

COHEN-MACAULAY MODULES OVER NOETHERIAN LOCAL RINGS

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ABSTRACT. Let (R, \mathfrak{m}) be a commutative Noetherian local ring. In this paper we show that a finitely generated R -module M of dimension d is Cohen-Macaulay if and only if there exists a proper ideal I of R such that $\text{depth}(M/I^n M) = d$ for $n \gg 0$. Also we show that, if $\dim(R) = d$ and $I_1 \subset \cdots \subset I_n$ is a chain of ideals of R such that R/I_k is maximal Cohen-Macaulay for all k , then $n \leq \ell_R(R/(a_1, \dots, a_d)R)$ for every system of parameters a_1, \dots, a_d of R . Also, in the case where $\dim(R) = 2$, we prove that the ideal transform $D_{\mathfrak{m}}(R/\mathfrak{p})$ is minimax balanced big Cohen-Macaulay, for every $\mathfrak{p} \in \text{Assh}_R(R)$, and we give some equivalent conditions for this ideal transform being maximal Cohen-Macaulay.

1. Introduction

Let R denote a commutative Noetherian ring (with identity), I an ideal of R , and M a finitely generated R -module. In [11] L. J. Ratliff, Jr., conjectured about the asymptotic behavior of $\text{Ass}_R(M/J^n M)$ when R is a Noetherian domain. Subsequently, M. Brodmann in [1] showed that if R is a Noetherian ring, then the sets $\text{Ass}_R(M/J^n M)$ and $\text{Ass}_R(J^n M/J^{n+1} M)$ are ultimately constant for large n . Also, based on this result, in [2] he showed that if R is a Noetherian ring, then the integers $\text{depth}(I, M/J^n M)$ and $\text{depth}(I, J^n M/J^{n+1} M)$ take constant values for large n . In particular, the integers $\text{depth}_R(M/J^n M)$ and $\text{depth}_R(J^n M/J^{n+1} M)$ take constant values for large n , whenever (R, \mathfrak{m}) is a Noetherian local ring. In Section 2 of this paper we show that if there exists an ideal J of R in which the constant value of $\text{depth}_R(M/J^n M)$ is equal to $\dim(M)$, then M is Cohen-Macaulay. More precisely, we shall prove the following characterization of Cohen-Macaulay modules over Noetherian local rings:

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Theorem 1.1. *Let (R, \mathfrak{m}) be a Noetherian local ring and M be a non-zero finitely generated R -module of dimension d . Then the following conditions are equivalent:*

- (i) *M is Cohen-Macaulay.*
- (ii) *There exists a proper ideal I of R such that $\text{depth}_R(M/I^n M) = d$ for all $n \gg 0$.*

Recall that a finitely generated module M over a Noetherian local ring (R, \mathfrak{m}) is said to be *maximal Cohen-Macaulay* if $\text{depth}_R(M) = \dim(M) = \dim(R)$. Note that if M is a maximal Cohen-Macaulay, then any system of parameters of R is also a system of parameters of M and therefore it is an M -regular sequence. In Section 2 we shall prove the following results about Cohen-Macaulay modules:

Theorem 1.2. *Let (R, \mathfrak{m}) be a Noetherian local ring of dimension $d \geq 1$ and*

$$I_1 \subset I_2 \subset \cdots \subset I_{n-1} \subset I_n$$

be a chain of ideals of R such that for each $1 \leq k \leq n$, the R -module R/I_k is maximal Cohen-Macaulay. Then for each system of parameters x_1, \dots, x_d of R , we have

$$n \leq \ell_R(R/(x_1, \dots, x_d)),$$

where $\ell_R(R/(x_1, \dots, x_d))$ denotes the length of the R -module $R/(x_1, \dots, x_d)$.

Theorem 1.3. *Let (R, \mathfrak{m}) be a Noetherian local ring of dimension $d \geq 1$ and I be an ideal of R such that $\dim(R/I) = n \geq 1$. Let x_1, \dots, x_n be a system of parameters for R/I . If R/I is a Cohen-Macaulay module and the ideal $Q := I + Rx_1 + \cdots + Rx_n$ is a parameter ideal of R , then there are elements $a_1, \dots, a_{d-n} \in I$ such that $I = Ra_1 + \cdots + Ra_{d-n}$ and $x_1, \dots, x_n, a_1, \dots, a_{d-n}$ is a system of parameters for R . In particular, if R is Cohen-Macaulay, then I is generated by an R -sequence.*

For proving these results we shall use the basic properties of regular and quasi regular sequences. Recall that if R is a commutative ring with identity, M is an R -module, X_1, \dots, X_n , n indeterminants and $Z_R(N)$ denotes to the set of all zero-divisors of N in R , for each non-zero R -module N , then an ordered sequence $a_1, \dots, a_n \in R$ is said to be an M -regular sequence if for all $1 \leq i \leq n$, $a_i \notin Z_R(M/(a_1, \dots, a_{i-1})M)$ and $(a_1, \dots, a_n)M \neq M$. Also a sequence (not necessary ordered) $a_1, \dots, a_n \in R$ is said to be an M -quasi-regular sequence if $(a_1, \dots, a_n)M \neq M$ and for each integer k and each homogeneous polynomial $P(X_1, \dots, X_n) \in M[X_1, \dots, X_n]$ of degree k , $P(a_1, \dots, a_n) \in (a_1, \dots, a_n)^{k+1}M$ implies that all the coefficients of P are in $(a_1, \dots, a_n)M$.

