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A NOTE ON $*$-PARANORMAL OPERATORS AND RELATED CLASSES OF OPERATORS

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Abstract. We shall show that the Riesz idempotent E_λ of every $*$-paranormal operator T on a complex Hilbert space $\mathcal{H}$ with respect to each isolated point λ of its spectrum $\sigma(T)$ is self-adjoint and satisfies $E_\lambda\mathcal{H} = \ker(T-\lambda) = \ker(T-\lambda)^*$. Moreover, Weyl's theorem holds for $*$-paranormal operators and more general for operators T satisfying the norm condition $\|Tx\|^n \leq \|T^n x\|\|x\|^{n-1}$ for all $x \in \mathcal{H}$. Finally, for this more general class of operators we find a sufficient condition such that $E_\lambda\mathcal{H} = \ker(T-\lambda) = \ker(T-\lambda)^*$ holds.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$. An operator T is said to be $*$-paranormal operator T if

$$\|T^*x\|^2 \leq \|T^2x\|\ \|x\|$$

for all $x \in \mathcal{H}$. The class of $*$-paranormal operators is a generalization of the class of hyponormal operators (i.e., operators satisfying $T^*T \geq TT^*$), and several interesting properties have been proved by many authors. For example, if T is a $*$-paranormal operator, then T is normaloid, i.e., $\|T\| = r(T) = \sup\{|z| \ : \ z \in \sigma(T)\}$, and $(T-\lambda)x = 0$ implies $(T-\lambda)^*x = 0$ ([1], [8]). There is another natural generalization of hyponormal operators called paranormal operators, which satisfy

$$\|Tx\|^2 \leq \|T^2x\|\ \|x\|$$

for all $x \in \mathcal{H}$. It is known that a paranormal operator T is normaloid and T^{-1} is also paranormal if T is invertible. Moreover $(T-\lambda)x = 0$ implies $(T-\lambda)^*x = 0$ if $\lambda \neq 0$ is an isolated point of spectrum of T. However it was not known whether T^{-1} must also be $*$-paranormal if T is an invertible $*$-paranormal operator. One of the main goals of this paper is to show that there

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exists an invertible $*$-paranormal operator T such that T^{-1} is not $*$-paranormal. We also show if T is an invertible $*$-paranormal operator, then

$$\|T^{-1}\| \leqq r(T^{-1})^3 r(T)^2.$$

Using this and a more general inequality, we shall show several properties of $*$-paranormal operators and class $\mathfrak{P}(n)$ operators, i.e., operators satisfying $\|Tx\|^n \leq \|T^n x\| \|x\|^{n-1}$ for all $x \in \mathcal{H}$ for $n \geq 2$.

We remark that an operator in $\mathfrak{P}(2)$ is called of class (N) by V. Istrăţescu, T. Saitō and T. Yoshino in [6] and paranormal by T. Furuta in [4], and an operator in $\mathfrak{P}(n)$ is called n-paranormal [2] and also called $(n-1)$-paranormal, e.g., [3], [7]. In order to avoid confusion we use the notation $\mathfrak{P}(n)$. S. M. Patel [8] proved that $*$-paranormal operators belong to the class $\mathfrak{P}(3)$. It is known that paranormal operators are in $\mathfrak{P}(n)$ for $n \geq 3$ (see the proof of Theorem 1 of [6]), but there is no inclusion relation between the class of paranormal operators and the class of $*$-paranormal operators.

The Riesz idempotent E_λ of an operator T with respect to an isolated point λ of $\sigma(T)$ is defined as

$$E_\lambda = \frac{1}{2\pi i} \int_{\partial D_\lambda} (z - T)^{-1} \, dz,$$

where the integral is taken in the positive direction and D_λ is a closed disk with center λ and small enough radius r such as $D_\lambda \cap \sigma(T) = \{\lambda\}$. Then $\sigma(T|_{E_\lambda \mathcal{H}}) = \{\lambda\}$ and $\sigma(T|_{(1-E_\lambda)\mathcal{H}}) = \sigma(T) \setminus \{\lambda\}$. In [9], Uchiyama proved that for every paranormal operator T and each isolated point λ of $\sigma(T)$ the Riesz idempotent E_λ satisfies that

$$E_0 \mathcal{H} = \ker T,$$
$$E_\lambda \mathcal{H} = \ker(T - \lambda) = \ker(T - \lambda)^* \text{ and } E_\lambda \text{ is self-adjoint if } \lambda \neq 0.$$

We shall show that for every $*$-paranormal operator T and each isolated point $\lambda \in \sigma(T)$ the Riesz idempotent E_λ of T with respect to λ is self-adjoint with the property that $E_\lambda \mathcal{H} = \ker(T - \lambda) = \ker(T - \lambda)^*$.

Let $w(T)$ be the Weyl spectrum of T, $\pi_{00}(T)$ the set of all isolated points of $\sigma(T)$ which are eigenvalues of T with finite multiplicities, i.e.,

$$w(T) = \{\lambda \in \sigma(T) \mid T - \lambda \text{ is not Fredholm with Fredholm index } 0\},$$
$$\pi_{00}(T) = \{\lambda \in \mathrm{iso}(\sigma(T)) \mid 0 < \dim \ker(T - \lambda) < \infty\}.$$

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to satisfy Weyl's theorem if

$$\sigma(T) \setminus w(T) = \pi_{00}(T),$$

also T is said to have the single valued extension property (SVEP) at λ if for any open neighborhood $\mathcal{U}$ of λ and analytic function $f : \mathcal{U} \to \mathcal{H}$ the zero function is only analytic solution of the equation

$$(T - z) f(z) = 0,$$

and T is said to have the SVEP if T has the SVEP at any $\lambda \in \mathbb{C}$ (or equivalently $\lambda \in \sigma(T)$).

It is well-known that every normal, hyponormal, p-hyponormal, w-hyponormal, class A, or paranormal operator satisfies Weyl's theorem and has the SVEP (see [9], [10] for definitions). We shall show that every $*$-paranormal operator satisfies Weyl's theorem. Y. M. Han and A. H. Kim [5] introduced totally $*$-paranormal operators T, i.e., operators for which $T-\lambda$ is $*$-paranormal for every $\lambda \in \mathbb{C}$, and they proved that every totally $*$-paranormal operator satisfies Weyl's theorem. Hence our result shows that the condition "totally" is not necessary. Also we shall show that every $*$-paranormal operator and every operator in the class $\mathfrak{P}(n)$ for $n \geq 2$ has the SVEP. The case of $\mathfrak{P}(n)$ for $n \geq 3$ was already proved by B. P. Duggal and C. S. Kubrusly [3] but we give another proof. We also show more general results for operators in the class $\mathfrak{P}(n)$ for $n \geq 2$.

