure that the Euler-lagrange functional associated to problem (P) has a mountain pass geometry, but also to guarantee that Palais-Smale sequence of the Euler-Lagrange functional is bounded. But this condition is very restrictive eliminating many nonlinearities. We recall that (f_2) implies a weaker condition

$$F(x,t) \geq c_3|t|^\theta - c_4, \quad c_3, c_4 > 0,\ (x,t) \in \Omega \times \mathbb{R} \text{ and } \theta > p^+.$$

The above condition implies another much weaker condition, which is a consequence of the superlinearity of f at infinity:

(f_4) $\lim\limits_{|t|\to\infty} \frac{F(x,t)}{|t|^{p^+}} = +\infty$, uniformly a.e. $x \in \Omega$.

Because the $p(x)$-Laplacian possesses more complicated nonlinearities than Laplacian and p-Laplacian, for example, it is inhomogeneous, thus our problem is much more difficult.

The main result of this paper is the following theorem.

Theorem 1.1 (Main Theorem)**.** *Under hypotheses* (f_1), (f_3), (f_4) *and*

(f_5) *there exists* $t_0 > 0$, *such that* $\frac{f(x,t)}{t^{2p^+-1}}$ *is increasing in* $t \geq t_0$ *and decreasing in* $t \leq -t_0$, $\forall x \in \Omega$.

Moreover, $f \in C(\overline{\Omega} \times \mathbb{R})$, *then problem* (P) *has a nontrivial weak solution, for all* $\lambda > 0$.

Example. Function $f(x,t) = (\beta(x)\ln(\frac{t}{3}) + 3)t^{\beta(x)-1}$ $(F(x,t) = t^{\beta(x)}\ln(\frac{t}{3}))$ where $\beta \in C_+(\overline{\Omega})$ satisfies condition (f_5), but it does not satisfy (f_2) if $2\beta^- > p^+ > \beta^+$.

Remark 1.2. In fact our result still holds if we consider a weaker condition than (f_5), which is,

$(f_5)'$ there is $C_* > 0$ such that $f(x,t)t - \theta F(x,t) \leq f(x,s)s - \theta F(x,s) + C_*$ for all $0 < t < s$ or $s < t < 0$.

This paper is organized in the following way: in Section 2, we recall some necessary preliminaries, which will be used in our investigation in Section 3; In Section 3, we prove the main result of the paper.

2. Preliminaries

In this part, we introduce some definitions and results which will be used in the next section.

Firstly, we introduce some theories of Lebesgue-Sobolev space with variable exponent. The detailed description can be found in [3, 4, 6, 8, 7, 10].

Write

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : h(x) > 1 \text{ for any } x \in \overline{\Omega}\},$$

$$h^- = \min_{x\in\overline{\Omega}} h(x),\ h^+ = \max_{x\in\overline{\Omega}} p(x) \text{ for any } h \in C_+(\overline{\Omega}).$$

Obviously, $1 < h^- \leq h^+ < +\infty$.

Denote by $\mathcal{U}(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $\mathcal{U}(\Omega)$ are considered to be one element of $\mathcal{U}(\Omega)$, when they are equal almost everywhere.

For $p \in C_+(\overline{\Omega})$, define

$$L^{p(x)}(\Omega) = \{u \in \mathcal{U}(\Omega) : \int_\Omega |u(x)|^{p(x)} dx < +\infty\},$$

with the norm $|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\}$, and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm $\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{p(x)} + |\nabla u|_{p(x)}$.

Denote $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

We remember that the variable exponent Lebesgue spaces are separable and reflexive Banach spaces. Denote by $L^{q(x)}(\Omega)$ the conjugate Lebesgue space of $L^{p(x)}(\Omega)$ with $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$, then the Hölder type inequality

$$\int_\Omega |uv| dx \leq (\tfrac{1}{p^-} + \tfrac{1}{q^-})|u|_{p(x)}|v|_{q(x)},\ \ u \in L^{p(x)}(\Omega),\ v \in L^{q(x)}(\Omega)$$

holds. Furthermore, define mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ by

$$\rho(u) = \int_\Omega |u(x)|^{p(x)} dx,$$

then the following relations hold

$$|\ u\ |_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+},$$

$$|\ u\ |_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}.$$

Proposition 2.1 ([7]). *In $W_0^{1,p(x)}(\Omega)$ the Poincare's inequality holds, that is, there exists a positive constant C_0 such that*

$$|u|_{p(x)} \leq C_0 |\nabla u|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

So $|\nabla u|_{p(x)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$. We will use the equivalent norm in the following discussion and write $\|u\| = |\nabla u|_{p(x)}$ for simplicity.