In the sequel let (R, \mathfrak{m}) be a Noetherian local ring of dimension $d \geq 1$. Let a_1, \dots, a_d be a system of parameters (s.o.p.) for R . An R -module M is said to be *big Cohen-Macaulay* R -module with respect to a_1, \dots, a_d if a_1, \dots, a_d is an M -regular sequence. Note that the definition does not require that M be

finitely generated. One of the major open problems in commutative algebra at the present time is to establish the existence of big Cohen-Macaulay modules with respect to a given s.o.p. in an arbitrary local Noetherian ring. In [7] M. Hochster has shown that if the existence of such modules could be established, then several conjectures in commutative algebra would be settled. Moreover, Hochster has established the existence of such big Cohen-Macaulay modules whenever the local ring R contains a field as a subring, or has dimension not exceeding 2 (see [7, Ch. 4, 5] and [6]). An R -module M for which every s.o.p. for R is an M -regular sequence is called a *balanced big Cohen-Macaulay R -module*. Hochster has shown in [8, §5] that such modules exist whenever R contains a field as a subring. Note that every finitely generated balanced big Cohen-Macaulay R -module is maximal Cohen-Macaulay. The existence of such finitely generated modules over complete Noetherian local rings is an open problem. In Section 3 we present some equivalent conditions for the existence of maximal Cohen-Macaulay modules over local Noetherian rings of dimension 2. Also we show that over an arbitrary local Noetherian ring of dimension 2 there exists a minimax balanced big Cohen-Macaulay module. Recall that a module is called *minimax module*, when it has a finitely generated submodule such that the quotient by it is an Artinian module [12].

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity and I will be an ideal of R . Note that for an R -module M , the i^{th} local cohomology module of M with support in $V(I)$ is defined as:

$$H_I^i(M) = \varinjlim_{n \geq 1} \text{Ext}_R^i(R/I^n, M).$$

We refer the reader to [3] for more details about local cohomology. Recall that, for each R -module M , all integers $j \geq 0$ and all prime ideals \mathfrak{p} of R , the j^{th} Bass number of M with respect to \mathfrak{p} is defined as $\mu_R^j(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \text{Ext}_{R_{\mathfrak{p}}}^j(k(\mathfrak{p}), M_{\mathfrak{p}})$, where $k(\mathfrak{p}) := R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Also, For an Artinian R -module A we denote by $\text{Att}_R(A)$ the set of attached prime ideals of A . For any ideal \mathfrak{a} of R , we denote $\{\mathfrak{p} \in \text{Spec } R : \mathfrak{p} \supseteq \mathfrak{a}\}$ by $V(\mathfrak{a})$. We denote the support of each R -module M by $\text{Supp}(M)$. Also, for each R -module M we denote by $\text{Ass}_R(M)$ (resp. $\text{Assh}_R(M)$) the set of associated prime ideals of M (resp. the set $\{\mathfrak{p} \in \text{Ass}_R(M) : \dim(R/\mathfrak{p}) = \dim(M)\}$). Moreover, for each R -module M , we denote by $\text{Ann}_R(M)$ the annihilator of M in R . Finally, for each R -module M , we denote by $E_R(M)$ the injective envelope (or injective hull) of M . For any other unexplained notation and terminology we refer the reader to [3] and [9].

2. A characterization of Cohen-Macaulay modules

In this section we shall prove some results concerning the Cohen-Macaulay modules. The main goals of this section are Theorems 2.3, 2.4 and 2.7. The following lemma, which is a generalization of [4, Proposition 1.4], is needed in the proof of Lemma 2.2.

Lemma 2.1. *Let R be a commutative ring, I an ideal of R , M a non-zero R -module, $a_1, \dots, a_n \in R$ and $J = (a_1, \dots, a_n)$. If $I \subseteq J$ and a_1, \dots, a_n is an M/IM -quasi-regular sequence, then a_1, \dots, a_n is an M -quasi-regular sequence.*

Proof. It follows easily from the definition that $JM \neq M$. Let $P(X_1, \dots, X_n) \in M[X_1, \dots, X_n]$ be homogeneous of degree k such that $P(a_1, \dots, a_n) \in J^{k+1}M$. Then $\overline{P}(X_1, \dots, X_n) \in M/IM[X_1, \dots, X_n]$ is homogeneous of degree k and $\overline{P}(a_1, \dots, a_n) \in J^{k+1}(M/IM)$, where $\overline{P} = \varphi(P)$, with φ is the canonical epimorphism. So, since a_1, \dots, a_n is a M/IM -quasi-regular sequence and $I \subseteq J$, it follows that the coefficients of \overline{P} are in $J(M/IM)$, and hence the coefficients of P are in JM , as required. \square

The following lemma is crucial in the proof of Theorem 2.3.

Lemma 2.2. *Let (R, \mathfrak{m}) be a Noetherian local ring and $M \neq 0$ be a finitely generated R -module of dimension $d \geq 1$. Let $x_1, \dots, x_d \in \mathfrak{m}$ be a system of parameters for M and I be an ideal of R such that $I \subseteq (x_1, \dots, x_d)$. If $\dim(M/IM) = d$ and M/IM is a Cohen-Macaulay R -module, then $I \subseteq \text{Ann}_R(M)$.*

Proof. Since $\dim(M/IM) = d = \dim(M)$ and x_1, \dots, x_d is a system of parameters for M , it is easy to see that x_1, \dots, x_d is a system of parameters for M/IM . Therefore, as M/IM is Cohen-Macaulay it follows that x_1, \dots, x_d is an M/IM -regular sequence. Hence in view of [9, Theorem 16.2], it follows that x_1, \dots, x_d is an M/IM -quasi-regular sequence. Now since $M/IM \cong (M/\mathfrak{m}IM)/(IM/\mathfrak{m}IM)$, it follows from Lemma 2.1 that x_1, \dots, x_d is an $M/\mathfrak{m}IM$ -quasi regular sequence. Whence, according to [9, Theorem 16.3], it follows that x_1, \dots, x_d is an $M/\mathfrak{m}IM$ -regular sequence. Since $d \geq 1$, it follows from the definition that $\mathfrak{m} \notin \text{Ass}_R(M/\mathfrak{m}IM)$. Therefore, we can deduce that $(0 :_{M/\mathfrak{m}IM} \mathfrak{m}) = 0$. But it is easy to see that $IM/\mathfrak{m}IM \subseteq (0 :_{M/\mathfrak{m}IM} \mathfrak{m})$. Thus, $IM/\mathfrak{m}IM = 0$ and so by NAK Lemma we have $IM = 0$. Which means $I \subseteq \text{Ann}_R(M)$, as required. \square

The following theorem is the first main result of this section.