2. $*$-paranormal operators

Let T be a $*$-paranormal operator, i.e., $\|T^*x\|^2 \leq \|T^2x\|\|x\|$ for all $x \in \mathcal{H}$. It is well-known that $T \in \mathfrak{P}(3)$. Indeed,

$$\|Tx\|^2 \leq \|T^*Tx\|\|x\| \leq \sqrt{\|T^3x\|\|Tx\|}\|x\| \quad \text{for all} \quad x \in \mathcal{H},$$

hence

$$\|Tx\|^3 \leq \|T^3x\|\|x\|^2 \quad \text{for all} \quad x \in \mathcal{H}. \tag{1}$$

Therefore every $*$-paranormal operator belongs to the class $\mathfrak{P}(3)$ (see [1], [8]). Proposition 1 and Lemmas 2, 3 and 4 are also well-known (see [1], [3], [7], [8]). For the convenience we give proofs of them.

Proposition 1. *Every $*$-paranormal operator T and every operator in the class $\mathfrak{P}(n)$ for $n \geq 2$ is normaloid, i.e., the operator norm $\|T\|$ is equal to the spectral radius $r(T)$.*

Proposition 1 follows from Lemma 1.

Lemma 1. *If T is $*$-paranormal or belongs to class $\mathfrak{P}(n)$ for $n \geq 2$ and $\{x_m\}$ is a sequence of unit vectors in $\mathcal{H}$ which satisfies $\lim_{m\to\infty} \|Tx_m\| = \|T\|$, then*

$$\lim_{m\to\infty} \|T^kx_m\| = \|T\|^k$$

for all $k \in \mathbb{N}$. Hence $\|T^k\| = \|T\|^k$ for all $k \in \mathbb{N}$.

Proof. Let T be $*$-paranormal. By the inequality (1), for every unit vector $x \in \mathcal{H}$ we have

$$\|Tx\|^3 \leq \|T^3x\| \leq \|T\|\|T^2x\| \leq \|T\|^3,$$

therefore if $\|Tx_m\| \to \|T\|$ as $m \to \infty$, then $\|T^2x_m\| \to \|T\|^2$ and $\|T^3x_m\| \to \|T\|^3$ as $m \to \infty$. Let $k \in \mathbb{N}$ satisfy $\lim_{m\to\infty} \|T^lx_m\| = \|T\|^l$ for all $l = 1, \ldots, k$. Since $T \in \mathfrak{P}(3)$, it follows that

$$\|T^kx_m\|^3 = \|T \cdot T^{k-1}x_m\|^3 \leq \|T^3 \cdot T^{k-1}x_m\|\|T^{k-1}x_m\|^2$$

$$\begin{aligned}&\le \|T^{k+2}x_m\|\|T^{k-1}x_m\|^2 \le \|T\|^{2(k-1)}\|T^{k+2}x_m\|\\&\le \|T\|^{2k-1}\|T^{k+1}x_m\| \le \|T\|^{3k}.\end{aligned}$$

This implies that $\lim_{m\to\infty}\|T^{k+1}x_m\| = \|T\|^{k+1}$. By the induction, the assertion follows.

Next, let $T \in \mathfrak{P}(n)$ for $n \ge 2$. The inequality

$$\|Tx_m\|^n \le \|T^n x_m\| \le \|T\|^{n-l}\|T^l x_m\| \le \|T\|^n$$

implies that $\lim_{m\to\infty}\|T^l x_m\| = \|T\|^l$ for all $l = 1,\dots,n$. By using same argument as above we also have $\lim_{m\to\infty}\|T^k x_m\| = \|T\|^k$ for all $k \in \mathbb{N}$. □

Theorem 1. *Let T be an invertible $*$-paranormal operator. Then*

$$\|T^{-1}\| \le r(T^{-1})^3 r(T)^2. \tag{2}$$

More generally, if $T \in \mathfrak{P}(n)$ for $n \ge 3$ is invertible, then

$$\|T^{-1}\| \le r(T^{-1})^{\frac{n(n-1)}{2}} r(T)^{\frac{(n+1)(n-2)}{2}}. \tag{3}$$

In particular, if T is $*$-paranormal or in the class $\mathfrak{P}(n)$ for $n \ge 3$ and $\sigma(T) \subset S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$, then T is unitary.

Proof. It is sufficient to consider the case where $T \in \mathfrak{P}(n)$ for $n \ge 3$. Since $S = T^{-1}$ satisfies

$$\|S^{n-1}x\|^n \le \|S^n x\|^{n-1}\|x\|$$

for $x \in \mathcal{H}$, we have $\|S^{n-1+k}x\|^n \le \|S^{n+k}x\|^{n-1}\|S^k x\|$ for every non-negative integer k. Then for any $x \ne 0$ we have

$$\prod_{k=0}^{l}\left(\frac{\|S^{n-1+k}x\|}{\|S^{n+k}x\|}\right)^{n-1} \le \prod_{k=0}^{l}\frac{\|S^k x\|}{\|S^{n-1+k}x\|}$$

and hence

$$\frac{\|S^{n-1}x\|^{n-1}}{\|S^{n+l}x\|^{n-1}} \le \|x\|\|Sx\|\cdots\|S^{n-2}x\|\frac{1}{\|S^{l+1}x\|\|S^{l+2}x\|\cdots\|S^{n-1+l}x\|}.$$

Then

$$\begin{aligned}&\prod_{l=0}^{L}\|S^{n-1}x\|^{n-1}\|S^{l+1}x\|\|S^{l+2}x\|\cdots\|S^{n-1+l}x\|\\ \le &\prod_{l=0}^{L}\|x\|\|Sx\|\cdots\|S^{n-2}x\|\|S^{n+l}x\|^{n-1}\end{aligned}$$

and

$$\begin{aligned}&\|S^{n-1}x\|^{(L+1)(n-1)}\|Sx\|\|S^2x\|^2\cdots\|S^{n-2}x\|^{n-2}\\&\Big(\|S^{n-1}x\|\|S^n x\|\cdots\|S^{L+1}x\|\Big)^{n-1}\|S^{L+2}x\|^{n-2}\|S^{L+3}x\|^{n-3}\cdots\|S^{L+n-1}x\|\\ \le &\Big(\|x\|\|Sx\|\cdots\|S^{n-2}x\|\Big)^{L+1}\Big(\|S^n x\|\|S^{n+1}x\|\cdots\|S^{n+L}x\|\Big)^{n-1}.\end{aligned}$$

