Bull. Korean Math. Soc. **51** (2014), No. 2, pp. 339–356
http://dx.doi.org/10.4134/BKMS.2014.51.2.339

# DING PROJECTIVE MODULES WITH RESPECT TO A SEMIDUALIZING MODULE

Chunxia Zhang, Limin Wang, and Zhongkui Liu

Abstract. In this paper, we introduce and discuss the notion of $D_C$-projective modules over commutative rings, where $C$ is a semidualizing module. This extends Gillespie and Ding, Mao's notion of Ding projective modules. The properties of $D_C$-projective dimensions are also given.

## Introduction

Throughout this paper all rings are commutative with identity, all modules are unitary modules. $C$ is a fixed semidualizing $R$-module, cf. Definition 0.1 below.

In basic homological algebra, projective, injective and flat modules play an important and fundamental role. It is by now the homological properties of the Gorenstein projective and injective modules have been studied by many authors, some references are [3, 4, 9, 12]. Over a commutative Noetherian ring, Holm and Jørgensen in [14] introduced the $C$-Gorenstein projective and $C$-Gorenstein injective modules using semidualizing modules and their associated projective, injective classes. White in [19] further considered these modules when $R$ is a commutative ring and she called $C$-Gorenstein projective as $\mathrm{G}_C$-projective and $C$-Gorenstein injective as $\mathrm{G}_C$-injective. In particular, many general results about the Gorenstein projectivity and Gorenstein injectivity in [7, 12, 13] were generalized in [19]. On the other hand, Ding et al. in [5] and [6] considered two special cases of the Gorenstein projective and Gorenstein injective modules using projective, flat classes and injective, FP-injective classes, which they called strongly Gorenstein flat and Gorenstein FP-injective modules, respectively. The same modules were studied by Gillespie in [10] with different names Ding projective and Ding injective modules, respectively. Thus, a natural question arises: What are the counterparts to Ding projective

Received February 13, 2012; Revised November 11, 2013.
2010 *Mathematics Subject Classification.* 16E05, 16E65, 18G20, 18G25.
*Key words and phrases.* semidualizing modules, $D_C$-projective modules, $D_C$-projective dimensions, $G_C$-projective modules.
This work is supported by National Natural Science Foundation of China (11261050, 11301240, 11326061) and by NWNU-lkqn-12-17.

and Ding injective modules using semidualizing modules and their associated projective, flat classes and injective, FP-injective classes?

In this paper, we shall introduce the notions of $D_C$-projective and $D_C$-injective modules which answer the question above. Also some properties of $D_C$-projective and $D_C$-injective modules and dimensions are given.

Next we shall recall some notions and definitions which we need in the later sections.

The study of semidualizing modules over commutative Noetherian rings was initiated independently (with different names) by Foxby [8], Golod [11], and Vasconcelos [18].

**Definition 0.1.** An $R$-module C is *semidualizing* if

(1) $C$ admits a degreewise finite projective resolution,
(2) the natural homothety morphism $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism,
(3) $\mathrm{Ext}_R^{\geqslant 1}(C, C) = 0$.

Let $C$ be a semidualizing $R$-module. We set

$$\mathcal{P}_C(R) = \text{the subcategory of modules } C \otimes_R P \text{ where } P \text{ is } R\text{-projective},$$
$$\mathcal{F}_C(R) = \text{the subcategory of modules } C \otimes_R F \text{ where } F \text{ is } R\text{-flat},$$
$$\mathcal{I}_C(R) = \text{the subcategory of modules } \mathrm{Hom}_R(C, I) \text{ where } I \text{ is } R\text{-injective},$$
$$\mathcal{FI}_C(R) = \text{the subcategory of modules } \mathrm{Hom}_R(C, E) \text{ where } E \text{ is } R\text{-FP-injective}.$$

Modules in $\mathcal{P}_C(R)$, $\mathcal{F}_C(R)$, $\mathcal{I}_C(R)$ and $\mathcal{FI}_C(R)$ are called *$C$-projective*, *$C$-flat*, *$C$-injective* and *$C$-FP-injective*, respectively. By setting $C = R$ in the definitions above we see that $\mathcal{P}_R(R)$, $\mathcal{F}_R(R)$, $\mathcal{I}_R(R)$ and $\mathcal{FI}_R(R)$ are the classes of ordinary *projective*, *flat*, *injective* and *FP-injective* $R$-modules, respectively, which we denote by $\mathcal{P}(R)$, $\mathcal{F}(R)$, $\mathcal{I}(R)$ and $\mathcal{FI}(R)$, respectively.

The following notions were introduced by Holm and Jørgensen in [14] over commutative Noetherian rings and White in [19] for commutative rings.

**Definition 0.2.** A *complete $\mathcal{PP}_C$-resolution* is a complex $X$ of $R$-modules satisfying the following conditions:

(1) $X$ is exact and $\mathrm{Hom}_R(-, \mathcal{P}_C(R))$-exact; and
(2) $X_i$ is projective for $i \geqslant 0$ and $X_i$ is $C$-projective for $i < 0$.

An $R$-module $M$ is *$G_C$-projective* if there exists a complete $\mathcal{PP}_C$-resolution $X$ such that $M \cong \mathrm{Coker}(\partial_1^X)$. Set

$$\mathcal{GP}_C(R) = \text{the subcategory of } G_C\text{-projective } R\text{-modules}.$$

In the case $C = R$ we use the more common terminology "complete projective resolution" and "Gorenstein projective module" and the notation $\mathcal{GP}(R)$.

A *complete $\mathcal{I}_C\mathcal{I}$-resolution* is a complex $Y$ of $R$-modules such that:

(1) $Y$ is exact and $\mathrm{Hom}_R(\mathcal{I}_C(R), -)$-exact; and
(2) $Y_i$ is injective for $i \leqslant 0$ and $Y_i$ is $C$-injective for $i > 0$.