Theorem 2.3. *Let (R, \mathfrak{m}) be a Noetherian local ring of dimension $d \geq 1$ and*

$$I_1 \subset I_2 \subset \dots \subset I_{n-1} \subset I_n$$

be a chain of ideals of R such that for each $1 \leq k \leq n$, the R -module R/I_k is maximal Cohen-Macaulay. Then for each system of parameters x_1, \dots, x_d of R , we have

$$n \leq \ell_R(R/(x_1, \dots, x_d)),$$

where $\ell_R(R/(x_1, \dots, x_d))$ denotes the length of the R -module $R/(x_1, \dots, x_d)$.

Proof. Let $2 \leq k \leq n$. If $I_k \subseteq (x_1, \dots, x_d) + I_{k-1}$, then as $x_1 + I_{k-1}, \dots, x_d + I_{k-1}$ is a system of parameters for local ring R/I_{k-1} and by hypothesis the ring $R/I_k \cong (R/I_{k-1})/(I_k/I_{k-1})$ is Cohen-Macaulay, it follows from Lemma

2.2 that $I_k \subseteq \text{Ann}_R(R/I_{k-1}) = I_{k-1}$, which is a contradiction. Therefore, $I_k \not\subseteq (x_1, \dots, x_d) + I_{k-1}$ and hence

$$(x_1, \dots, x_d) + I_k \not\subseteq (x_1, \dots, x_d) + I_{k-1}.$$

Consequently, $\ell_R(((x_1, \dots, x_d) + I_k)/((x_1, \dots, x_d) + I_{k-1})) \geq 1$. So, we have

$$\begin{aligned} \ell_R(R/(x_1, \dots, x_d)) &\geq \ell_R(R/(I_1 + (x_1, \dots, x_d))) \\ &\geq 1 + \ell_R((I_n + (x_1, \dots, x_d))/(I_1 + (x_1, \dots, x_d))) \\ &= 1 + \sum_{k=2}^n \ell_R(((x_1, \dots, x_d) + I_k)/((x_1, \dots, x_d) + I_{k-1})) \\ &\geq 1 + n - 1 = n. \end{aligned}$$

This completes the proof. □

The following theorem is another application of Lemma 2.2.

Theorem 2.4. *Let (R, \mathfrak{m}, k) be a Noetherian local ring of dimension $d \geq 1$ and I be an ideal of R such that $\dim(R/I) = n \geq 1$. Let x_1, \dots, x_n be a system of parameters for R/I . If R/I is a Cohen-Macaulay module and the ideal $Q := I + Rx_1 + \dots + Rx_n$ is a parameter ideal of R , i.e., Q is generated by a system of parameters for R , then there are elements $a_1, \dots, a_{d-n} \in I$ such that $I = Ra_1 + \dots + Ra_{d-n}$ and $x_1, \dots, x_n, a_1, \dots, a_{d-n}$ is a system of parameters for R . In particular, if R is Cohen-Macaulay, then I is generated by an R -sequence.*

Proof. By definition we have

$$\dim_k(Q/\mathfrak{m}Q) = d.$$

Therefore, as $\dim_k(Q/I)/\mathfrak{m}(Q/I) = \dim_k(Q/\mathfrak{m}Q + I) = n$ it follows from the exact sequence

$$0 \rightarrow (I + \mathfrak{m}Q)/\mathfrak{m}Q \rightarrow Q/\mathfrak{m}Q \rightarrow Q/(I + \mathfrak{m}Q) \rightarrow 0$$

that $\dim_k((I + \mathfrak{m}Q)/\mathfrak{m}Q) = d - n$. Therefore, there are elements a_1, \dots, a_{d-n} in I such that

$$(I + \mathfrak{m}Q)/\mathfrak{m}Q = (Ra_1 + \dots + Ra_{d-n} + \mathfrak{m}Q)/\mathfrak{m}Q.$$

Now we have

$$Q/(I + \mathfrak{m}Q) = (x_1 + (I + \mathfrak{m}Q), \dots, x_n + (I + \mathfrak{m}Q)),$$

and

$$I + \mathfrak{m}Q = Ra_1 + \dots + Ra_{d-n} + \mathfrak{m}Q,$$

which implies $Q = Ra_1 + \dots + Ra_{d-n} + Rx_1 + \dots + Rx_n + \mathfrak{m}Q$. So by NAK lemma it follows that $Q = Ra_1 + \dots + Ra_{d-n} + Rx_1 + \dots + Rx_n$. Whence, we can deduce that $a_1, \dots, a_{d-n} \in I$ is a part of a system of parameters for R so $\dim(R/(Ra_1 + \dots + Ra_{d-n})) = d - (d - n) = n = \dim(R/I)$. Now it follows from Lemma 2.2 that $I \subseteq (Ra_1 + \dots + Ra_{d-n})$, and therefore $I = Ra_1 + \dots + Ra_{d-n}$, as required. □

The following lemma and its corollary play key role in the proof of second main result of this section.

Lemma 2.5. *Let (R, \mathfrak{m}) be a Noetherian local ring and $M \neq 0$ be a finitely generated R -module of dimension $d \geq 1$. Let I be an ideal of R such that $I \not\subseteq \text{Rad}(\text{Ann}_R(M))$ and $\dim(M/IM) = d$. Then there exists a positive integer t such that the following statements hold:*

- (i) $\dim(I^n M/I^{n+1}M) < d$ for all integers $n \geq t$.
- (ii) $\text{Ass}_R(M/I^n M) \neq \text{Assh}_R(M/I^n M)$ for all integers $n > t$.
- (iii) The R -module $M/I^n M$ is not Cohen-Macaulay for all integers $n > t$.