So we have

$$\|S^{n-1}x\|^{(L+2)(n-1)} \le \|x\|^{L+1}\|Sx\|^{L}\cdots\|S^{n-2}x\|^{L-n+3}\cdot\|S^{L+2}x\|\|S^{L+3}x\|^{2}\cdots\|S^{L+n}x\|^{n-1},$$

and

(4)

$$\|S^{n-1}x\|^{\frac{(L+2)(n-1)}{L+1}} \le \|x\|\|Sx\|^{\frac{L}{L+1}}\cdots\|S^{n-2}x\|^{\frac{L-n+3}{L+1}}\cdot\|S^{L+2}x\|^{\frac{1}{L+1}}\|S^{L+3}x\|^{\frac{2}{L+1}}\cdots\|S^{L+n}x\|^{\frac{n-1}{L+1}}.$$

By letting $L\to\infty$ in (4) we have

$$\|S^{n-1}x\|^{n-1} \le \|x\|\|Sx\|\cdots\|S^{n-2}x\|r(S)r(S)^2\cdots r(S)^{n-1}.$$

Therefore

$$\left\|S\frac{S^{n-2}x}{\|S^{n-2}x\|}\right\| \le \left(\prod_{k=2}^{n-1}\left\|T^k\frac{S^{n-1}x}{\|S^{n-1}x\|}\right\|\right)r(S)^{\frac{n(n-1)}{2}} \le \left(\prod_{k=2}^{n-1}\|T\|^k\right)r(S)^{\frac{n(n-1)}{2}} = r(T)^{\frac{(n+1)(n-2)}{2}}r(S)^{\frac{n(n-1)}{2}},$$

and

$$\|T^{-1}\| \le r(T)^{\frac{(n+1)(n-2)}{2}}r(T^{-1})^{\frac{n(n-1)}{2}}.$$

Since every $*$-paranormal operator T belongs to the class $\mathfrak{P}(3)$, so if T is invertible, then

$$\|T^{-1}\| \le r(T)^2 r(T^{-1})^3.$$

Finally, if T is $*$-paranormal or belongs to the class $\mathfrak{P}(n)$ such that $\sigma(T)\subset S^1$, then $r(T)=r(T^{-1})=1$. Hence, $\|T\|=r(T)=1$ and $1=r(T^{-1})\le \|T^{-1}\|\le r(T)^{\frac{(n+1)(n-2)}{2}}r(T^{-1})^{\frac{n(n-1)}{2}}=1$ implies $\|T^{-1}\|=1$. It follows that T is invertible and an isometry because

$$\|x\| = \|T^{-1}Tx\| \le \|Tx\| \le \|x\|$$

for all $x\in\mathcal{H}$, so T is unitary. □

Remark 1. Theorem 1 also holds for $n=2$. If T is in the class $\mathfrak{P}(2)$, then T is paranormal and normaloid. Hence if T is invertible, then T^{-1} is also paranormal and normaloid, i.e., $r(T)=\|T\|$ and $r(T^{-1})=\|T^{-1}\|$. Hence if $\sigma(T)\subset S^1$, then T is unitary.

Corollary 1. *Let T be $*$-paranormal or belong to the class $\mathfrak{P}(n)$ for $n\ge 2$. If $\sigma(T)=\{\lambda\}$, then $T=\lambda$.*

Proof. If $\lambda=0$, then $\|T\|=r(T)=0$ by Theorem 1. Hence $T=0$.

If $\lambda\ne 0$, then $\frac{1}{\lambda}T$ is unitary with $\sigma(\frac{1}{\lambda}T)=\{1\}$. Hence $T=\lambda$. □

Lemma 2. *If T is $*$-paranormal and $\mathcal{M}$ is a T-invariant closed subspace, then the restriction $T|_{\mathcal{M}}$ of T to $\mathcal{M}$ is also $*$-paranormal.*

Proof. Let P be the orthogonal projection onto $\mathcal{M}$. Since $TP = PTP$, we have

$$\|(T|_{\mathcal{M}})^*x\|^2 = \|PT^*Px\|^2 = \|PT^*x\|^2 \leq \|T^*x\|^2 \leq \|T^2x\|\|x\| = \|(T|_{\mathcal{M}})^2x\|\|x\|$$

for all $x \in \mathcal{M}$. Thus $T|_{\mathcal{M}}$ is $*$-paranormal. □

Similarly, the following is proved by C. S. Kubrusly and B. P. Duggal [7].

Lemma 3 ([7])**.** *If $T \in \mathfrak{P}(n)$ for $n \geq 2$ and $\mathcal{M}$ is a T-invariant closed subspace, then the restriction $T|_{\mathcal{M}}$ of T to $\mathcal{M}$ also belongs to the class $\mathfrak{P}(n)$.*

Lemma 4 ([1])**.** *If T is $*$-paranormal, $\lambda \in \sigma_p(T)$ and a vector $x \in \mathcal{H}$ satisfies $(T-\lambda)x = 0$, then $(T-\lambda)^*x = 0$.*

Proof. Without loss of generality we may assume $\|x\| = 1$.

$$\|T^*x\|^2 \leq \|T^2x\|\|x\| = |\lambda|^2\|x\|^2 = |\lambda|^2$$

implies that $\|T^*x\| \leq |\lambda|$. Hence

$$\begin{aligned} 0 \leq \|(T-\lambda)^*x\|^2 &= \|T^*x\|^2 - 2\text{Re}\langle T^*x, \overline{\lambda}x\rangle + |\lambda|^2 \\ &\leq |\lambda|^2 - 2\text{Re}\langle x, \overline{\lambda}Tx\rangle + |\lambda|^2 \\ &= 2|\lambda|^2 - 2|\lambda|^2 = 0. \end{aligned}$$

□

Lemma 5. *Let T be $*$-paranormal or belong to the class $\mathfrak{P}(n)$ for $n \geq 2$, $\lambda \in \mathbb{C}$ an isolated point of $\sigma(T)$ and E_λ the Riesz idempotent with respect to λ. Then*

$$(T-\lambda)E_\lambda = 0.$$

Thus λ is an eigenvalue of T. Therefore T is isoloid, i.e., every isolated point of $\sigma(T)$ is an eigenvalue of T.

Proof. The Riesz idempotent E_λ satisfies $\sigma(T|_{E_\lambda\mathcal{H}}) = \{\lambda\}$ and $\sigma(T|_{(1-E_\lambda)\mathcal{H}}) = \sigma(T) \setminus \{\lambda\}$.