An $R$-module $N$ is $G_C$*-injective* if there exists a complete $\mathcal{I}_C\mathcal{I}$-resolution $Y$ such that $N \cong \mathrm{Ker}(\partial_0^Y)$. Set

$$\mathcal{GI}_C(R) = \text{the subcategory of } G_C\text{-injective } R\text{-modules.}$$

In the case $C = R$ we use the more common terminology "complete injective resolution" and "Gorenstein injective module" and the notation $\mathcal{GI}(R)$.

**Definition 0.3.** An $R$-module $M$ is called *Ding projective* if there exists a $\mathrm{Hom}_R(-, \mathcal{F}(R))$-exact exact sequence of projective $R$-modules

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

with $M = \mathrm{Coker}(P_1 \to P_0)$. Set

$$\mathcal{DP}(R) = \text{the subcategory of Ding projective } R\text{-modules.}$$

An $R$-module $N$ is called *Ding injective* if there exists a $\mathrm{Hom}_R(\mathcal{FI}(R), -)$-exact exact sequence of injective $R$-modules

$$\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$$

with $N = \mathrm{Coker}(I_1 \to I_0)$. Set

$$\mathcal{DI}(R) = \text{the subcategory of Ding injective } R\text{-modules.}$$

Note that every Ding projective (respectively, Ding injective) module is Gorenstein projective (respectively, Gorenstein injective). If $R$ is Noetherian, then any FP-injective module is injective by [16, Thm. 1.6], and so any Gorenstein injective module is Ding injective. Clearly, any Gorenstein projective module over perfect ring is Ding projective. Also, it follows from [10, Cor. 4.6] that any Gorenstein projective (respectively, Gorenstein injective) module over a Gorenstein ring is Ding projective (respectively, Ding injective).

**Definition 0.4.** Let $\mathcal{X}$ be a class of $R$-modules and $M$ an $R$-module. An $\mathcal{X}$*-resolution* of $M$ is a complex of $R$-modules in $\mathcal{X}$ of the form

$$X = \cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to 0$$

such that $\mathrm{H}_0(X) \cong M$ and $\mathrm{H}_n(X) = 0$ for $n \geqslant 1$, and the following exact sequence is the *augmented* $\mathcal{X}$*-resolution* of $M$ associated to $X$:

$$X^+ = \cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0.$$

The $\mathcal{X}$*-projective dimension* of $M$ is the quantity

$$\mathcal{X}\text{-pd}_R(M) = \inf\{\sup\{n \geqslant 0 \mid X_n \neq 0\} \mid X \text{ is an } \mathcal{X}\text{-resolution of } M\}.$$

In particular, one has $\mathcal{X}\text{-pd}_R(0) = -\infty$. The modules of $\mathcal{X}$-projective dimension 0 are the nonzero modules of $\mathcal{X}$. We set

$$\overline{\mathcal{X}} = \text{the subcategory of } R\text{-modules with } \mathcal{X}\text{-pd}_R(M) < \infty.$$

An $\mathcal{X}$-resolution $X$ of $M$ is *proper* if the augmented resolution $X^+$ is $\mathrm{Hom}_R(\mathcal{X}, -)$-exact.

We define (*proper*) $\mathcal{X}$*-coresolutions* and $\mathcal{X}$*-injective dimensions* dually. And the $\mathcal{X}$*-injective dimension* of $M$ is denoted by $\mathcal{X}\text{-id}_R(M)$.

## 1. $D_C$-projective modules

The Ding projective $R$-modules of interest in this paper are built from semidualizing modules and their associated projective, $C$-projective and $C$-flat classes, defined next.

**Definition 1.1.** A *Ding $\mathcal{PP}_C$-resolution* is a complex $X$ of $R$-modules satisfying the following

(1) The complex $X$ is exact and $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact.

(2) The $R$-module $X_i$ is projective if $i \geqslant 0$ and $X_i$ is $C$-projective if $i < 0$.

An $R$-module $M$ is called *$D_C$-projective* if there exists a Ding $\mathcal{PP}_C$-resolution $X$ such that $M \cong \mathrm{Coker}\partial_1^X$, in which case $X$ is a Ding $\mathcal{PP}_C$-resolution of $M$.

A *Ding $\mathcal{I}_C\mathcal{I}$-resolution* is a complex $Y$ of $R$-modules such that

(1) $Y$ is exact and $\mathrm{Hom}_R(\mathcal{FI}_C(R), -)$-exact.

(2) The $R$-module $Y_i \in \mathcal{I}_C(R)$ for all $i \geqslant 0$ and $Y_i \in \mathcal{I}(R)$ for all $i < 0$.

An $R$-module $N$ is called *$D_C$-injective* if there exists a Ding $\mathcal{I}_C\mathcal{I}$-resolution $Y$ such that $N \cong \mathrm{Im}\partial_0^Y$, in which case $Y$ is a Ding $\mathcal{I}_C\mathcal{I}$-resolution of $N$. We set

$$\mathcal{DP}_C(R) = \text{the subcategory of } D_C\text{-projective } R\text{-modules},$$
$$\mathcal{DI}_C(R) = \text{the subcategory of } D_C\text{-injective } R\text{-modules}.$$

Note that when $C = R$, the definitions above correspond to the definitions of Ding projective and Ding injective modules. By definitions, every $D_C$-projective ($D_C$-injective, respectively) module is $G_C$-projective ($G_C$-injective, respectively).

*Remark* 1.2. In the following, we only deal with $D_C$-projective modules. But it should be pointed out that all of the obtained results have $D_C$-injective counterparts by using dual arguments.