Proposition 2.2 ([5])**.** *If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous.*

Consider the following function:

$$J(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + \sqrt{1 + |\nabla u|^{2p(x)}} \right) dx, u \in W_0^{1,p(x)}(\Omega).$$

We denote $A= J' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$, then

$$\langle A(u), v \rangle = \int_\Omega \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) (\nabla u, \nabla v)_{\mathbb{R}^N} dx$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$.

Proposition 2.3 ([1])**.** *Set $X = W_0^{1,p(x)}(\Omega)$, A is as above, then*

(1) *$A : X \to X^*$ is a convex, bounded and strictly monotone operator*;

(2) *$A : X \to X^*$ is a mapping of type $(S)_+$, i.e., if $u_n \stackrel{w}{\to} u$ in X and $\limsup_{n\to\infty} \langle A(u_n), u_n - u \rangle \leq 0$, implies $u_n \to u$ in X*;

(3) *$A : X \to X^*$ is a homeomorphism.*

3. Existence theorems

Now we introduce the energy functional $\varphi : W_0^{1,p(x)}(\Omega) \to \mathrm{R}$ associated with problem (P), defined by

$$\varphi_\lambda(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + \sqrt{1 + |\nabla u|^{2p(x)}} \right) dx - \lambda \int_\Omega F(x, u) dx, u \in W_0^{1,p(x)}(\Omega),$$

which is due to [11].

From the hypotheses on f, it is standard to check that $\varphi_\lambda \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R}))$ and its Gateaux derivative is

$$\langle \varphi_\lambda'(u), v \rangle = \int_\Omega \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) (\nabla u, \nabla v)_{\mathbb{R}^N} dx - \lambda \int_\Omega f(x, u) v dx, u, v \in W_0^{1,p(x)}(\Omega).$$

Lemma 3.1. (a) *Under the condition (f_4), the functional φ_λ is unbounded from below.*

(b) *Under the conditions (f_1) and (f_3), $u = 0$ is a strict local minimum for φ_λ.*

Proof. (a) From (f_4), it is follows, for all $M > 0$ there exists $C_M > 0$, such that

$$F(x,t) \geq Mt^{p^+} - C_M, \forall x \in \Omega, t \geq 0. \tag{3.1}$$

Take $\phi \in C_0^\infty(\Omega) \setminus \{0\}$ with $\phi(x) > 0$. For $t > 1$, from (3.1) we have

$$\begin{aligned}
\varphi_\lambda(t\phi) = &\int_\Omega \frac{1}{p(x)} \left(|\nabla t\phi|^{p(x)} + \sqrt{1 + |\nabla t\phi|^{2p(x)}} \right) dx - \lambda \int_\Omega F(x, t\phi) dx \\
&\leq t^{p^+} \int_\Omega \left(|\nabla \phi|^{p(x)} + \sqrt{1 + |\nabla \phi|^{2p(x)}} \right) dx - \lambda M t^{p^+} \int_\Omega |\phi|^{p^+} dx \\
&\quad + \lambda C_M |\Omega| \\
&\leq t^{p^+} \int_\Omega \left(|\nabla \phi|^{p(x)} + \sqrt{1 + |\nabla \phi|^{2p(x)}} \right) dx - \lambda M t^{p^+} \int_\Omega |\phi|^{p^+} dx \\
&\quad + \lambda C_M |\Omega| \\
&= t^{p^+} \left[\int_\Omega \left(|\nabla \phi|^{p(x)} + \sqrt{1 + |\nabla \phi|^{2p(x)}} \right) dx - \lambda M \int_\Omega |\phi|^{p^+} dx \right] \\
&\quad + \lambda C_M |\Omega|,
\end{aligned}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. If M is large, then

$$\lim_{t \to \infty} \varphi_\lambda(t\phi) = -\infty.$$

This proves (a).