Proof. (i) Let $\text{Assh}_R(M) = \{\mathfrak{q}_1, \dots, \mathfrak{q}_k\}$. Then for each $1 \leq j \leq k$ the $R_{\mathfrak{q}_j}$ -module $M_{\mathfrak{q}_j}$ has finite length. Let $t := \max\{\ell_{R_{\mathfrak{q}_1}}(M_{\mathfrak{q}_1}), \dots, \ell_{R_{\mathfrak{q}_k}}(M_{\mathfrak{q}_k})\}$. Then we claim that $\text{Supp}_R(I^n M/I^{n+1}M) \cap \text{Assh}_R(M) = \emptyset$ for all integers $n \geq t$. Suppose that the contrary be true. Then there is $1 \leq j \leq k$ such that $\mathfrak{q}_j \in \text{Supp}_R(I^n M/I^{n+1}M)$. So by definition we have $I^n M_{\mathfrak{q}_j} \neq I^{n+1}M_{\mathfrak{q}_j}$. Therefore,

$$\begin{aligned} t &\geq \ell_{R_{\mathfrak{q}_j}}(M_{\mathfrak{q}_j}) \geq \ell_{R_{\mathfrak{q}_j}}(M_{\mathfrak{q}_j}/I^{n+1}M_{\mathfrak{q}_j}) \\ &= \ell_{R_{\mathfrak{q}_j}}(M_{\mathfrak{q}_j}/IM_{\mathfrak{q}_j}) + \sum_{s=1}^n \ell_{R_{\mathfrak{q}_j}}(I^s M_{\mathfrak{q}_j}/I^{s+1}M_{\mathfrak{q}_j}) \geq n + 1, \end{aligned}$$

and so $t \geq n + 1$, which is a contradiction. Therefore,

$$\text{Supp}_R(I^n M/I^{n+1}M) \cap \text{Assh}_R(M) = \emptyset$$

for all integers $n \geq t$. Now it is easy to see that $\dim(I^n M/I^{n+1}M) < d$ for all integers $n \geq t$.

(ii) Since $\text{Ass}_R(I^{n-1}M/I^n M) \subseteq \text{Ass}_R(M/I^n M)$ and $\dim(M/I^n M) = \dim(M/IM) = d$, the assertion follows from (i).

(iii) Follows from (ii) using [9, Theorem 17.2]. \square

Corollary 2.6. *Let (R, \mathfrak{m}) be a Noetherian local ring and $M \neq 0$ be a finitely generated R -module of dimension d . Then for any proper ideal I of R the following statements are equivalent:*

- (i) M is Cohen-Macaulay and $I \subseteq \text{Rad}(\text{Ann}_R(M))$.
- (ii) $\text{depth}_R(M/I^n M) = d$ for all $n \gg 0$.

Proof. The assertion follows from Lemma 2.5. \square

Now we are ready to give a characterization of Cohen-Macaulay modules over Noetherian local rings, which is the second main result of this section.

Theorem 2.7. *Let (R, \mathfrak{m}) be a Noetherian local ring and M be a non-zero finitely generated R -module of dimension d . Then the following conditions are equivalent:*

- (i) M is Cohen-Macaulay.
- (ii) There exists a proper ideal I of R such that $\text{depth}_R(M/I^n M) = d$ for all $n \gg 0$.

Proof. The assertion follows from Corollary 2.6. \square

3. Cohen-Macaulay modules over Noetherian local rings of small dimension

In this section we study the existence of Cohen-Macaulay and balanced big Cohen-Macaulay modules over Noetherian local ring of dimension 2. We shall use the following lemma in the proof of some results of this section.

Proposition 3.1. *Let (R, \mathfrak{m}) be a Noetherian local domain of dimension d and M be a finitely generated R -module such that $\text{Ass}_R(M) = \{0\}$ and $\mu_R^0(0, M) = n \geq 1$. Then there exists an exact sequence*

$$0 \rightarrow M \rightarrow \bigoplus_{i=1}^n R.$$

Proof. First let $d = 0$. Then by hypothesis R is a field and so M is a vector space over R of dimension of n and hence $M \cong \bigoplus_{i=1}^n R$. Now let $d \geq 1$. By definition it is easy to see that there exists an exact sequence

$$0 \rightarrow E_R(M) \rightarrow \bigoplus_{i=1}^n E_R(R).$$

Hence by assumption $F := \bigoplus_{i=1}^n R$ we can assume that, M is a finitely generated submodule of $E_R(F)$.

Let

$$0 \rightarrow F \rightarrow E \rightarrow E_1 \rightarrow \dots$$

be a minimal injective resolution for F . Then in view of the definition we have $E_R(E/F) = E_1$ and so $\text{Ass}_R(E/F) = \text{Ass}_R(E_1)$. But $\mathfrak{p} \in \text{Ass}_R(E_1)$ if and only if $E_R(R/\mathfrak{p})$ is a direct summand of E_1 if and only if $\mu_R^1(\mathfrak{p}, F) \neq 0$ if and only if $\mu_R^1(\mathfrak{p}, R) \neq 0$ (Because we have $F = \bigoplus_{i=1}^n R$ and so $\mu_R^1(\mathfrak{p}, F) = n\mu_R^1(\mathfrak{p}, R)$). Since $\mathfrak{q} = 0$ is a prime ideal of R it follows that the zero dimensional local ring $R_{\mathfrak{q}}$ is a field and hence is a Gorenstein ring. Therefore $\mu_R^1(\mathfrak{q}, F) = n\mu_R^1(\mathfrak{q}, R) = 0$. Hence $\text{height}(\mathfrak{p}) \geq 1$ for every $\mathfrak{p} \in \text{Ass}_R(E/F) = \text{Ass}_R(E_1)$.