**Notation 1.3.** Let $C$ be a semidualizing $R$-module. We use the following abbreviations.

$$\mathrm{pd}_R(-) = \mathcal{P}(R)\text{-pd}(-)$$
$$\mathcal{P}_C\text{-pd}_R(-) = \mathcal{P}_C(R)\text{-pd}(-)$$
$$\mathcal{F}_C\text{-pd}_R(-) = \mathcal{F}_C(R)\text{-pd}(-)$$
$$\mathcal{DP}_C\text{-pd}_R(-) = \mathcal{DP}_C(R)\text{-pd}(-).$$

The following result which is an immediate consequence of the definition of $D_C$-projective modules.

**Proposition 1.4.** *An $R$-module $M \in \mathcal{DP}_C(R)$ if and only if* $\mathrm{Ext}_R^{\geqslant 1}(M, C \otimes_R F) = 0$ *and $M$ admits a $\mathcal{P}_C$-coresolution $X$ with* $\mathrm{Hom}_R(X, C \otimes_R G)$ *exact for any flat $R$-modules $F$ and $G$.*

By dimension shifting, one can get the following result. When $C = R$, which is contained in [5, Lem. 2.4].

**Proposition 1.5.** *If $X$ is a Ding $\mathcal{PP}_C$-resolution and $L$ is an $R$-module with finite $\mathcal{F}_C$-projective dimension, then the complex* $\mathrm{Hom}_R(X, L)$ *is exact. Consequently, if $M$ is $D_C$-projective, then* $\mathrm{Ext}^i_R(M, L) = 0$ *for all $i > 0$ and all $R$-modules $L$ with $\mathcal{F}_C$-$\mathrm{pd}_R(L) < \infty$.*

The next three results provide ways to create $D_C$-projective modules.

**Proposition 1.6.** *Let $P$ be a finitely generated projective $R$-module. If $M \in \mathcal{DP}_C(R)$, then so is* $\mathrm{Hom}_R(P, M)$.

*Proof.* Since $M \in \mathcal{DP}_C(R)$, there exists a Ding $\mathcal{PP}_C$-resolution

$$X = \cdots \to P_1 \to P_0 \to C \otimes_R P_{-1} \to C \otimes_R P_{-2} \to \cdots$$

with $M = \mathrm{Coker}(P_1 \to P_0)$. Then

$$\begin{aligned}\mathrm{Hom}_R(P, X) = \cdots \to \mathrm{Hom}_R(P, P_0) &\to C \otimes_R \mathrm{Hom}_R(P, P_{-1}) \\ &\to C \otimes_R \mathrm{Hom}_R(P, P_{-2}) \to \cdots\end{aligned}$$

is exact by [1, Prop. 20.10] and all $\mathrm{Hom}_R(P, P_i) \in \mathcal{P}(R)$ (see [3, p. 14]) for all $i \in \mathbb{Z}$. Let $F$ be any flat $R$-module. Then

$$\mathrm{Hom}_R(\mathrm{Hom}_R(P, X), C \otimes_R F) \cong P \otimes_R \mathrm{Hom}_R(X, C \otimes_R F)$$

is exact by [1, Prop. 20.11]. It follows that $\mathrm{Hom}_R(P, M)$ is $D_C$-projective. $\square$

**Proposition 1.7.** *Let $Q \in \mathcal{P}(R)$. If $M \in \mathcal{DP}_C(R)$, then so is $M \otimes_R Q$. The converse holds when $Q$ is faithfully projective.*

*Proof.* Since $M \in \mathcal{DP}_C(R)$, by Proposition 1.4, there exists a $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact exact sequence $X = 0 \to M \to C \otimes_R P_{-1} \to C \otimes_R P_{-2} \to \cdots$, where each $P_i \in \mathcal{P}(R)$. Then

$$X \otimes_R Q = 0 \to M \otimes_R Q \to C \otimes_R (P_{-1} \otimes_R Q) \to C \otimes_R (P_{-2} \otimes_R Q) \to \cdots$$

is exact with $P_i \otimes_R Q \in \mathcal{P}(R)$ for all $i < 0$. Let $F$ be any flat $R$-module. Then

$$\mathrm{Hom}_R(X \otimes_R Q, C \otimes_R F) \cong \mathrm{Hom}_R(Q, \mathrm{Hom}_R(X, C \otimes_R F))$$

is exact by adjunction and $Q \in \mathcal{P}(R)$. It follows from [17, p. 258, 9.20] that

$$\mathrm{Ext}^{\geqslant 1}_R(M \otimes_R Q, C \otimes_R F) \cong \mathrm{Hom}_R(Q, \mathrm{Ext}^{\geqslant 1}_R(M, C \otimes_R F)) = 0.$$

So $M \otimes_R Q \in \mathcal{DP}_C(R)$ by Proposition 1.4 again.

If $Q$ is faithfully projective, then the complex $\mathrm{Hom}_R(Q, \mathrm{Hom}_R(X, C \otimes_R F))$ is exact if and only if the complex $\mathrm{Hom}_R(X, C \otimes_R F)$ is exact and $\mathrm{Hom}_R(Q, \mathrm{Ext}^{\geqslant 1}_R(M, C \otimes_R F)) = 0$ if and only if $\mathrm{Ext}^{\geqslant 1}_R(M, C \otimes_R F) = 0$. $\square$

**Proposition 1.8.** *If $P \in \mathcal{P}(R)$, then $P$, $C \otimes_R P \in \mathcal{DP}_C(R)$. Thus, every $R$-module admits a $D_C$-projective resolution.*

*Proof.* By Proposition 1.7, it suffices to construct Ding $\mathcal{PP}_C$-resolutions of $C$ and $R$. By definition, $C$ admits an augmented degreewise finite free resolution

$$X = \cdots \to R^{\alpha_1} \to R^{\alpha_0} \to C \to 0$$

and this is a Ding $\mathcal{PP}_C$-resolution of $C$. Indeed, the complex $X$ is exact by definition and $C \cong \mathrm{Coker}(R^{\alpha_1} \to R^{\alpha_0})$. Furthermore, since $\mathrm{Ext}_R^{\geqslant 1}(C, C) = 0$, the complex $\mathrm{Hom}_R(X, C \otimes_R F)$ is exact for any flat $R$-module $F$ by [19, Lem. 1.11(b)]. Thus, $C$ is $D_C$-projective.