(b) From (f_3), for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that

$$F(x,t) \leq \varepsilon |t|^{p^+}, \quad \forall t \in (-\delta, \delta), x \in \Omega.$$

On the other hard, by (f_1) and the mean value theorem, there exists $c_5 > 0$ such that,

$$F(x,t) \leq c_5 |t|^{\beta(x)} \quad \text{for a.e. } x \in \Omega \text{ and } |t| \geq \delta.$$

Therefore, it is follows that

$$F(x,t) \leq \varepsilon |t|^{p^+} + c_5 |t|^{\beta(x)} \quad \text{for a.e. } x \in \Omega \text{ and all } t \in \mathrm{R}.$$

For $u \in W_0^{1,p}(\Omega)$ and $\|u\| < 1$, we have

$$\begin{aligned}
\varphi_\lambda(u) = &\int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + \sqrt{1 + |\nabla u|^{2p(x)}} \right) dx - \lambda \int_\Omega F(x,u) dx \\
&\geq \frac{2}{p^+} \int_\Omega |\nabla u|^{p(x)} dx - \lambda \varepsilon \int_\Omega |u|^{p^+} dx - \lambda c_5 \int_\Omega |u|^{\beta(x)} dx \\
&\geq \frac{2}{p^+} \|u\|^{p^+} - \lambda \varepsilon C_0^{p^+} \|u\|^{p^+} - c_5 C_0^{\beta^-} \|u\|^{\beta^-},
\end{aligned}$$

here we use the continuity(in fact, compactness) embedding $W_0^{1,p(x)}(\Omega)$ into $L^{\beta(x)}(\Omega)$ (recall that $1 < \beta(x) < p^*(x)$) and Poincare's inequality. Then

$$\varphi_\lambda(x) \geq (\frac{2}{p^+} - \lambda \varepsilon C_0^{p^+}) \|u\|^{p^+} - \lambda c_5 C_0^{\beta^-} \|u\|^{\beta^-}. \tag{3.2}$$

For given $\lambda > 0$, we choose $\varepsilon = \varepsilon(\lambda) > 0$, such that $\varepsilon < \frac{1}{\lambda C_0^{p^+} p^+}$. Then from (3.2) and Poincare's inequality we have

$$\varphi_\lambda(x) \geq \frac{1}{p^+}\|u\|^{p^+} - c_6\|u\|^{\beta^-} \tag{3.3}$$

for some $c_6 > 0$ and all $u \in W_0^{1,p(x)}(\Omega)$.

Since $p^+ < \beta^-$, if we choose $\rho = \rho(\lambda) > 0$ small, from (3.3), we see that $\{\varphi_\lambda(u) : \|u\| = \rho\} \geq d(\lambda) > 0$. So far, we complete the proof. □

Fix $0 < \lambda_0 < \mu_0$. Now we can see that geometry on φ_λ works uniformly on $[\lambda_0, \mu_0]$. By choosing $\varepsilon > 0$ such that $\frac{2}{p^+} - \mu_0 \varepsilon C_0^{p^+} \geq \frac{1}{p^+}$, we obtain that

$$\varphi_\lambda(u) \geq \frac{1}{p^+}\|u\|^{p^+} - c_7\|u\|^{\beta^-},\ \forall u \in W_0^{1,p(x)}(\Omega),\ 0 < \lambda \leq \mu_0,\ c_7 > 0.$$

That is, there exist $\rho > 0$ and $r > 0$, such that

$$\varphi_\lambda(u) \geq r,\ \|u\| = \rho,\ \forall \lambda \leq \mu_0. \tag{3.4}$$

By choosing $e \in W_0^{1,p(x)}(\Omega)$, such that $\varphi_{\lambda_0}(e) < 0$, we infer that

$$\frac{\varphi_\lambda(e)}{\lambda} < \frac{\varphi_{\lambda_0}(e)}{\lambda_0} < 0,\ \lambda_0 \leq \lambda \leq \mu_0.$$

Also we have

$$\frac{\varphi_\lambda(u)}{\lambda} \leq \frac{\varphi_\mu(u)}{\mu},\ \forall u \in W_0^{1,p(x)}(\Omega),\ \mu < \lambda. \tag{3.5}$$

Define

$$\mathcal{T} =: \{\gamma : [0,1] \to W_0^{1,p(x)}(\Omega) \mid \gamma \text{ is continuous and } \gamma(0) = 0 \text{ and } \gamma(1) = e\}$$

and for $\lambda_0 \leq \lambda \leq \mu_0$, let $c_\lambda =: \inf_{\gamma \in \mathcal{T}} \max_{t \in [0,1]} \varphi_\lambda(\gamma(t))$.

We recall that the map $c : [\lambda_0, \mu_0] \to \mathrm{R}_+$, given by $c(\lambda) = c_\lambda$, is bounded from below by $c_{\mu_0} > 0$.