Since $T|_{E_\lambda\mathcal{H}}$ is also $*$-paranormal or belongs to the class $\mathfrak{P}(n)$ it follows that $(T-\lambda)E_\lambda = (T|_{E_\lambda\mathcal{H}} - \lambda)E_\lambda = 0$ by Corollary 1. Hence $\lambda \in \sigma_p(T)$. □

Theorem 2. *Let $T \in \mathfrak{P}(n)$ for $n \geq 2$, λ an isolated point of $\sigma(T)$ and E_λ the Riesz idempotent with respect to λ. Then*

$$E_\lambda\mathcal{H} = \ker(T-\lambda).$$

Proof. In Lemma 5, we have already shown $E_\lambda\mathcal{H} \subset \ker(T-\lambda)$. Let $x \in \ker(T-\lambda)$. Then

$$E_\lambda x = \frac{1}{2\pi i}\int_{\partial D_\lambda} (z-T)^{-1}x\,dz = \left(\frac{1}{2\pi i}\int_{\partial D_\lambda} \frac{1}{z-\lambda}\,dz\right)x = x,$$

so $x \in E_\lambda\mathcal{H}$. This completes the proof of $E_\lambda\mathcal{H} = \ker(T-\lambda)$. □

Theorem 3. *Let T be a $*$-paranormal operator, $\lambda \in \sigma(T)$ an isolated point and E_λ the Riesz idempotent with respect to λ. Then*

$$E_\lambda \mathcal{H} = \ker(T-\lambda) = \ker(T-\lambda)^*.$$

In particular, E_λ is self-adjoint, i.e., it is an orthogonal projection.

Proof. It suffices to show that $\ker(T-\lambda) = \ker(T-\lambda)^*$. The inclusion $\ker(T-\lambda) \subset \ker(T-\lambda)^*$ holds by Lemma 4 and hence $E_\lambda \mathcal{H} = \ker(T-\lambda)$ reduces T. Put $T = \lambda \oplus T_2$ on $\mathcal{H} = E_\lambda \mathcal{H} \oplus (E_\lambda \mathcal{H})^\perp$. If $\lambda \in \sigma(T_2)$, then λ is an isolated point of $\sigma(T_2)$. Since T_2 is also $*$-paranormal by Lemma 2, $\lambda \in \sigma_p(T_2)$ by Lemma 5. Since $\ker(T_2-\lambda) \subset \ker(T-\lambda)$, we have

$$\{0\} \neq \ker(T_2-\lambda) \subset \ker(T-\lambda) \cap (\ker(T-\lambda))^\perp = \{0\},$$

and it is a contradiction. Hence $\lambda \notin \sigma(T_2)$ and $T_2 - \lambda$ is invertible. This implies $\ker(T-\lambda)^* \subset \ker(T-\lambda)$ and $\ker(T-\lambda)^* = \ker(T-\lambda)$. Finally, we show $E_\lambda = E_\lambda^*$. Consider the E_λ on $\mathcal{H} = E_\lambda \mathcal{H} \oplus (E_\lambda \mathcal{H})^\perp$ in its block operator form $\left(\begin{smallmatrix} I & B \\ 0 & 0 \end{smallmatrix}\right)$. Observe that E_λ and $T = \lambda \oplus T_2$ commute and that $(T_2 - \lambda)$ is invertible. This implies $B = 0$. Hence E_λ is self-adjoint. □

Theorem 4. *Weyl's theorem holds for $*$-paranormal operator, i.e.,*

$$\sigma(T) \setminus w(T) = \pi_{00}(T).$$

Proof. Let $\lambda \in \sigma(T) \setminus w(T)$. Then $T-\lambda$ is Fredholm with $\operatorname{ind}(T-\lambda) = 0$ and is not invertible. Hence $\lambda \in \sigma_p(T)$ and $0 < \dim\ker(T-\lambda) < \infty$. By Theorem 3, $\ker(T-\lambda)$ reduces T, so $T = \lambda \oplus T_2$ on $\mathcal{H} = \ker(T-\lambda) \oplus (\ker(T-\lambda))^\perp$. If $\lambda \notin \operatorname{iso}(\ \sigma(T)\ )$, then $\lambda \in \sigma(T_2)$. Since $T-\lambda$ is a Fredholm operator with $\operatorname{ind}(T-\lambda) = 0$ and $\ker(T-\lambda)$ is finite dimensional subspace the operator $T_2 - \lambda$ is also Fredholm with $\operatorname{ind}(T_2-\lambda) = 0$. Hence, $\ker(T_2-\lambda) \neq \{0\}$. However, this is a contradiction since

$$\{0\} \neq \ker(T_2-\lambda) \subset (\ker(T-\lambda))^\perp \cap \ker(T-\lambda) = \{0\}.$$

Therefore $\lambda \in \operatorname{iso}(\ \sigma(T)\ )$ and $\lambda \in \pi_{00}(T)$.

Conversely, let $\lambda \in \pi_{00}(T)$. Then $0 < \dim\ker(T-\lambda) < \infty$. Since $\ker(T-\lambda)$ reduces T, the operator T is of the form $T = \lambda \oplus T_2$ on $\mathcal{H} = \ker(T-\lambda) \oplus (\ker(T-\lambda))^\perp$. If $\lambda \in \sigma(T_2)$, then λ is an isolated point of $\sigma(T_2)$ and hence $\lambda \in \sigma_p(T_2)$ by Lemma 5. However $\ker(T_2-\lambda) \subset \ker(T-\lambda) \cap (\ker(T-\lambda))^\perp = \{0\}$ implies $\ker(T_2-\lambda) = \{0\}$, contradiction. So, $T_2 - \lambda$ is invertible and $\operatorname{ind}(T-\lambda) = \operatorname{ind}(T_2-\lambda) = 0$. Hence $\lambda \in \sigma(T) \setminus w(T)$. □

For an operator T, we denote the approximate point spectrum of T by $\sigma_a(T)$, i.e., $\sigma_a(T)$ is the set of all $\lambda \in \mathbb{C}$ such that there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ which satisfies

$$\|(T-\lambda)x_n\| \to 0 \quad (\text{as } n \to \infty).$$

In [10], the authors defined spectral properties (I) and (II) as follows and proved that each property implies SVEP.

(I) if $\lambda \in \sigma_a(T)$ and $\{x_n\}$ is a sequence of bounded vectors of $\mathcal{H}$ satisfying $\|(T-\lambda)x_n\| \to 0$ (as $n \to \infty$), then $\|(T-\lambda)^* x_n\| \to 0$ (as $n \to \infty$),

(II) if λ, $\mu \in \sigma_a(T)$ $(\lambda \neq \mu)$ and sequences of bounded vectors $\{x_n\}$ and $\{y_n\}$ of $\mathcal{H}$ satisfy $\|(T-\lambda)x_n\| \to 0$ and $\|(T-\mu)y_n\| \to 0$ (as $n \to \infty$), then $\langle x_n, y_n \rangle \to 0$ (as $n \to \infty$), where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{H}$.