Now, we show that

$$\mathrm{Hom}_R(X, C) = 0 \to R \to C^{\alpha_0} \to C^{\alpha_1} \to \cdots$$

is a Ding $\mathcal{PP}_C$-resolution of $R$. First, left exactness of $\mathrm{Hom}_R(-, C)$ and the equality $\mathrm{Ext}_R^{\geqslant 1}(C, C) = 0$ imply $\mathrm{Hom}_R(X, C)$ is exact. Moreover, since $\mathrm{Hom}_R(X, C)$ consists of finitely presented modules, for any flat $R$-module $F$, tensor evaluation provides the first isomorphism of complexes

$$\mathrm{Hom}_R(\mathrm{Hom}_R(X, C), C \otimes_R F) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(X, C), C) \otimes_R F \cong X \otimes_R F.$$

The second isomorphism follows from the fact that $\mathrm{Hom}_R(C, C) \cong R$ and [1, Prop. 20.11]. These complexes are exact since the complex $X$ is exact and $F$ is flat.

Finally, since the class $\mathcal{DP}_C(R)$ contains the class $\mathcal{P}(R)$, every $R$-module admits a $D_C$-projective resolution. □

**Lemma 1.9** ([13, Prop. 5.2])**.** *Let $X = 0 \to W' \to W \to W'' \to 0$ be an exact sequence of $R$-modules. If $W'$, $W'' \in \mathcal{P}_C(R)$, then $X$ splits and $W \in \mathcal{P}_C(R)$.*

**Proposition 1.10.** *The class $\mathcal{DP}_C(R)$ is closed under extensions.*

*Proof.* We consider any short exact sequence of $R$-modules $0 \to M' \to M \to M'' \to 0$ where $M'$ and $M''$ are $D_C$-projective with Ding $\mathcal{PP}_C$-resolutions $P'$ and $P''$, respectively. We claim that $M$ is also $D_C$-projective. Use Horseshoe lemmas in [12, (1.7)] and [17, Lem. (6.20)]), together with the fact that the classes $\mathcal{P}(R)$ and $\mathcal{P}_C(R)$ are closed under extensions by Lemma 1.9, we can construct an exact sequence of complexes $0 \to P' \to P \to P'' \to 0$ such that $M \cong \mathrm{Coker}(P_1 \to P_0)$, where $P_n = P'_n \oplus P''_n$ for all $n \in \mathbb{Z}$. To show that $M$ is $D_C$-projective, we only have to prove the complex $\mathrm{Hom}_R(P, C \otimes_R F)$ is exact for all flat $R$-modules $F$. Since each $0 \to P'_n \to P_n \to P''_n \to 0$ is split exact by Lemma 1.9, we have that

$$0 \to \mathrm{Hom}_R(P'', C \otimes_R F) \to \mathrm{Hom}_R(P, C \otimes_R F) \to \mathrm{Hom}_R(P', C \otimes_R F) \to 0$$

is an exact sequence of complexes. By hypothesis, both $\mathrm{Hom}_R(P', C \otimes_R F)$ and $\mathrm{Hom}_R(P'', C \otimes_R F)$ are exact, then $\mathrm{Hom}_R(P, C \otimes_R F)$ is also exact. □

**Proposition 1.11.** *The class $\mathcal{DP}_C(R)$ is closed under direct sums.*

*Proof.* Note that if $X_\lambda$ is a collection of Ding $\mathcal{PP}_C$-resolutions, then $\oplus_\Lambda X_\lambda$ is also a Ding $\mathcal{PP}_C$-resolution. In fact, for any flat $R$-module $F$ there is an isomorphism

$$\mathrm{Hom}_R(\oplus_\Lambda X_\lambda, C \otimes_R F) \cong \prod_\Lambda \mathrm{Hom}_R(X_\lambda, C \otimes_R F).$$

Thus, if the complex $\mathrm{Hom}_R(X_\lambda, C \otimes_R F)$ is exact for all $\lambda$, then so is the complex $\mathrm{Hom}_R(\oplus_\Lambda X_\lambda, C \otimes_R F)$. $\square$

Recall that a class of modules is called *projectively resolving* if it is closed under extensions, kernels of surjections and it contains all projective modules.

**Theorem 1.12.** *The class $\mathcal{DP}_C(R)$ is projectively resolving and closed under direct summands.*

*Proof.* By Propositions 1.8 and 1.10, it suffices to show that the class $\mathcal{DP}_C(R)$ is closed under kernels of surjections. Consider an exact sequence of $R$-modules $0 \to M' \to M \to M'' \to 0$ with $M$, $M'' \in \mathcal{DP}_C(R)$. Let $P$ and $P''$ be Ding $\mathcal{PP}_C$-resolutions of $M$ and $M''$, respectively. "Comparison Lemma" for resolutions (see [12, Prop. 1.8] and [17, Prop. 6.9]) provides a morphism of chain complexes $\varphi : P \to P''$ inducing $M \to M''$ on the degree 0 cokernels. By adding complexes of the form $0 \to P''_i \xrightarrow{id} P''_i \to 0$ for $i \geqslant 0$ and $0 \to C \otimes_R P''_i \xrightarrow{id} C \otimes_R P''_i \to 0$ for $i < 0$ to $P$, one can assume $\varphi$ is surjective. Since both the class $\mathcal{P}(R)$ and $\mathcal{P}_C(R)$ are closed under kernels of epimorphisms, see [13, Cor. 6.4], the complex $P' = \mathrm{Ker}(\varphi)$ has the form

$$P' = \cdots \to P'_1 \to P'_0 \to C \otimes_R P'_{-1} \to C \otimes_R P'_{-2} \to \cdots$$

with $P'_i \in \mathcal{P}(R)$ for any $i \in \mathbb{Z}$. The exact sequence $0 \to P' \to P \to P'' \to 0$ is degreewise split by Lemma 1.9. So an argument similar to that of the previous Proposition 1.10 implies that $P'$ is a Ding $\mathcal{PP}_C$-resolution and $M'$ is $D_C$-projective.