In fact, (3.5) implies the monotonicity of $\frac{c_\lambda}{\lambda}$, while the estimate (3.4) implies $c_\lambda \geq r > 0$.

Now, we are in the position to check the left semi-continuity of $\frac{c_\lambda}{\lambda}$. Fix $\mu \in [\lambda_0, \mu_0]$ and $\varepsilon > 0$. Then fix $\gamma \in \mathcal{T}$ such that

$$c(\mu) \leq \max_{t \in [0,1]} \varphi_\mu(\gamma(t)) \leq c(\mu) + \frac{\varepsilon\mu}{8}.$$

Let $R_0 = \max_{t \in [0,1]} |\int_\Omega F(x, \gamma(t))dx|$. Then, for $\lambda > \frac{\mu}{2}$ such that $\frac{1}{\lambda} < \frac{1}{\mu} + \frac{\varepsilon}{2c_\mu}$, we have

$$\begin{aligned}\varphi_\lambda(\gamma(t)) &= (\varphi_\lambda(\gamma(t)) - \varphi_\mu(\gamma(t))) + \varphi_\mu(\gamma(t)) \\ &\leq \varphi_\mu(\gamma(t)) + (\mu - \lambda)\int_\Omega F(x, \gamma(t))dx \\ &\leq R_0|\lambda - \mu| + c_\mu + \frac{\varepsilon\mu}{8}, \forall t \in [0,1].\end{aligned}$$

That is, $c_\lambda \leq c_\mu + \frac{\varepsilon\mu}{4}$, if $|\mu-\lambda| < \frac{\varepsilon\mu}{8R_0}$. Hence, if $\mu > \lambda$, it follows that

$$\frac{c_\mu}{\mu} - \varepsilon < \frac{c_\mu}{\mu} \leq \frac{c_\lambda}{\lambda} \leq \frac{c_\mu}{\lambda} + \frac{\varepsilon}{2} \leq \frac{c_\mu}{\mu} + \varepsilon.$$

Lemma 3.2 ([9])**.** *There exists $K > 0$, such that*

$$\|\varphi'_\mu(u) - \varphi'_\lambda(u)\|_{(W_0^{1,p(x)}(\Omega))^*} \leq K(1 + \|u\|^{\beta^+-1})|\mu-\lambda|, \ \forall\lambda, \ \mu > 0.$$

Proof. For $\beta \in C_+(\overline{\Omega})$, define $\beta'(x)$ such that $\frac{1}{\beta(x)} + \frac{1}{\beta'(x)} = 1, \forall x \in \overline{\Omega}$. By (f_1), we have

$$|f(x,t)|^{\beta'(x)} = |f(x,t)|^{\frac{\beta(x)}{\beta(x)-1}} \leq d_1 + d_2|t|^{\beta(x)}, \ \forall x \in \Omega, \ t \in \mathbb{R}$$

for some constants $d_1, d_2 > 0$, and then

$$\int_\Omega |f(x,t)|^{\beta'(x)} dx \leq d_1|\Omega| + d_2 \int_\Omega |u|^{\beta(x)} dx.$$

Therefore, for any $u \in W_0^{1,p(x)}(\Omega)$, we have

$$\int_\Omega |f(x,t)|^{\beta'(x)} dx \leq d_3 + d_4\|u\|^{\beta^+},$$

where d_3 and d_4 are positive constants.

Thus, for all $v \in W_0^{1,p(x)}(\Omega)$ with $\|v\| \leq 1$, we have

$$|\langle\varphi'_\mu(u), v\rangle - \langle\varphi'_\lambda(u), v\rangle| \leq (\lambda-\mu)\int_\Omega f(x,u)v dx.$$

Thus,

$$\begin{aligned}|\langle\varphi'_\mu(u), v\rangle - \langle\varphi'_\lambda(u), v\rangle| &\leq |\lambda-\mu|\int_\Omega |f(x,u)||v| dx \\ &\leq 2|\lambda-\mu||f(x,u)|_{\beta'(x)}|v|_{\beta(x)} \\ &\leq 2C_0|\lambda-\mu|(d_3 + d_4\|u\|^{\beta^+})^{\frac{\beta^+-1}{\beta^+}}\|v\|.\end{aligned}$$

So there exists constant $K > 0$ such that

$$\|\varphi'_\mu(u) - \varphi'_\lambda(u)\|_{(W_0^{1,p(x)}(\Omega))^*} \leq K(1 + \|u\|^{\beta^+-1})|\mu-\lambda|, \forall\lambda, \mu > 0. \qquad \square$$

Remark 3.3. We recall that the map $b : [\lambda_0, \mu_0] \to \mathrm{R}_+$, given by $b(\lambda) = \frac{c_\lambda}{\lambda}$, is monotone decreasing. Thus, $b(\lambda)$ and $c(\lambda)$ are differentiable at almost all value $\lambda \in (\lambda_0, \mu_0)$.