Theorem 5. *Every $*$-paranormal operator T has the spectral property* (I)*, so T has SVEP.*

Proof. Let $\lambda \in \sigma_a(T)$ and $\{x_n\}$ be a sequence of bounded vectors of $\mathcal{H}$ satisfying $\|(T-\lambda)x_n\| \to 0$ (as $n \to \infty$). Then

$$\begin{aligned}\|(T-\lambda)^* x_n\|^2 &= \|T^* x_n\|^2 - 2\mathrm{Re}\langle \overline{\lambda} x_n, T^* x_n \rangle + |\lambda|^2 \|x_n\|^2 \\ &= \|T^2 x_n\| \|x_n\| - 2\mathrm{Re}\langle \overline{\lambda} T x_n, x_n \rangle + |\lambda|^2 \|x_n\|^2 \\ &= |\lambda|^2 \|x_n\|^2 - 2|\lambda|^2 \|x_n\|^2 + |\lambda|^2 \|x_n\|^2 + O(\|(T-\lambda)x_n\|) \\ &\to 0 \quad (n \to \infty).\end{aligned}$$ □

Remark 2. According to [7], it is still unknown whether the inverse of an invertible operator in $\mathfrak{P}(n)$ is normaloid and so whether an operator in $\mathfrak{P}(n)$ is totally hereditarily normaloid. Example 1 shows that there exists a $*$-paranormal operator which is not paranormal. Example 2 shows there are invertible $*$-paranormal operators T such that T^{-1} are not normaloid, so not $*$-paranormal. Hence a $\mathfrak{P}(n)$ operator is not totally hereditarily normaloid in general. Moreover, these examples show that the inequality (2) is sharp.

Example 1. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal base of $\mathcal{H}$ and T be a weighted shift operator defined by

$$Te_n = \begin{cases} \sqrt{2} e_2 & (n=1), \\ e_3 & (n=2), \\ 2e_{n+1} & (n \geq 3). \end{cases}$$

Then $T^{2*}T^2 = 2 \oplus 4 \oplus \Big(\overset{\infty}{\underset{n=3}{\oplus}} 16 \Big)$, $T^*T = 2 \oplus 1 \oplus \Big(\overset{\infty}{\underset{n=3}{\oplus}} 4 \Big)$ and $TT^* = 0 \oplus 2 \oplus 1 \oplus \Big(\overset{\infty}{\underset{n=4}{\oplus}} 4 \Big)$. It is well-known that an operator S is $*$-paranormal if and only if $S^{2*}S^2 - 2kSS^* + k^2 \geq 0$ for all $k > 0$ and also well-known that S is paranormal if and only if $S^{2*}S^2 - 2kS^*S + k^2 \geq 0$ for all $k > 0$. We shall show that T is $*$-paranormal but not paranormal. Since

$$\begin{aligned}&T^{2*}T^2 - 2kTT^* + k^2 \\ &= (2+k^2) \oplus (4 - 4k + k^2) \oplus (16 - 2k + k^2) \oplus \Big(\overset{\infty}{\underset{n=4}{\oplus}} (16 - 8k + k^2) \Big) \\ &= (2+k^2) \oplus (k-2)^2 \oplus \{(k-1)^2 + 15\} \oplus \Big(\overset{\infty}{\underset{n=4}{\oplus}} (k-4)^2 \Big) \geq 0\end{aligned}$$

for all $k > 0$, T is $*$-paranormal. However, since

$$T^{2*}T^2 - 2kT^*T + k^2$$

$$= (2-4k+k^2) \oplus (4-2k+k^2) \oplus \Big(\bigoplus_{n=3}^{\infty} (16-8k+k^2) \Big)$$
$$= \{(k-2)^2-2\} \oplus \{(k-1)^2+3\} \oplus \Big(\bigoplus_{n=3}^{\infty} (k-4)^2 \Big) \not\geq 0$$

for $k = 2$, T is not paranormal.

Example 2. Let $a > 1$, $\{e_n\}_{n=1}^{\infty}$ be an orthonormal base of $\mathcal{H}$ and T_a a weighted shift defined by

$$T_a e_n = \begin{cases} \sqrt{a} e_{n+1} & (n \leq -2), \\ a e_0 & (n = -1), \\ e_1 & (n = 0), \\ a^2 e_{n+1} & (n \geq 1). \end{cases}$$

Then

$$(T_a)^{2*}(T_a)^2 = (\bigoplus_{n=-3}^{-\infty} a^2) \oplus a^3 \oplus a^2 \oplus \overset{(0)}{a^4} \oplus (\bigoplus_{n=1}^{\infty} a^8),$$
$$T_a(T_a)^* = (\bigoplus_{n=-1}^{-\infty} a) \oplus \overset{(0)}{a^2} \oplus 1 \oplus (\bigoplus_{n=2}^{\infty} a^4).$$

Thus,

$$\begin{aligned} &(T_a)^{2*}(T_a)^2 - 2kT_a(T_a)^* + k^2 \\ &= (\bigoplus_{n=-3}^{-\infty} (a-k)^2) \oplus \{(a-k)^2 + \underline{a^3 - a^2}\} \oplus (a-k)^2 \\ &\quad \oplus \overset{(0)}{(a^2-k)^2} \oplus \{(1-k)^2 + \underline{a^8 - 1}\} \oplus (\bigoplus_{n=2}^{\infty} (a^4-k)^2) \geq 0 \end{aligned}$$

for all $k > 0$. Therefore T_a is ∗-paranormal. Since $\|T_a^{-1}\| = 1, r(T_a) = a^2$ and $r(T_a^{-1}) = \frac{1}{\sqrt{a}}$, we have that T_a^{-1} is not normaloid and not paranormal. Since

$$r(T_a^{-1})^3 \cdot r(T_a)^2 = a^{\frac{5}{2}} \to 1 \quad (a \downarrow 1),$$

the inequality $\|T^{-1}\| \leq r(T^{-1})^3 \cdot r(T)^2$ is sharp in the sense that the least constant c which satisfies

$$\|T^{-1}\| \leq c \cdot r(T^{-1})^3 r(T)^2$$

for every ∗-paranormal operator T which is not paranormal is $c = 1$.