Since the class $\mathcal{DP}_C(R)$ is projectively resolving and closed under arbitrary direct sums by Proposition 1.11, it follows from Eilenberg's swindle [12, 1.4] that it is also closed under direct summands. $\square$

**Proposition 1.13.** *Every cokernel in a Ding $\mathcal{PP}_C$-resolution is $D_C$-projective.*

*Proof.* Consider a Ding $\mathcal{PP}_C$-resolution

$$X = \cdots \xrightarrow{\partial_2^X} P_1 \xrightarrow{\partial_1^X} P_0 \xrightarrow{\partial_0^X} C \otimes_R P_{-1} \xrightarrow{\partial_{-1}^X} C \otimes_R P_{-2} \xrightarrow{\partial_{-2}^X} \cdots .$$

Set $M = \mathrm{Coker}\partial_1^X$ and $K = \mathrm{Coker}\partial_2^X$. Since $M$, $P_0 \in \mathcal{DP}_C(R)$, the exact sequence $0 \to K \to P_0 \to M \to 0$ shows that $K \in \mathcal{DP}_C(R)$, see Theorem 1.12. Inductively, one can show that $\mathrm{Coker}\partial_i^X \in \mathcal{DP}_C(R)$ for every $i \geqslant 1$.

Set $N_1 = M$, $N_0 = \mathrm{Coker}\partial_0^X$ and $N_i = \mathrm{Coker}\partial_i^X$ for $i \leqslant -1$. Using Proposition 1.4, we will be done once we verify that $\mathrm{Ext}_R^{\geqslant 1}(N_i, C \otimes_R F) = 0$ for any flat $R$-module $F$ for all $i \leqslant 0$. Consider the exact sequence

$$Y_i = \ 0 \to N_i \to C \otimes_R P_{i-2} \to N_{i-1} \to 0.$$

By induction, one has $\mathrm{Ext}_R^{\geqslant 1}(N_i, C \otimes_R F) = 0$. Proposition 1.8 implies that $C \otimes_R P_i \in \mathcal{DP}_C(R)$ for each $i \leqslant -1$, so $\mathrm{Ext}_R^{\geqslant 1}(C \otimes_R P_i, C \otimes_R F) = 0$. The long exact sequence in $\mathrm{Hom}_R(-, C \otimes_R F)$ associated to $Y_i$ provides $\mathrm{Ext}_R^{\geqslant 2}(N_{i-1}, C \otimes_R F) = 0$. Furthermore, since $\mathrm{Hom}_R(X, C \otimes_R F)$ is exact, so is the complex $\mathrm{Hom}_R(Y_i, C \otimes_R F)$. Since $\mathrm{Ext}_R^1(C \otimes_R P_i, C \otimes_R F) = 0$, we have $\mathrm{Ext}_R^1(N_{i-1}, C \otimes_R F) = 0$ by "Five Lemma". □

**Corollary 1.14.** *An $R$-module $M \in \mathcal{DP}_C(R)$ if and only if there exists an exact sequence of $R$-modules $0 \to M \to C \otimes_R P \to N \to 0$ such that $P \in \mathcal{P}(R)$ and $N \in \mathcal{DP}_C(R)$.*

*Proof.* Just use Proposition 1.8, Theorem 1.12 and Proposition 1.13. □

Holm proved in [12] that if $0 \to M \to N \to L \to 0$ is a short exact sequence of $R$-modules with $M$, $N \in \mathcal{GP}(R)$, then $L \in \mathcal{GP}(R)$ if and only if $\mathrm{Ext}_R^1(L, P) = 0$ for all projective $R$-modules $P$. Using the same methods, Mahdou and Tamekkante in [15] have proved the similar result holds for Ding projective modules. In the following, we will give a new proof to $D_C$-projective modules.

**Corollary 1.15.** *Let $0 \to M \to N \to L \to 0$ be a short exact sequence of $R$-modules. If $M$, $N \in \mathcal{DP}_C(R)$, then $L \in \mathcal{DP}_C(R)$ if and only if $\mathrm{Ext}_R^1(L, C \otimes_R F) = 0$ for all flat $R$-modules $F$.*

*Proof.* The necessity follows from Proposition 1.4. We now prove the sufficiency. Let

$$0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0$$

be a short exact sequence of $R$-modules with $M$, $N \in \mathcal{DP}_C(R)$, and $\mathrm{Ext}_R^1(L, C \otimes_R F) = 0$ for all flat $R$-modules $F$. Then there exist $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact exact sequences of $R$-modules

$$\mathcal{X} =: \quad 0 \to M \to C \otimes_R P^0 \to C \otimes_R P^1 \to \cdots,$$
$$\mathcal{Y} =: \quad 0 \to N \to C \otimes_R Q^0 \to C \otimes_R Q^1 \to \cdots$$