Now if $M \subseteq F$, then there is nothing to prove. But, if $M \not\subseteq F$, then $M + F/F$ is a non-zero finitely generated submodule of E_1 . Hence, for every $\mathfrak{p} \in \text{Ass}_R(M + F/F)$, we have $\text{height}(\mathfrak{p}) \geq 1$, which implies $\text{height}(L) \geq 1$, where $L := \text{Ann}_R(M + F/F)$. Therefore, $L \neq 0$ and so there is an element $x \in L$ such that $x \neq 0$. But in this situation it is easy to see that x is an M -regular element. (Note that by hypothesis we have $\text{Ass}_R(M) = \{0\}$.) Therefore, $M \cong xM \subseteq F$. This completes the proof. \square

The following theorem, which is the first main result of this section, gives some equivalent conditions for the existence of maximal Cohen-Macaulay modules over local Noetherian rings of dimension 2.

Theorem 3.2. *Let (R, \mathfrak{m}) be a local Noetherian ring of dimension 2. Then the following conditions are equivalent:*

- (i) *There exists a finitely generated R -module M of dimension 2 such that the R -module $H_{\mathfrak{m}}^1(M)$ is finitely generated.*
- (ii) *There is $\mathfrak{p} \in \text{Assh}_R(R)$ such that the R -module $H_{\mathfrak{m}}^1(R/\mathfrak{p})$ is finitely generated.*

(iii) There is $\mathfrak{p} \in \text{Assh}_R(R)$ such that the R -module $D_m(R/\mathfrak{p})$ is a maximal Cohen-Macaulay R -module.

(iv) There exists $\mathfrak{p} \in \text{Assh}_R(R)$ such that for some ideals $I \supset \mathfrak{p}$ of R the R -module $M = I/\mathfrak{p}$ is maximal Cohen-Macaulay and $\text{ara}(I/\mathfrak{p}) = 1$.

(v) There exists a maximal Cohen-Macaulay R -module.

Proof. (i) \Rightarrow (ii) Let $\mathfrak{p} \in \text{Assh}_R(M)$. Then it is easy to see that $\mathfrak{p} \in \text{Assh}_R(R)$. As, $\mathfrak{p} \in \text{Ass}_R(M)$, it follows from definition that there is an exact sequence

$$0 \rightarrow R/\mathfrak{p} \rightarrow M \rightarrow T \rightarrow 0$$

for some finitely generated R -module T . This exact sequence induces the following exact sequence

$$H_m^0(T) \rightarrow H_m^1(R/\mathfrak{p}) \rightarrow H_m^1(M).$$

Since by hypothesis the R -module $H_m^1(M)$ is finitely generated, and the R -module $H_m^0(T)$ is finitely generated it follows that the R -module $H_m^1(R/\mathfrak{p})$ is finitely generated.

(ii) \Rightarrow (i) Is clear.

(ii) \Rightarrow (iii) In view of [3, Theorem 2.2.4], there is an exact sequence

$$(\dagger) \quad 0 \rightarrow R/\mathfrak{p} \rightarrow D_m(R/\mathfrak{p}) \rightarrow H_m^1(R/\mathfrak{p}) \rightarrow 0,$$

which implies that the R -module $D_m(R/\mathfrak{p})$ is finitely generated. As $\text{Ass}_R(M) = \{\mathfrak{p}\}$ we can find an element $x \in (\mathfrak{m} \setminus \mathfrak{p})$. It is clear that x is an R/\mathfrak{p} -regular element and so the sequence $0 \rightarrow R/\mathfrak{p} \xrightarrow{x} R/\mathfrak{p}$, is exact. Therefore, for each $k \geq 1$, there is an exact sequence

$$0 \rightarrow \text{Hom}_R(\mathfrak{m}^k, R/\mathfrak{p}) \xrightarrow{x} \text{Hom}_R(\mathfrak{m}^k, R/\mathfrak{p}).$$

By definition we have

$$D_m(M) = \varinjlim_{k \geq 1} \text{Hom}_R(\mathfrak{m}^k, R/\mathfrak{p}),$$

hence it follows that x is a $D_m(R/\mathfrak{p})$ -regular element. In particular,

$$\Gamma_m(D_m(R/\mathfrak{p})) = 0.$$

Now the exact sequence (\dagger) induces an exact sequence

$$(\dagger\dagger) \quad 0 \rightarrow H_m^1(R/\mathfrak{p}) \xrightarrow{f} H_m^1(R/\mathfrak{p}) \rightarrow H_m^1(D_m(R/\mathfrak{p})) \rightarrow H_m^1(H_m^1(R/\mathfrak{p})).$$

But, from the Grothendieck's Vanishing Theorem, [3, Theorem 6.1.2], we deduce that $H_m^1(H_m^1(R/\mathfrak{p})) = 0$. Consequently, the exact sequence $(\dagger\dagger)$ gives the following exact sequence:

$$(\ddagger) \quad 0 \rightarrow H_m^1(R/\mathfrak{p}) \xrightarrow{f} H_m^1(R/\mathfrak{p}) \rightarrow H_m^1(D_m(R/\mathfrak{p})) \rightarrow 0.$$

Since by hypothesis the R -module $H_m^1(R/\mathfrak{p})$ is of finite length and so is Artinian, it follows from the exact sequence (\ddagger) that f is an epimorphism and so $H_m^1(D_m(R/\mathfrak{p})) = 0$. Now using [3, Theorem 6.2.7] and the fact that

$H_m^0(D_m(R/\mathfrak{p})) = 0 = H_m^1(D_m(R/\mathfrak{p}))$ it follows that the R -module $D_m(R/\mathfrak{p})$ is a maximal Cohen-Macaulay R -module.