3. The class $\mathfrak{P}(n)$

Lemma 6. *Let $T \in \mathfrak{P}(n)$ for $n \geq 2$, $\lambda \in \sigma_p(T)$. Put $T = \left(\begin{smallmatrix} \lambda & S \\ 0 & T_2 \end{smallmatrix}\right)$ on $\mathcal{H} = \ker(T-\lambda) \oplus (\ker(T-\lambda))^{\perp}$. Then $\lambda \notin \sigma_p(T_2)$. In particular, if λ is isolated in $\sigma(T)$, then $T_2 - \lambda$ is invertible.*

Proof. If $\lambda \in \sigma_p(T_2)$, then $\mathcal{M} := \ker(T-\lambda) \oplus \ker(T_2 - \lambda)$ is an invariant subspace of T and $(T-\lambda)^2\mathcal{M} = \{0\}$. The operator $T|_{\mathcal{M}}$ belongs to the class

$\mathfrak{P}(n)$ by Lemma 3 and $\sigma(T|_{\mathcal{M}}) = \{\lambda\}$, so $T|_{\mathcal{M}} = \lambda$ by Corollary 1. This means that

$$\{0\} \neq \ker(T_2 - \lambda) \subset \ker(T - \lambda) \cap (\ker(T - \lambda))^{\perp} = \{0\},$$

which is a contradiction. Hence $\lambda \notin \sigma_p(T_2)$.

Next, we shall show the remaining assertion. Assume λ is isolated in $\sigma(T)$. Suppose $\lambda \in \sigma(T_2)$. Then λ is an isolated point of $\sigma(T_2)$. Let F be the Riesz idempotent of T_2 with respect to λ. Then $\mathcal{M}' := \ker(T - \lambda) \oplus F(\ker(T - \lambda))^{\perp}$ is an invariant subspace of T and $T|_{\mathcal{M}'}$ is of the form $\left(\begin{smallmatrix} \lambda & S_1 \\ 0 & T_3 \end{smallmatrix}\right)$ with $\sigma(T_3) = \{\lambda\}$. Thus $T|_{\mathcal{M}'} \in \mathfrak{P}(n)$ and $\sigma(T|_{\mathcal{M}'}) = \{\lambda\}$, so $T|_{\mathcal{M}'} = \lambda$ and hence $T_3 = \lambda$ by Corollary 1 and hence $S_1 = 0$. This implies $\lambda \in \sigma_p(T_2)$, a contradiction. Therefore $T_2 - \lambda$ is invertible. □

Theorem 6. *Weyl's theorem holds for operators in $\mathfrak{P}(n)$ for $n \geq 2$.*

Proof. Let $\lambda \in \sigma(T) \setminus w(T)$. Then $0 < \dim\ker(T - \lambda) < \infty$ and $T - \lambda$ is Fredholm with $\operatorname{ind}(T - \lambda) = 0$. Put $T = \left(\begin{smallmatrix} \lambda & S \\ 0 & T_2 \end{smallmatrix}\right)$ on $\mathcal{H} = \ker(T - \lambda) \oplus (\ker(T - \lambda))^{\perp}$. Since $\ker(T - \lambda)$ is finite dimensional, the operator S is a finite rank operator and the operator $T_2 - \lambda$ is Fredholm with $\operatorname{ind}(T_2 - \lambda) = 0$. By Lemma 6, $\ker(T_2 - \lambda) = \{0\}$ so $T_2 - \lambda$ is invertible and hence λ is isolated in $\sigma(T)$. Thus $\lambda \in \pi_{00}(T)$.

Conversely, if $\lambda \in \pi_{00}(T)$, then λ is isolated in $\sigma(T)$ and $0 < \dim\ker(T-\lambda) < \infty$. Put $T = \left(\begin{smallmatrix} \lambda & S \\ 0 & T_2 \end{smallmatrix}\right)$ on $\mathcal{H} = \ker(T - \lambda) \oplus (\ker(T - \lambda))^{\perp}$. Then $T_2 - \lambda$ is invertible by Lemma 6, so it is Fredholm with index 0. Since $\ker(T-\lambda)$ is finite dimensional, $T - \lambda$ is also Fredholm with index 0. Hence $\lambda \in \sigma(T) \setminus w(T)$. □

In [9], Uchiyama showed that if T is paranormal, i.e., $T \in \mathfrak{P}(2)$, then Weyl's theorem holds for T, and if λ is a non-zero isolated point of $\sigma(T)$, then the Riesz idempotent E_λ of T with respect to λ is self-adjoint and

$$E_\lambda \mathcal{H} = \ker(T - \lambda) = \ker(T - \lambda)^*.$$

In the case of $\lambda = 0$, it is well-known that the Riesz idempotent E_0 is not necessarily self-adjoint.

The following example is a paranormal operator having zero as an isolated point of the spectrum, but the Riesz idempotent is not self-adjoint.

Example 3. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal base of $\mathcal{H}$ and $\{a_n\}_{n=-\infty}^{\infty} \subset [1, 2]$ satisfy $a_n < a_{n+1}$ for all $n \in \mathbb{Z}$. Let A be the weighted bilateral shift defined by

$$Ae_n = a_n e_{n+1} \quad (n \in \mathbb{Z}),$$

$S = (A^*A - AA^*)^{\frac{1}{2}}$. Then the operator $T = \left(\begin{smallmatrix} A & S \\ 0 & 0 \end{smallmatrix}\right)$ on $\mathcal{H} \oplus \mathcal{H}$ satisfies $T^{2*}T^2 = (T^*T)^2$ which means that T is paranormal. Observe that $\sigma(T) = \sigma(A) \cup \{0\}$ and that A is invertible. This implies that 0 is an isolated point of $\sigma(T)$. Let E_0 the Riesz idempotent with respect to 0. Then

$$E_0 = \frac{1}{2\pi i} \int_{D_0} (z - T)^{-1}\, dz = \frac{1}{2\pi i} \int_{D_0} \begin{pmatrix} (z - A)^{-1} & \frac{1}{z}(z - A)^{-1} S \\ 0 & \frac{1}{z} \end{pmatrix} dz$$

$$= \begin{pmatrix} 0 & A^{-1}S \\ 0 & 1 \end{pmatrix}.$$

This E_0 satisfies $E_0\mathcal{H} = \ker T$, but E_0 is not self-adjoint since $A^{-1}S \neq 0$.

For operators in $\mathfrak{P}(n)$ $(n \geq 3)$, it is still not known whether the Riesz idempotent E_λ with respect to a non-zero isolated point λ of the spectrum is self-adjoint or not.

Next, we shall show that the self-adjointness of the Riesz idempotent E_λ for a $\mathfrak{P}(n)$ operator $(n \geq 3)$ with respect to a non-zero isolated point λ of its spectrum under some additional assumptions.

Let $n \in \mathbb{N}, \lambda \in \mathbb{C}$. The polynomial

$$F_{n,\lambda}(z) := -(n-1)\lambda^{n-1} + \lambda^{n-2}z + \lambda^{n-3}z^2 + \cdots + \lambda z^{n-2} + z^{n-1}$$

is important to study the class $\mathfrak{P}(n)$.