The proof of the next lemma is done by adapting some arguments employed in the proof of Lemma 3.3 in [13] and Lemma 2.5 in [14].

Lemma 3.4. *Suppose the map $c : [\lambda_0, \mu_0] \to \mathbb{R}_+$, given by $c(\lambda) = c_\lambda$, is differentiable in μ, then there exists a sequence $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ such that*

$$\varphi_\mu(u_n) \to c_\mu, \varphi'_\mu(u_n) \to 0, \ \textit{and} \ \|u_n\|^{p^0} \leq \overline{C} \ \ \textit{as}\ n \to \infty,$$

where $p^0 = p^+$ if $\|u_n\| \le 1, p^0 = p^-$ if $\|u_n\| > 1,$ *and* $\overline{C} = p^+c_\mu + p^+\mu(2 - c'(\mu)) + 1.$

Proof. Assume, by contradiction, that the lemma was false. Then $\|\varphi'_\mu(u_n)\|_* \ge 2\delta$ for all $u \in N^\mu_\delta = \{u \in W_0^{1,p(x)}(\Omega) : \|u\|^{p^0} \le \overline{C},\ |\varphi_\mu(u) - c_\mu| \le \delta\}$.

Let c_8 be such that

(3.6)
$$\begin{aligned}\left|\int_\Omega F(x,u(x))dx\right| &= \left|\varphi_\mu(u) - \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}}\right)dx\right| \\ &\le |\varphi_\mu(u)| + \left|\int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}}\right)dx\right| \\ &\le \frac{1}{\mu}\left[c_\mu + \delta + \frac{1}{p^-}\int_\Omega (2|\nabla u|^{p(x)} + 1)dx\right] \\ &\le \frac{1}{\mu}\left[c_\mu + \delta + \frac{1}{p^-}(|\Omega| + 2\|u\|^{p^0})\right] \\ &\le c_8,\ \forall u \in N^\mu_\delta.\end{aligned}$$

Set $V : N^\mu_\delta \to W_0^{1,p}(\Omega)$ to be a locally Lipschtz pseudo-gradient vector field, $\|V\| \le 1$ and

$$\langle \varphi'_\mu(u), V(u)\rangle \le -\delta, \forall u \in N^\mu_\delta \quad \text{(see [10])}.$$

Now, fix $\{\lambda_n\}$ a sequence in (λ_0, μ_0) such that $\mu < \lambda_{n+1} < \lambda_n$, and λ_n converges to μ, $|\lambda_n - \mu| \le \min\{\frac{\delta}{2c}, \frac{\delta}{2}\}$, and $|c_\mu - c_{\lambda_n}| \le \frac{\delta}{4}$.

For each n, let $\gamma_n \in \mathcal{T}$ be such that

$$\max_{t\in[0,1]} \varphi_\mu(\gamma_n(t)) \le c_\mu + (\lambda_n - \mu). \tag{3.7}$$

Consider the open set

$$A_n = \{t \in [0,1] : \varphi_{\lambda_n}(\gamma_n(t)) > c_{\lambda_n} - (\lambda_n - \mu)\}.$$

By the definition of c_{λ_n}, we know that A_n is nonempty.

If $v \in \gamma_n(A_n)$, then from (3.7), we have,

$$\int_\Omega F(x,v)dx = \frac{\varphi_\mu(v) - \varphi_{\lambda_n}(v)}{\lambda_n - \mu} \le \frac{c_\mu - c_{\lambda_n}}{\lambda_n - \mu} + 2 = -c'(\mu) + 2 + o_n(1),$$

where we have used $c_\mu - c_{\lambda_n} = (c'(\mu) + o_n(1))(\mu - \lambda_n)$.