(iii) \Rightarrow (iv) Let $\mathfrak{p} \in \text{Assh}_R(R)$ such that $D_m(R/\mathfrak{p})$ is a maximal Cohen-Macaulay R -module. Then the exact sequence

$$(\dagger) \quad 0 \rightarrow R/\mathfrak{p} \rightarrow D_m(R/\mathfrak{p}) \rightarrow H_m^1(R/\mathfrak{p}) \rightarrow 0$$

implies

$$\mu_{R/\mathfrak{p}}^0(0, D_m(R/\mathfrak{p})) = \mu_{R/\mathfrak{p}}^0(0, R/\mathfrak{p}) = 1.$$

Therefore, in view of Proposition 3.1 there is an exact sequence $0 \rightarrow D_m(R/\mathfrak{p}) \rightarrow R/\mathfrak{p}$. Whence, it follows that $D_m(R/\mathfrak{p}) \cong I/\mathfrak{p}$ for some ideals $I \supset \mathfrak{p}$ of R . Thus the R -module $M = I/\mathfrak{p}$ is maximal Cohen-Macaulay. Also, as $I/\mathfrak{p} \cong D_m(R/\mathfrak{p})$ it follows from the exact sequence (\dagger) that I/\mathfrak{p} has a submodule $(Rx + \mathfrak{p})/\mathfrak{p} \cong R/\mathfrak{p}$ such that $I/(Rx + \mathfrak{p})$ has finite length. Consequently, $\mathfrak{m}^n I \subseteq (Rx + \mathfrak{p})$ for some positive integer n . If $(Rx + \mathfrak{p})/\mathfrak{p} = R/\mathfrak{p}$, then there is nothing to prove. But in other cases if \mathfrak{q} is a minimal prime divisor of $Rx + \mathfrak{p}$, then using Principal Ideal Theorem we can deduce that $\text{height}(\mathfrak{q}/\mathfrak{p}) = 1$. In particular, $\mathfrak{q} \neq \mathfrak{m}$. But as $\mathfrak{m}^n I \subseteq (Rx + \mathfrak{p}) \subseteq \mathfrak{q}$, it follows that $I \subseteq \mathfrak{q}$. Now it is easy to see that $\text{Rad}(I/\mathfrak{p}) = \text{Rad}((Rx + \mathfrak{p})/\mathfrak{p})$. So from the definition we have $\text{ara}(I/\mathfrak{p}) = 1$.

(iv) \Rightarrow (v) Is clear.

(v) \Rightarrow (i) The assertion follows from [3, Theorem 6.2.7]. □

Before bringing the next Corollary, note that if (R, \mathfrak{m}) is a Noetherian local ring of dimension 2 and R is universally catenary with all Cohen-Macaulay formal fibers, then $D_m(R/\mathfrak{p})$ is finitely generated for all $\mathfrak{p} \in \text{Assh}_R(R)$.

Corollary 3.3. *Let (R, \mathfrak{m}) be a local Noetherian ring of dimension 2. Then the following statements hold:*

(i) *For each finitely generated R -module M of dimension 2, if the R -module $H_m^1(M)$ is finitely generated, then the R -module $D_m(M/\Gamma_{\mathfrak{m}}(M))$ is a maximal Cohen-Macaulay R -module.*

(ii) *If R is universally catenary with all Cohen-Macaulay formal fibers, then $D_m(R/\mathfrak{p})$ is a maximal Cohen-Macaulay R -module for all $\mathfrak{p} \in \text{Assh}_R(R)$.*

Proof. (i) As the R -module $H_m^1(M/\Gamma_{\mathfrak{m}}(M)) \cong H_m^1(M)$ is finitely generated, $H_m^0(M/\Gamma_{\mathfrak{m}}(M)) = 0$ and $\dim(M/\Gamma_{\mathfrak{m}}(M)) = 2$, the assertion follows from the method used in the proof of Theorem 3.2.

(ii) Follows from (i), using [3, Theorem 2.2.4]. □

Proposition 3.4. *Let (R, \mathfrak{m}) be a Cohen-Macaulay local (Noetherian) ring of dimension $d \geq 1$ and \mathfrak{p} a prime ideal of R . Then the following statements hold:*

(i) $H_m^1(R/\mathfrak{p})$ is not finitely generated if and only if $\dim(R/\mathfrak{p}) = 1$.

(ii) *For every finitely generated R -module N and each $\mathfrak{p} \in \text{Att}_R(H_m^1(N))$ we have $\dim(R/\mathfrak{p}) \leq 1$.*

Proof. (i) “ \Leftarrow ” is clear. “ \Rightarrow ” Since $H_m^1(R/\mathfrak{p})$ is not finitely generated, it follows that $\dim R/\mathfrak{p} \geq 1$. If $\dim R/\mathfrak{p} \geq 2$, then \mathfrak{p} contains an R -regular sequence x_1, \dots, x_n of length $n = \text{height}(\mathfrak{p})$. We have

$$\text{depth}(R/(x_1, \dots, x_n)) = \dim R - n = \dim R - \text{height}(\mathfrak{p}) = \dim R/\mathfrak{p} \geq 2$$

and so $H_m^1(R/(x_1, \dots, x_n)) = 0$. Since $\text{height}((x_1, \dots, x_n)) = n = \text{height}(\mathfrak{p})$ and $(x_1, \dots, x_n) \subseteq \mathfrak{p}$ it follows that \mathfrak{p} is a minimal prime ideal of (x_1, \dots, x_n) and hence $\mathfrak{p} \in \text{Ass}_R(R/(x_1, \dots, x_n))$. Therefore, there exists an exact sequence

$$0 \rightarrow R/\mathfrak{p} \rightarrow R/(x_1, \dots, x_n) \rightarrow T \rightarrow 0$$

for some finitely generated R -module T . This exact sequence implies that $H_m^1(R/\mathfrak{p}) \cong H_m^0(T)$. Since T is finitely generated, it follows that $H_m^1(R/\mathfrak{p})$ is finitely generated, which is a contradiction.