with $P^i$, $Q^i \in \mathcal{P}(R)$ for $i \geqslant 0$. So the homomorphism $f : M \to N$ can be lifted to a chain map $\alpha : \mathcal{X} \to \mathcal{Y}$ and the mapping cone $\mathcal{D}$ of $\alpha : \mathcal{X} \to \mathcal{Y}$ is exact by [7, Prop. 1.4.14]. Also, the sequence $\mathcal{D}$ of $R$-modules is $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact since both $\mathcal{X}$ and $\mathcal{Y}$ are so. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
\mathcal{W} =: & 0 \to & M & = & M & \to & 0 & \to & \cdots \\
\downarrow & & \| & & \downarrow & & \downarrow & & \\
\mathcal{D} =: & 0 \to & M & \to & N \oplus (C \otimes_R P^0) & \to & C \otimes_R (Q^0 \oplus P^1) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \| & & \\
\mathcal{Z} =: & 0 \to & 0 & \to & K & \to & C \otimes_R (Q^0 \oplus P^1) & \to & \cdots,
\end{array}$$

where $K = \mathrm{Coker}(M \to N \oplus (C \otimes_R P^0))$. Since the sequence $\mathcal{D}$ is $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact, breaking $\mathcal{D}$ into short exact ones, we have that the sequence $0 \to M \to N \oplus (C \otimes_R P^0) \to K$ is also $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact. Clearly, the sequence

$$0 \to \mathcal{W} \to \mathcal{D} \to \mathcal{Z} \to 0$$

is exact. Since both $\mathcal{W}$ and $\mathcal{D}$ are exact, that $\mathcal{Z}$ is exact. Moreover, $\mathcal{Z}$ is $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact since both $\mathcal{W}$ and $\mathcal{D}$ are so.

It is straightforward to see that there exists a homomorphism $h : K \to L$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N \oplus (C \otimes_R P^0) & \longrightarrow & K & \longrightarrow & 0 \\ & & \Vert & & \downarrow \pi & & \downarrow h & & \\ 0 & \longrightarrow & M & \xrightarrow{\ f\ } & N & \xrightarrow{\ g\ } & L & \longrightarrow & 0, \end{array}$$

where $\pi : N \oplus (C \otimes_R P^0) \to N$ is the canonical projection. By "Five Lemma", we get that $h$ is an epimorphism. By "Snake Lemma", we have $\mathrm{Ker}(h) \cong \mathrm{Ker}(\pi) = C \otimes_R P^0$. Thus the sequence of $R$-modules

$$0 \to C \otimes_R P^0 \to K \xrightarrow{h} L \to 0$$

is exact. Moreover, this short exact sequence splits by assumption that $\mathrm{Ext}_R^1(L, C \otimes_R P^0) = 0$. Hence $K \cong L \oplus (C \otimes_R P^0)$. On the other hand, one can check that $\mathrm{Ext}_R^i(L, C \otimes_R F) = 0$ for all flat $R$-modules $F$ and all $i \geqslant 1$, and so $\mathrm{Ext}_R^i(K, C \otimes_R F) \cong \mathrm{Ext}^i(L \oplus (C \otimes_R P^0), C \otimes_R F) = 0$ for all flat $R$-modules $F$ and all $i \geqslant 1$, thus the projective resolution

$$\mathcal{K} = \quad \cdots \to P_1 \to P_0 \to K \to 0$$

of $K$ is $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact. Assembling the sequences $\mathcal{K}$ and $\mathcal{Z}$, we get the Ding $\mathcal{PP}_C$-resolution of $K$, and so $K \cong L \oplus (C \otimes_R P^0)$ is $D_C$-projective. Then, $L$ is $D_C$-projective by Theorem 1.12, as desired. □

## 2. $D_C$-projective dimensions of modules

In this section, we investigate some properties of $D_C$-projective dimensions of modules. To prove the main result, we need the following three results.

**Lemma 2.1.** *Let* $0 \to H \to D_1 \xrightarrow{f} D_0 \to M \to 0$ *be an exact sequence of* $R$*-modules with* $D_0$, $D_1 \in \mathcal{DP}_C(R)$. *Then the following conclusions hold*:

(1) *We have the following exact sequences of* $R$*-modules*:

$$0 \to H \to W \to Q \to M \to 0, \tag{1}$$

*and*

$$0 \to H \to C \otimes_R P \to V \to M \to 0 \tag{2}$$

*such that* $P$, $Q \in \mathcal{P}(R)$ *and* $W$, $V \in \mathcal{DP}_C(R)$.

(2) *If the exact sequence* $0 \to H \to D_1 \xrightarrow{f} D_0 \to M \to 0$ *is* $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$*-exact, then so are* (1) *and* (2).

*Proof.* (1) Since $D_0 \in \mathcal{DP}_C(R)$, there is an exact sequence of $R$-modules $0 \to D' \to Q \to D_0 \to 0$ with $Q \in \mathcal{P}(R)$ and $D' \in \mathcal{DP}_C(R)$. Then we have the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & D' & = & D' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & A & \to & Q & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathrm{Im} f & \to & D_0 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

And consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & H & = & H & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & D' & \to & W & \to & D_1 & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & D' & \to & A & \to & \mathrm{Im} f & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Both $D_1$ and $D'$ are $D_C$-projective, then so is $W$ by Proposition 1.10. Connecting the middle column in the second diagram and the middle row in the first diagram, we get the first desired exact sequence.

Dually, since $D_1 \in \mathcal{DP}_C(R)$, there is an exact sequence of $R$-modules $0 \to D_1 \to C \otimes_R P \to D'' \to 0$ with $P \in \mathcal{P}(R)$ and $D'' \in \mathcal{DP}_C(R)$. Then we have the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & H & \to & D_1 & \to & \mathrm{Im} f & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & H & \to & C \otimes_R P & \to & B & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & D'' & = & D'' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

And consider the following pullback diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{Im}f & \longrightarrow & D_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & B & \longrightarrow & V & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & D'' & = & D'' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}
$$

Since both $D_0$ and $D''$ are $D_C$-projective, so is $V$ by Proposition 1.10. Connecting the middle rows in the above two diagrams, so we get the second desired exact sequence.