Since

$$\int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}}\right)dx = \varphi_\mu(v) + \mu\int_\Omega F(x,v)dx$$

and

$$\int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}}\right)dx \ge \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}dx \ge \frac{1}{p^+}\|v\|^{p^0},$$

we obtain for $v \in \gamma_n(A_n)$,

$$\begin{aligned}\|v\|^{p^0} &\leq c_\mu + (\lambda_n - \mu) + \mu(-c'(\mu) + 2 + o_n(1)) \\ &= p^+ c_\mu + p^+ \mu(2 - c'(\mu)) + p^+(\lambda_n - \mu) + \mu o_n(1) \\ &\leq \overline{C}\end{aligned}$$

for n large.

It is easy to see that inequality (3.7) is satisfied for $v \in \gamma_n(A_n)$. Thus $\gamma_n(A_n) \subseteq N^\mu_\delta$, since,

$$c_{\lambda_n} - (\lambda_n - \mu) \leq \varphi_{\lambda_n}(v),\ \varphi_\mu(v) \leq c_\mu + (\lambda_n - \mu).$$

$$|\varphi_{\lambda_n}(v) - \varphi_\mu(v)| = (\lambda_n - \mu)|\int_\Omega F(x,v)dx| \leq c_8|\lambda_n - \mu|, \tag{3.8}$$

for n large

$$c_\mu - \delta < \varphi_\mu(v) < c_\mu + \delta,\ \forall v \in \gamma_n(A_n).$$

So,

$$\langle \varphi'_{\lambda_n}(u), V(u)\rangle \leq -\frac{\delta}{2} \text{ for all } u \in N^\mu_\delta.$$

Now consider a Lipschitz continuous cut-off function η such that $0 \leq \eta \leq 1; \eta(x) = 0,\ x \notin N^\mu_\delta; \eta(x) = 1,\ x \in N^\mu_{\frac{\delta}{2}}$.

Let ϕ be the flow generated by ηV, that is

$$\begin{cases} \dfrac{\partial \phi(u,r)}{\partial r} = \eta(\phi(u,r))V(\phi(u,r)), \quad \text{on } \mathrm{R}_+, \\ \phi(u,0) = u. \end{cases}$$

From the uniqueness result of ODE we have:

If $u \notin N^\mu_\delta$, then $\phi(u,r) = u, \forall r \geq 0$;

If $u \in N^\mu_\delta$, then $\phi(u,r) \in N^\mu_\delta, \forall r \geq 0$.

$\Rightarrow$ (i) If $u \in W^{1,p}_0(\Omega)$, then $\langle \varphi'_{\lambda_n}(\phi(u,r)), \frac{\partial\phi(u,r)}{\partial r}\rangle \leq 0$,

$\Rightarrow$ (ii) If $\phi(x,r) \in N^\mu_{\frac{\delta}{2}}, \forall r \in [0, r_0]$, then

$$\varphi_{\lambda_n}(\phi(u,r)) \leq \varphi_{\lambda_n}(u) - \frac{\delta}{2} r_0.$$

Since $e \notin N^\mu_\delta$, we have $\phi(e,r) = e$ and $\phi(0,r) = 0$, for all $r \geq 0$, and then $\phi(\gamma, r) \in \mathcal{T}$, for all real r and $\gamma \in \mathcal{T}$.

This implies that $h_n(t) = \phi(\gamma_n(t), 1)$ is a continuous path in $\mathcal{T}$ such that $\varphi_{\lambda_n}(h_n(t)) \leq \varphi_{\lambda_n}(\gamma_n(t))$, and then for its maximum point $s_n \in [0,1]$, we should have $s_n \in A_n$, and

$$c_\mu - o_n(1) = c_{\lambda_n} \leq \max_{t\in[0,1]} \varphi_{\lambda_n}(h_n(t)) = \varphi_{\lambda_n}(h_n(s_n)) \leq \varphi_\lambda(\gamma_n(s_n)) - \frac{\delta}{2}.$$

On the other hand, from (3.7) and (3.8), we have

$$\varphi_\lambda(\gamma_n(s_n)) \leq \varphi_\mu(\gamma_n(s_n)) + c_3|\lambda_n - \mu| \leq c_\mu + (1 + c_3)|\lambda_n - \mu|,$$

which is a contradiction. □

The next theorem follows directly from Lemma 3.4.

Theorem 3.5. *For almost all $\lambda > 0$, c_λ is a critical value for φ_λ.*

Proof. From Lemma 3.4, $\{u_n\}_{n\geq 1}$ is bounded and so we may assume that $u_n \rightharpoonup u$(weak convergence) in $W_0^{1,p(x)}(\Omega)$ and $u_n \to u$ in $L^{\beta(x)}(\Omega)$.