(ii) In view of [9, Theorem 6.4] N has a prime filtration as $0 = N_0 \subset N_1 \subset \dots \subset N_n = N$ of submodules such that for each i we have $N_i/N_{i-1} \cong R/\mathfrak{p}_i$ with $\mathfrak{p}_i \in \text{Spec}(R)$. We use induction on n . For $n = 1$ the assertion follows from (i) and [3, Theorem 7.3.2]. Suppose that $n > 1$ and the case $n - 1$ is settled. Then by inductive hypothesis we have $\dim(R/\mathfrak{q}) \leq 1$ for each $\mathfrak{q} \in \text{Att}_R(H_m^1(N_{n-1})) \cup \text{Att}_R(H_m^1(N_n/N_{n-1}))$ and so $\dim(R/\text{Ann}_R(H_m^1(N_{n-1}))) \leq 1$ and $\dim(R/\text{Ann}_R(H_m^1(N_n/N_{n-1}))) \leq 1$. The exact sequence

$$0 \rightarrow N_{n-1} \rightarrow N_n \rightarrow N_n/N_{n-1} \rightarrow 0$$

induces the following exact sequence

$$H_m^1(N_{n-1}) \xrightarrow{f_1} H_m^1(N_{n-1}) \xrightarrow{f_2} H_m^1(N_n/N_{n-1}).$$

From the exact sequence

$$0 \rightarrow \text{im}(f_1) \rightarrow H_m^1(N_{n-1}) \rightarrow \text{im}(f_2) \rightarrow 0,$$

it follows that

$$\dim(R/\text{Ann}_R(H_m^1(N_n))) \leq 1.$$

Let $J := \text{Ann}_R(H_m^1(N_n))$. Then for each $\mathfrak{q} \in \text{Att}_R(H_m^1(N_n))$ we have $\mathfrak{q} \in V(J)$ and so $\dim(R/\mathfrak{q}) \leq 1$. This completes the inductive step. \square

Corollary 3.5. *Let (R, \mathfrak{m}) be a local Noetherian ring of dimension 2 such that R is a homomorphic image of a Cohen-Macaulay local ring. Then there exists a maximal Cohen-Macaulay R -module.*

Proof. Let $\mathfrak{p} \in \text{Assh}_R(R)$. Then using [3, Theorem 4.2.1] and Proposition 3.4 it follows that the R -module $H_m^1(R/\mathfrak{p})$ is finitely generated. Therefore the assertion follows from Theorem 3.2. \square

Corollary 3.6. *Let (R, \mathfrak{m}) be a local Noetherian domain of dimension 2. Then the following conditions are equivalent:*

- (i) *The R -module $H_m^1(R)$ is finitely generated.*
- (ii) *There exists a maximal Cohen-Macaulay R -module.*

(iii) *The R -module $M = I$ is maximal Cohen-Macaulay for some ideals I of R with $\text{ara}(I) = 1$.*

Proof. Since R is a domain it follows that $\text{Assh}_R(R) = \{0\}$. Now the assertion follows from Theorem 3.2. \square

Remark 3.7. There exists a two-dimensional Noetherian local domain (R, \mathfrak{m}) which does not have a maximal Cohen-Macaulay module (see [6, §1] and [5]). Obviously, R is a catenary domain and by Corollary 3.5 R is not a homomorphic image of a Cohen-Macaulay local ring. This shows that in general an arbitrary catenary Noetherian local domain is not a homomorphic image of a Cohen-Macaulay local ring.

The following result, which is the second main result of this section, shows that over an arbitrary Noetherian local ring of dimension 2 there exists a minimax balanced big Cohen-Macaulay module. Recall that, a module is called *minimax module*, when it has a finitely generated submodule, such that the quotient by it is an Artinian module [12].

Theorem 3.8. *Let (R, \mathfrak{m}) be a local Noetherian ring of dimension 2 and I be an ideal of R such that $\dim(R/I) = 2$ and $\text{Ass}_R(R/I) = \text{Assh}_R(R/I)$. Then there exists a minimax balanced big Cohen-Macaulay R -module M such that $\text{Ann}_R(M) = I$.*

Proof. In view of [3, Theorem 2.2.4], there is an exact sequence

$$0 \rightarrow R/I \rightarrow D_{\mathfrak{m}}(R/I) \rightarrow H_{\mathfrak{m}}^1(R/I) \rightarrow 0,$$

which implies that the R -module $D_{\mathfrak{m}}(R/I)$ is minimax. Let $M := D_{\mathfrak{m}}(R/I)$. Then it is easy to see that $\text{Ann}_R(M) = I$. By method used in the proof of Theorem 3.2 we can deduce that $H_{\mathfrak{m}}^0(M) = 0 = H_{\mathfrak{m}}^1(M)$. Also it follows from [3, Theorem 6.1.2] that $H_{\mathfrak{m}}^2(M) \cong H_{\mathfrak{m}}^2(R/I)$. So by [3, Theorem 7.3.2] we have $H_{\mathfrak{m}}^2(M) \neq 0$. By [3, Exercise 6.1.9] we have $H_{\mathfrak{m}}^2(M) \cong H_{\mathfrak{m}}^2(R) \otimes_R M$. Since $H_{\mathfrak{m}}^2(R)$ can be viewed as the direct limit of its finitely generated submodules and the functor $-\otimes_R M$ commutes with direct limits, it follows from the hypothesis $H_{\mathfrak{m}}^2(R) \otimes_R M \neq 0$ that $H_{\mathfrak{m}}^2(R)$ has a finitely generated non-zero submodule A such that $A \otimes_R M \neq 0$. Since the R -module A is Artinian and finitely generated it follows that A has finite length. Therefore, there is a positive integer n such that $\mathfrak{m}^n A = 0$. Hence there exists an exact sequence $\bigoplus_{i=1}^t R/\mathfrak{m}^n \rightarrow A \rightarrow 0$ for some positive integer t . Now the exact sequence

$$\bigoplus_{i=1}^t M/\mathfrak{m}^n M \rightarrow A \otimes_R M \rightarrow 0$$

implies that $M/\mathfrak{m}^n M \neq 0$. Whence, it follows that $\mathfrak{m} M \neq M$. On the other hand, we have $\text{Ass}_R(R/I) \subseteq \text{Ass}_R(M) \subseteq \text{Ass}_R(R/I) \cup \text{Ass}_R(H_{\mathfrak{m}}^1(R/I))$ and $\Gamma_{\mathfrak{m}}(M) = 0$ which implies $\mathfrak{m} \notin \text{Ass}_R(M)$. But $\text{Ass}_R(H_{\mathfrak{m}}^1(R/I)) \subseteq \{\mathfrak{m}\}$. Thus we have

$$\text{Ass}_R(M) = \text{Ass}_R(R/I) = \text{Assh}_R(R/I) \subseteq \text{Assh}_R(R).$$