(2) Let $F$ be any flat $R$-module. Note that $\mathrm{Ext}_R^{\geqslant 1}(G, C\otimes_R F) = 0$ for any $D_C$-projective $R$-module $G$. If the exact sequence $0 \to H \to D_1 \xrightarrow{f} D_0 \to M \to 0$ is $\mathrm{Hom}_R(-,\mathcal{F}_C(R))$-exact, then $\mathrm{Ext}_R^1(M, C\otimes_R F) = 0 = \mathrm{Ext}_R^2(M, C\otimes_R F)$ and $\mathrm{Ext}_R^1(\mathrm{Im}f, C\otimes_R F) = 0$. So in the proof of (1), both $\mathrm{Ext}_R^1(A, C\otimes_R F) = 0$ and $\mathrm{Ext}_R^1(B, C\otimes_R F) = 0$. Hence the exact sequences (1) and (2) are $\mathrm{Hom}_R(-,\mathcal{F}_C(R))$-exact. This completes the proof. □

**Proposition 2.2.** *Let $n$ be a positive integer and*

$$0 \to H \to D_{n-1} \to D_{n-2} \to \cdots \to D_1 \to D_0 \to M \to 0 \tag{3}$$

*an exact sequence of $R$-modules with all $D_i \in \mathcal{DP}_C(R)$. Then we have the following*:

(1) *There exist exact sequences of $R$-modules*

$$0 \to H \to C\otimes_R P_{n-1} \to C\otimes_R P_{n-2} \to \cdots \to C\otimes_R P_0 \to N \to 0 \tag{4}$$

*and $0 \to M \to N \to W \to 0$ with all $P_i \in \mathcal{P}(R)$ and $W \in \mathcal{DP}_C(R)$.*

(2) *There exist exact sequences of $R$-modules*

$$0 \to L \to Q_{n-1} \to Q_{n-2} \to \cdots \to Q_0 \to M \to 0 \tag{5}$$

*and $0 \to V \to L \to H \to 0$ with all $Q_i \in \mathcal{P}(R)$ and $V \in \mathcal{DP}_C(R)$.*

(3) *If the exact sequence* (3) *is* $\mathrm{Hom}_R(-,\mathcal{F}_C(R))$*-exact, then so are* (4) *and* (5).

*Proof.* We use an induction argument on $n$.

(1) If $n = 1$, then we have an exact sequence of $R$-modules $0 \to H \to D_0 \to M \to 0$. Since $D_0 \in \mathcal{DP}_C(R)$, there is a $\mathrm{Hom}_R(-,\mathcal{F}_C(R))$-exact exact sequence of $R$-modules $0 \to D_0 \to C\otimes_R P_0 \to W \to 0$ with $P_0 \in \mathcal{P}(R)$ and

$W \in \mathcal{DP}_C(R)$. Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H & \longrightarrow & D_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H & \longrightarrow & C \otimes_R P_0 & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & W & = & W & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

The middle row and the rightmost column in the above diagram are the desired two exact sequences.

Now suppose that $n \geqslant 2$ and we have an exact sequence of $R$-modules $0 \to H \to D_{n-1} \to D_{n-2} \to \cdots \to D_0 \to M \to 0$ with $D_i \in \mathcal{DP}_C(R)$ for $0 \leqslant i \leqslant n-1$. Set $K = \mathrm{Coker}(D_{n-1} \to D_{n-2})$. By Lemma 2.1, we get an exact sequence of $R$-modules

$$0 \to H \to C \otimes_R P_{n-1} \to D'_{n-2} \to K \to 0 \tag{6}$$

with $P_{n-1} \in \mathcal{P}(R)$ and $D'_{n-2} \in \mathcal{DP}_C(R)$. Put $H' = \mathrm{Im}(C \otimes_R P_{n-1} \to D'_{n-2})$. Then we have an exact sequence of $R$-modules

$$0 \to H' \to D'_{n-2} \to D_{n-3} \to \cdots \to D_0 \to M \to 0.$$

So, by the induction hypothesis, we get the assertion.

(2) When $n = 1$, we have an exact sequence of $R$-modules $0 \to H \to D_0 \to M \to 0$. Since $D_0 \in \mathcal{DP}_C(R)$, there is a $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact exact sequence of $R$-modules $0 \to V \to Q_0 \to D_0 \to 0$ with $Q_0 \in \mathcal{P}(R)$ and $V \in \mathcal{DP}_C(R)$, then we have the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & V & = & V & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L & \longrightarrow & Q_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & H & \longrightarrow & D_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

The middle row and the leftmost column in the above diagram are the desired two exact sequences.

Now suppose that $n \geqslant 2$ and we have an exact sequence of $R$-modules $0 \to H \to D_{n-1} \to D_{n-2} \to \cdots \to D_0 \to M \to 0$ with $D_i \in \mathcal{DP}_C(R)$ for

$0 \leqslant i \leqslant n-1$. Set $K = \mathrm{Ker}(D_1 \to D_0)$. By Lemma 2.1, we get an exact sequence of $R$-modules

$$0 \to K \to D'_1 \to Q_0 \to M \to 0 \tag{7}$$

with $Q_0 \in \mathcal{P}(R)$ and $D'_1 \in \mathcal{DP}_C(R)$. Put $M' = \mathrm{Im}(D'_1 \to Q_0)$. Then we have an exact sequence of $R$-modules

$$0 \to H \to D_{n-1} \to \cdots \to D_2 \to D'_1 \to M' \to 0.$$

So, by the induction hypothesis, we get the assertion.