As above from the choice of $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p(x)}(\Omega)$, we have $\int_\Omega f(x,u_n)(u_n - u)dx \to 0$, that is

$$\langle A(u_n), u_n - u\rangle \to 0 \Rightarrow u_n \to u \text{ in } W_0^{1,p(x)}(\Omega).$$

So, $\varphi_\mu(u) = c_\mu, \varphi'_\mu(u) = 0$. That is u is a critical point of φ_μ. □

Theorem 3.6. *For almost all $\lambda > 0$, problem (P) has a nontrivial weak solution.*

Proof. As c_λ is left continuous, from Lemma 3.4, for each $\mu > 0$, we can fix sequences $\{u_n\} \subseteq W_0^{1,p(x)}(\Omega)$, and $\{\lambda_n\} \subseteq$ R, such that

$$\lambda_n \to \mu,\ c_{\lambda_n} \to c_\mu \text{ as } n \to \infty.$$

$$\varphi_{\lambda_n}(u_n) = c_{\lambda_n},\ \varphi'_{\lambda_n}(u_n) = 0.$$

We claim that $\{u_n\}$ is bounded. Suppose that this is not true. Then we can assume that $\|u_n\| \to +\infty$, as $n \to +\infty$. Set $w_n = \frac{u_n}{\|u_n\|},\ n \geq 1$.

We may assume that

$$\begin{aligned}
&w_n \to w \text{ weakly in } W_0^{1,p(x)}(\Omega);\\
&w_n \to w \text{ in } L^{p(x)}(\Omega);\\
&w_n \to w \text{ in } L^{\beta(x)}(\Omega);\ (\text{by } 1 < \beta(x) < p^*(x))\\
&w_n(x) \to w(x) \text{ a.e. on } \Omega;
\end{aligned}$$

and $|w_n(x)| \leq h(x)$ a.e. on Ω, for $n \geq 1$ and $h \in L^{p(x)}(\Omega)$.

Let $\Omega_0 = \{x \in \Omega : w(x) \neq 0\}$. If $x \in \Omega_0$, then

$$\lim_{n\to\infty} \frac{F(x,u_n(x))}{|u_n(x)|^{p^+}} \frac{|u_n(x)|^{p^+}}{\|u_n\|^{p^+}} = \lim_{n\to\infty} \frac{F(x,u_n(x))}{|u_n(x)|^{p^+}} |w_n(x)|^{p^+} = +\infty \text{ (by } (f_4)).$$

Applying the Fatou's lemma, we have

$$\lim_{n\to\infty} \int_\Omega \frac{F(x,u_n(x))}{|u_n(x)|^{p^+}} |w_n(x)|^{p^+} \leq \frac{1}{\mu p^-}.$$

We conclude that Ω_0 has zero measure and $w = 0$ a.e. in Ω.

Let $\varphi_{\lambda_n}(t_n u_n) = \max\limits_{t\in[0,1]} \varphi_{\lambda_n}(t u_n)$, we have

$$\varphi_{\lambda_n}(t x_n) \leq \varphi_{\lambda_n}(t_n x_n)$$

and $\varphi'_{\lambda_n}(t_n u_n) = 0$, hence $\langle \varphi'_{\lambda_n}(t_n u_n), t_n u_n \rangle = 0$, that is

$$\int_\Omega \left[|t_n \nabla u_n|^{p(x)} + \frac{|t_n \nabla u_n|^{2p(x)}}{\sqrt{1 + |t_n \nabla u_n|^{2p(x)}}} \right] dx = \lambda_n \int_\Omega t_n u_n f(x, t_n u_n) dx.$$

Next, we show that

$$\begin{aligned} &\frac{1}{p(x)} \sqrt{1 + |t_n \nabla u_n|^{2p(x)}} - \frac{1}{2p^+} \frac{|t_n \nabla u_n|^{2p(x)}}{\sqrt{1 + |t_n \nabla u_n|^{2p(x)}}} \\ &\leq \frac{1}{p(x)} \sqrt{1 + |\nabla u_n|^{2p(x)}} - \frac{1}{2p^+} \frac{|\nabla u_n|^{2p(x)}}{\sqrt{1 + |\nabla u_n|^{2p(x)}}}. \end{aligned} \tag{3.9}$$