So $\text{Ass}_R(M) \subseteq \text{Assh}_R(R)$. Next, let x_1, x_2 be an arbitrary system of parameters for R . Then as $x_1 \notin \mathfrak{p}$ for each $\mathfrak{p} \in \text{Assh}_R(R)$ it follows from the fact $\text{Ass}_R(M) \subseteq \text{Assh}_R(R)$ that x_1 is a non-zero-divisor on M . The exact sequence

$$0 \rightarrow M \xrightarrow{x_1} M \rightarrow M/x_1M \rightarrow 0$$

induces the following exact sequence:

$$0 = H_m^0(M) \rightarrow H_m^0(M/x_1M) \rightarrow H_m^1(M) = 0,$$

which implies that $H_m^0(M/x_1M) = 0$. Therefore, $\mathfrak{m} \notin \text{Ass}_R(M/x_1M)$. Since $\dim R/x_1R = 1$, this implies that $\text{Ass}_R(M/x_1M) \subseteq \text{Assh}_R(R/x_1R)$. Therefore, x_2 is a non-zero-divisor on M/x_1M . Since $(x_1, x_2)M \subseteq \mathfrak{m}M \neq M$, it follows from the definition that x_1, x_2 is an M -regular sequence and so M is a minimax balanced big Cohen-Macaulay R -module with $\text{Ann}_R(M) = I$. \square

Corollary 3.9. *Let (R, \mathfrak{m}) be a local Noetherian ring of dimension 2. Then for each $\mathfrak{p} \in \text{Assh}_R(R)$, there exists a minimax balanced big Cohen-Macaulay R -module M such that $\text{Ann}_R(M) = \mathfrak{p}$.*

Proof. As $\text{Ass}(R/\mathfrak{p}) = \text{Assh}(R/\mathfrak{p})$ and $\dim(R/\mathfrak{p}) = 2$ by Theorem 3.8 the assertion holds with $M := D_m(R/\mathfrak{p})$. \square

The following result verifies the existence of the maximal Cohen-Macaulay modules with certain annihilators.

Theorem 3.10. *Let (R, \mathfrak{m}) be a local Noetherian ring of dimension 2 which is a homomorphic image of a Gorenstein local ring. Let I be an ideal of R such that $\dim(R/I) = 2$. Then the following conditions are equivalent:*

- (i) $\text{Ass}_R(R/I) = \text{Assh}_R(R/I)$.
- (ii) *There exists a Maximal Cohen-Macaulay R -module M such that*

$$\text{Ann}_R(M) = I.$$

Proof. (i) \Rightarrow (ii) In view of Theorem 3.8 the R -module $M := D_m(R/I)$ is balanced big Cohen-Macaulay and $\text{Ann}_R(M) = I$. So it is enough to prove that M is finitely generated. To do this considering the exact sequence

$$0 \rightarrow R/I \rightarrow D_m(R/I) \rightarrow H_m^1(R/I) \rightarrow 0,$$

it is enough to prove that the R -module $H_m^1(R/I)$ is finitely generated. In view of [3, Corollary 7.2.12] we must show that $\text{Att}_R(H_m^1(R/I)) \subseteq \{\mathfrak{m}\}$. Suppose that the contrary is true. Then there is $\mathfrak{p} \in \text{Att}_R(H_m^1(R/I))$ such that $\mathfrak{p} \neq \mathfrak{m}$. Then by [3, Corollary 11.3.5] we have $\dim(R/\mathfrak{p}) = 1$. So it follows from [3, Theorem 11.3.2] that $\mathfrak{p} \in \text{Ass}_R(R/I)$. Therefore, $\text{Ass}_R(R/I) \neq \text{Assh}_R(R/I)$, which is a contradiction.

(i) \Rightarrow (ii) As $\text{Ann}_R(M) = I$ and M is finitely generated it follows that there is an exact sequence

$$0 \rightarrow R/I \rightarrow \bigoplus_{i=1}^n M$$

for some positive integer n . Therefore, $\text{Ass}_R(R/I) \subseteq \text{Ass}_R(M) \subseteq \text{Assh}_R(R)$. Now it is easy to see that $\text{Ass}_R(R/I) = \text{Assh}_R(R/I)$. \square

Corollary 3.11. *Let (R, \mathfrak{m}) be an analytically irreducible local ring of dimension 2. Then there exists a maximal Cohen-Macaulay R -module.*

Proof. Since by definition \widehat{R} is a domain it follows that R is a domain. In view of Cohen's structure theorem \widehat{R} is a homomorphic image of a regular local ring. Since \widehat{R} is a domain it follows from Corollary 3.5 and Corollary 3.6 that the \widehat{R} -module $H_{\mathfrak{m}\widehat{R}}^1(\widehat{R})$ is finitely generated. Since the \widehat{R} -module

$$H_{\mathfrak{m}}^1(R) \cong H_{\mathfrak{m}}^1(R) \otimes_R \widehat{R} \cong H_{\mathfrak{m}\widehat{R}}^1(\widehat{R})$$

has finite length as an \widehat{R} -module, it follows that this R -module has finite length. Now the assertion follows from Corollary 3.6. \square

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