Theorem 7. *Let $T \in \mathfrak{P}(n)$ for an $n \geq 3$ and λ be a non-zero isolated point $\sigma(T)$. Put $T = \left(\begin{smallmatrix} \lambda & S \\ 0 & A \end{smallmatrix}\right)$ on $\mathcal{H} = \ker(T-\lambda) \oplus (\ker(T-\lambda))^{\perp}$. Then*

$$S(\lambda^{n-1} + \lambda^{n-2}A + \cdots + \lambda A^{n-2} + A^{n-1}) = n\lambda^{n-1}S.$$

In particular, if $\sigma(T) \cap \{z \in \mathbb{C} \mid F_{n,\lambda}(z) = 0\} = \{\lambda\}$, then the Riesz idempotent E_λ with respect to λ is self-adjoint and

$$E_\lambda\mathcal{H} = \ker(T-\lambda) = \ker(T-\lambda)^*.$$

Proof. We remark that $\lambda \in \sigma_p(T)$ by Lemma 5. Without loss of generality, we may assume $\lambda = 1$. Let $x \in \ker(T-1)$ and $y \in (\ker(T-1))^{\perp}$ be arbitrary unit vectors and let $0 < \epsilon < 1$ be arbitrary. It follows that $T^n = \left(\begin{smallmatrix} 1 & S_n \\ 0 & A^n \end{smallmatrix}\right)$ where $S_n = S(1 + A + \cdots + A^{n-1})$. Since $T \in \mathfrak{P}(n)$, the inequality

$$\|T\big((\sqrt{1-\epsilon}x) \oplus (\sqrt{\epsilon}y)\big)\|^n \leq \|T^n\big((\sqrt{1-\epsilon}x) \oplus (\sqrt{\epsilon}y)\big)\|$$

implies that

$$\big(\|\sqrt{1-\epsilon}x + \sqrt{\epsilon}Sy\|^2 + \|\sqrt{\epsilon}Ay\|^2\big)^n \leq \|\sqrt{1-\epsilon}x + \sqrt{\epsilon}S_ny\|^2 + \|\sqrt{\epsilon}A^ny\|^2.$$

Hence

$$\begin{aligned} &\Big\{(1-\epsilon) + 2\sqrt{\epsilon(1-\epsilon)}\mathrm{Re}\langle x, Sy\rangle + \epsilon(\|Sy\|^2 + \|Ay\|^2)\Big\}^n \\ &\leq (1-\epsilon) + 2\sqrt{\epsilon(1-\epsilon)}\mathrm{Re}\langle x, S_ny\rangle + \epsilon(\|S_ny\|^2 + \|A^ny\|^2), \end{aligned}$$

and

$$\begin{aligned} &(1-\epsilon)^n + 2n(1-\epsilon)^{n-1}\sqrt{\epsilon(1-\epsilon)}\mathrm{Re}\langle x, Sy\rangle + O(\epsilon) \\ &\leq (1-\epsilon) + 2\sqrt{\epsilon(1-\epsilon)}\mathrm{Re}\langle x, S_ny\rangle + O(\epsilon). \end{aligned}$$

Since $(1-\epsilon) - (1-\epsilon)^n = (1-\epsilon)\epsilon\{1 + (1-\epsilon) + \cdots + (1-\epsilon)^{n-2}\} = O(\epsilon)$, we have

$$n(1-\epsilon)^{n-1}\mathrm{Re}\langle x, Sy\rangle - \mathrm{Re}\langle x, S_ny\rangle \leq \frac{1}{2}\sqrt{\frac{\epsilon}{1-\epsilon}}\frac{1}{\epsilon}O(\epsilon).$$

Letting $\epsilon \to 0$, we have

$$\mathrm{Re}\langle x, (nS - S_n)y\rangle \le 0,$$

and hence $S_n = nS$.

Next, let $\sigma(T) \cap \{z \in \mathbb{C} \,|\, F_{n,1}(z) = 0\} = \{1\}$. Since $1 \notin \sigma(A)$ by Lemma 6 and

$$\sigma(A) \cap \{z \in \mathbb{C} \,|\, F_{n,1}(z) = 0\} \subset \sigma(T) \cap \{z \in \mathbb{C} \,|\, F_{n,1}(z) = 0\} = \{1\},$$

it follows $\sigma(A) \cap \{z \in \mathbb{C} \,|\, F_{n,1}(z) = 0\} = \emptyset$ and $F_{n,1}(A) = 1 - n + A + \cdots + A^{n-1}$ is invertible. Since $S_n = nS$, we have $S(1 - n + A + \cdots + A^{n-1}) = 0$ and hence $S = 0$. Then T is of the form $1 \oplus A$ with $1 \notin \sigma(A)$, this implies that

$$E_1 = \frac{1}{2\pi i}\int_{\partial D_1} ((z-1)^{-1} \oplus (z-A)^{-1})\, dz = 1 \oplus 0,$$

so E_1 is self-adjoint and $E_1\mathcal{H} = \ker(T-1) = \ker(T-1)^*$. □

If we assume that $-n + 1 + A + \cdots + A^{n-1}$ has a dense range instead of the assumption $\sigma(T) \cap \{z \in \mathbb{C} \,|\, f_\lambda(z) = n\lambda^{n-1}\} = \{\lambda\}$ in Theorem 7, we also have the same conclusions.

Lemma 7. *Let $T \in \mathfrak{P}(n)$ for $n \ge 3$ and $\lambda, \mu \in \sigma_p(T)$ such as $\lambda \ne \mu$. Then* $\ker(T-\lambda) \perp \ker(T-\mu)$.

Proof. Without loss of generality, we may assume $\lambda = 1$ and $|\mu| \le 1$. Consider the subspace $\mathcal{M} = \ker(T-1) \vee \ker(T-\mu)$, the closed subspace generated by $\ker(T-1)$ and $\ker(T-\mu)$, then $\mathcal{M}$ is invariant under T and $\sigma(T|_{\mathcal{M}}) = \{1, \mu\}$. Since $T|_{\mathcal{M}}$ belongs to the class $\mathfrak{P}(n)$ by Lemma 3 we have $\|T|_{\mathcal{M}}\| \le r(T|_{\mathcal{M}}) = 1$. For any $u \in \mathcal{M}$

$$\|T|_{\mathcal{M}}u\|^n \le \|(T|_{\mathcal{M}})^n u\|\|u\|^{n-1} \le \|T|_{\mathcal{M}}\|\|u\|^n \le \|u\|^n.$$