In order to prove this, we define the following functional $f : [0, 1] \to \infty$:

$$f(t) = p_1 \sqrt{1 + at^p} - p_2 \frac{at^p}{\sqrt{1 + at^p}},$$

where p, p_1, p_2, a are positive constants with $p_1 \geq 2p_2$ and $p > 1$. Obviously, $f'(t) \geq 0, \forall t \in [0, 1]$. Thus we deduce that $f(t_n) \leq f(1)$, that is

$$p_1 \sqrt{1 + at_n^p} - p_2 \frac{at_n^p}{\sqrt{1 + at_n^p}} \leq p_1 \sqrt{1 + a} - p_2 \frac{a}{\sqrt{1 + a}}. \tag{3.10}$$

Finally, we notice that by taking in (3.10) $p_1 = \frac{1}{p(x)}$, $p_2 = \frac{1}{2p^+}$ and $a = |\nabla u_n|^{2p(x)}$, we deduce that (3.9) holds true.

Therefore, from (f_5) and (3.9), we have

$$\begin{aligned} \varphi_{\lambda_n}(tu_n) &\leq \varphi_{\lambda_n}(t_n u_n) - \frac{1}{2p^+} \langle \varphi'_{\lambda_n}(t_n u_n), t_n u_n \rangle \\ &= \int_\Omega \left[\frac{1}{p(x)} - \frac{1}{2p^+} \right] |t_n \nabla u_n|^{p(x)} dx + \int_\Omega \frac{1}{p(x)} \sqrt{1 + |t_n \nabla u_n|^{2p(x)}} dx \\ &\quad + \lambda_n \int_\Omega \left[\frac{1}{2p^+} t_n u_n f(x, t_n u_n) - F(x, t_n u_n) \right] dx \\ &\quad - \frac{1}{2p^+} \int_\Omega \frac{|t_n \nabla u_n|^{2p(x)}}{\sqrt{1 + |t_n \nabla u_n|^{2p(x)}}} dx \\ &\leq \int_\Omega \left[\frac{1}{p(x)} - \frac{1}{2p^+} \right] |\nabla u_n|^{p(x)} dx + \int_\Omega \frac{1}{p(x)} \sqrt{1 + |\nabla u_n|^{2p(x)}} dx \\ &\quad + \lambda_n \int_\Omega \left[\frac{1}{2p^+} t_n u_n f(x, t_n u_n) - F(x, t_n u_n) \right] dx \\ &\quad - \frac{1}{2p^+} \int_\Omega \frac{|\nabla u_n|^{2p(x)}}{\sqrt{1 + |\nabla u_n|^{2p(x)}}} dx \\ &\leq \int_\Omega \left[\frac{1}{p(x)} - \frac{1}{2p^+} \right] |\nabla u_n|^{p(x)} dx + \int_\Omega \frac{1}{p(x)} \sqrt{1 + |\nabla u_n|^{2p(x)}} dx \\ &\quad + \lambda_n \int_\Omega \left[\frac{1}{2p^+} u_n f(x, u_n) - F(x, u_n) + \frac{C_*}{2p^+} \right] dx \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2p^+}\int_\Omega \frac{|\nabla u_n|^{2p(x)}}{\sqrt{1+|\nabla u_n|^{2p(x)}}}dx \\
&= c_{\lambda_n} + \frac{\lambda_n C_*}{2p^+}|\Omega|
\end{aligned}
$$

for all $t \in [0,1]$.

On the other hard, for all $R > 1$, set $R_1 = (\theta R)^{\frac{1}{p^-}}$,

$$
\begin{aligned}
\varphi_{\lambda_n}(R_1 w_n) = & \int_\Omega \frac{1}{p(x)}\left(|R_1 \nabla w_n|^{p(x)} + \sqrt{1+|R_1\nabla w_n|^{2p(x)}}\right)dx \\
& - \lambda_n \int_\Omega F(x, R_1 w_n)dx \\
& \geq \frac{2}{\theta} R_1^{p^-}\|w_n\|^{p^0} - \lambda_n \int_\Omega F(x, R_1 w_n)dx \\
& = 2R - \lambda_n \int_\Omega F(x, R_1 w_n)dx \\
& \geq R,
\end{aligned}
$$

which contradicts $\varphi_{\lambda_n}(R_1 w_n) \leq c_{\lambda_n} + \frac{\lambda_n C_*}{\theta}|\Omega|$, for n large.

Now we have a bounded sequence $\{x_n\}$ such that

$$
\varphi_\mu(x_n) \to c_\mu \quad \text{and} \quad \varphi'_\mu(x_n) = 0, \quad n \to \infty.
$$

The proof is done. □

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