Therefore, for any $x \in \ker(T-1)$ and any $y \in \ker(T-\mu)$,

$$\begin{aligned}\|T|_{\mathcal{M}}(x+y)\|^n = \|x + \mu y\|^n &= \left(\sqrt{\|x\|^2 + |\mu|^2\|y\|^2 + 2\mathrm{Re}\langle x, \mu y\rangle}\right)^n \\ &\le \|x+y\|^n = \left(\sqrt{\|x\|^2 + \|y\|^2 + 2\mathrm{Re}\langle x, y\rangle}\right)^n,\end{aligned}$$

so we have

$$2\mathrm{Re}\Big((1-\overline{\mu})\langle x, y\rangle\Big) \le (1 - |\mu|^2)\|y\|^2.$$

If necessary, replace x by $ne^{i\theta}x$ for a $\theta \in \mathbb{R}$ and any $n \in \mathbb{N}$ so that $(1 - \overline{\mu})\langle ne^{i\theta}x, y\rangle = n|(1-\overline{\mu})\langle x, y\rangle|$ it follows that

$$|\langle x, y\rangle| \le \frac{(1-|\mu|^2)\|y\|^2}{2n|1-\overline{\mu}|}$$

for any $n \in \mathbb{N}$ and hence $\langle x, y\rangle = 0$. □

Theorem 8. *If $T \in \mathfrak{P}(n)$ for $n \ge 2$, then T has the SVEP.*

Proof. If $T \in \mathfrak{P}(2)$, T has SVEP by [9]. Let $T \in \mathfrak{P}(n)$ for $n \geq 3$. Let $\lambda \in \mathbb{C}$ be arbitrary, $\mathcal{U}$ any neighborhood of λ and $f : \mathcal{U} \to \mathcal{H}$ an analytic function which is a solution of the equation

$$(T - z)f(z) = 0 \quad \text{for all} \quad z \in \mathcal{U}.$$

Since $f(z) \in \ker(T - z)$ for all $z \in \mathcal{U}$ and $\ker(T - z) \perp \ker(T - w)$ for all $z, w \in \mathcal{U}$ such as $z \neq w$, we have

$$\|f(z)\|^2 = \lim_{w \to z} \langle f(z), f(w) \rangle = 0,$$

and $f = 0$. □

In [10], the authors show that every paranormal operator, i.e., any operator in $\mathfrak{P}(2)$, has the spectral property (II). We extend this result as follows.

Theorem 9. *If $T \in \mathfrak{P}(n)$ for $n \geq 3$, then T satisfies the spectral property* (II).

Proof. Let $\lambda, \mu \in \sigma_a(T)$ such as $\lambda \neq \mu$ with $|\mu| \geq |\lambda|$, $\{x_m\}$ and $\{y_m\}$ be arbitrary sequences of unit vectors such that

$$\|(T - \lambda)x_m\| \to 0, \quad \|(T - \mu)y_m\| \to 0 \quad (m \to \infty).$$

We shall show that $\langle x_m, y_m \rangle \to 0$ as $m \to \infty$. Suppose $\langle x_m, y_m \rangle \not\to 0$. By considering subsequence we may assume that $\langle x_m, y_m \rangle$ converges to some number a. Also, we may assume $a > 0$, if necessary replace x_m by $e^{it_m} x_m$ for some $t_m \in \mathbb{R}$ such as $\langle e^{it_m} x_m, y_m \rangle = |\langle x_m, y_m \rangle|$. Let $0 < \epsilon < 1$ and $c \in S^1$ be arbitrary. Then

$$\begin{aligned} &\|T(\sqrt{\epsilon} c x_m + \sqrt{1-\epsilon} y_m)\|^{2n} \\ &\leq \|T^n(\sqrt{\epsilon} c x_m + \sqrt{1-\epsilon} y_m)\|^2 \|(\sqrt{\epsilon} c x_m + \sqrt{1-\epsilon} y_m)\|^{2(n-1)}. \end{aligned}$$

Letting $m \to \infty$, we have

$$\begin{aligned} &\left(\epsilon|\lambda|^2 + (1-\epsilon)|\mu|^2 + 2a\sqrt{\epsilon(1-\epsilon)}\mathrm{Re}(c\lambda\overline{\mu})\right)^n \\ \leq\ &\left(\epsilon|\lambda|^{2n} + (1-\epsilon)|\mu|^{2n} + 2a\sqrt{\epsilon(1-\epsilon)}\mathrm{Re}\{c(\lambda\overline{\mu})^n\}\right)(1 + 2a\sqrt{\epsilon(1-\epsilon)}\mathrm{Re}(c))^{n-1}. \end{aligned}$$

Hence

$$\begin{aligned} &|\mu|^{2n} + n|\mu|^{2n-2} 2a\sqrt{\epsilon(1-\epsilon)}\mathrm{Re}(c\lambda\overline{\mu}) + O(\epsilon) \\ &\leq |\mu|^{2n} + |\mu|^{2n}(n-1)2a\sqrt{\epsilon(1-\epsilon)}\mathrm{Re}(c) + 2a\sqrt{\epsilon(1-\epsilon)}\mathrm{Re}\left\{c(\lambda\overline{\mu})^n\right\} + O(\epsilon), \end{aligned}$$

and

$$\begin{aligned} (5) \qquad &n|\mu|^{2n-2} 2a\sqrt{1-\epsilon}\mathrm{Re}(c\lambda\overline{\mu}) + O(\sqrt{\epsilon}) \\ &\leq |\mu|^{2n}(n-1)2a\sqrt{1-\epsilon}\mathrm{Re}(c) + 2a\sqrt{1-\epsilon}\mathrm{Re}\left\{c(\lambda\overline{\mu})^n\right\} + O(\sqrt{\epsilon}). \end{aligned}$$

Letting $\epsilon \downarrow 0$ in (5), we have

$$(n-1)\mathrm{Re}\left\{c|\mu|^{2n}\right\} + \mathrm{Re}\{c(\lambda\overline{\mu})^n\} - n\mathrm{Re}\left\{c\lambda\overline{\mu}|\mu|^{2(n-1)}\right\} \geq 0,$$

and

$$\mathrm{Re}\left\{c\left((n-1)-n\frac{\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^n\right)\right\}=0$$

for all $c \in S^1$. Hence

$$(n-1)-n\frac{\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^n=0. \tag{6}$$

If $\lambda = 0$, then $n-1=0$ by (6). This is a contradiction. Hence $\lambda \neq 0$. Let $z = \dfrac{\lambda}{\mu}$. Then $0 < |z| \leq 1, z \neq 1$ and

$$(n-1)-nz+z^n=(z-1)F_{n,1}(z)=0.$$

Hence

$$F_{n,1}(z)=1+z+z^2+\cdots+z^{n-1}-n=0.$$

Then

$$n=1+z+\cdots+z^{n-1}\leq 1+|z|+\cdots+|z^{n-1}|\leq n.$$

This implies $z=1$. This is a contradiction. □

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