(3) If the exact sequence (3) is $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact, then the middle rows in the above two commutative diagrams are also $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact. On the other hand, we can choose both (6) and (7) to be $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact by Lemma 2.1. Then we get the assertion by the induction hypothesis. □

Here is a version of Schanuel's Lemma for $\mathcal{DP}_C$-resolutions.

**Proposition 2.3.** *Let $M$ be an $R$-module. Consider two exact sequences,*

$$0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0,$$
$$0 \to \widetilde{K}_n \to \widetilde{P}_{n-1} \to \cdots \to \widetilde{P}_1 \to \widetilde{P}_0 \to M \to 0,$$

*where each $P_i$, $\widetilde{P}_i \in \mathcal{DP}_C(R)$. Then $K_n \in \mathcal{DP}_C(R)$ if and only if $\widetilde{K}_n \in \mathcal{DP}_C(R)$.*

*Proof.* Since the class $\mathcal{DP}_C(R)$ is projectively resolving and closed under arbitrary sums, direct summands by Theorem 1.12, the stated result is a direct consequence of [2, Lem. 3.12]. □

The main result of this section gives both functorial descriptions of the $D_C$-projective dimensions of modules and some criterions for computing the $D_C$-projective dimensions of modules.

**Theorem 2.4.** *Let $M$ be an $R$-module with finite $D_C$-projective dimension and let $n$ be an integer. Then the following conditions are equivalent*:

1. $\mathcal{DP}_C\text{-pd}_R(M) \leqslant n$.
2. *$\mathrm{Ext}^i_R(M, L) = 0$ for all $i > n$ and all $R$-modules $L$ with $\mathcal{F}_C\text{-pd}_R(L) < \infty$.*
3. *$\mathrm{Ext}^i_R(M, C \otimes_R F) = 0$ for all $i > n$ and all flat $R$-modules $F$.*
4. *There is an exact sequence $0 \to K_n \to D_{n-1} \to \cdots \to D_0 \to M \to 0$ such that $D_i \in \mathcal{DP}_C(R)$ for $0 \leqslant i \leqslant n-1$ and $K_n \in \mathcal{DP}_C(R)$.*
5. *For every non-negative integer $t$ such that $0 \leqslant t \leqslant n$, there is an exact sequence of $R$-modules $0 \to Q_n \to Q_{n-1} \to \cdots \to Q_{t+1} \to D \to Q_{t-1} \to \cdots \to Q_0 \to M \to 0$ such that $D \in \mathcal{DP}_C(R)$ and $Q_i \in \mathcal{P}(R) \cup \mathcal{P}_C(R)$ for $0 \leqslant i \leqslant n$, $i \neq t$.*

*Consequently, the $\mathcal{DP}_C$-projective dimension of $M$ is determined by the formulas:*

$$\mathcal{DP}_C\text{-pd}_R(M) \quad = \quad \sup\{\ i \in \mathbb{N}_0 \mid \exists\ L \in \overline{\mathcal{F}_C(R)} : \mathrm{Ext}^i_R(M, L) \neq 0\}$$

$$= \sup\{\ i \in \mathbb{N}_0 \mid \exists\ Q \in \mathcal{F}_C(R) : \mathrm{Ext}^i_R(M, Q) \neq 0\}.$$

*Proof.* Note that the equivalence of (1) and (4) is simply obtained by Proposition 2.3 and by the definition of the $\mathcal{DP}_C$-projective dimension. Then it remain to prove the equivalences of (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5).

(1) $\Rightarrow$ (2). We assume that $\mathcal{DP}_C\text{-pd}_R(M) \leqslant n$. By definition there is an exact sequence $0 \to D_n \to \cdots \to D_0 \to M \to 0$ with $D_i \in \mathcal{DP}_C(R)$ for $0 \leqslant i \leqslant n$. By dimension shifting and Proposition 1.4, for any $R$-module $L$ with finite $\mathcal{F}_C$-projective dimension, we have $\mathrm{Ext}^i_R(M, L) \cong \mathrm{Ext}^{i-n}_R(D_n, L) = 0$ for all $i > n$.

(2) $\Rightarrow$ (3) and (5) $\Rightarrow$ (1) are obvious.

(3) $\Rightarrow$ (1). Since $\mathcal{DP}_C\text{-pd}_R(M) < \infty$ and by Proposition 2.3, we may pick, for some positive integer $m > n$, an exact sequence

$$0 \to D_m \to D_{m-1} \to \cdots \to D_0 \to M \to 0,$$

where $D_i \in \mathcal{DP}_C(R)$ for $0 \leqslant i \leqslant m$. Set $K_n = \mathrm{Ker}(D_{n-1} \to D_{n-2})$. Our aim to prove that $K_n \in \mathcal{DP}_C(R)$. We decompose the sequence $0 \to D_m \to \cdots \to D_n \to K_n \to 0$ into short exact sequences $0 \to H_{i+1} \to D_i \to H_i \to 0$ for $i = n, \ldots, m-1$, where $H_n = K_n$ and $H_m = D_m$. Consider the short exact sequence

$$0 \to H_m(= D_m) \to D_{m-1} \to H_{m-1} \to 0.$$

We claim that $H_{m-1} \in \mathcal{DP}_C(R)$. By the exact sequence $0 \to H_{m-1} \to D_{m-2} \to \cdots \to D_0 \to M \to 0$, using assumption and dimension shifting, we have $\mathrm{Ext}^i_R(H_{m-1}, C \otimes_R F) \cong \mathrm{Ext}^{i+m-1}_R(M, C \otimes_R F) = 0$ for every integer $i > 0$ and every flat $R$-module $F$. Thus, by Corollary 1.15, $H_{m-1} \in \mathcal{DP}_C(R)$. Now we repeat successively this last argument to conclude that $H_{m-2}, \ldots, H_n = K_n$ are $D_C$-projective.

(1) $\Rightarrow$ (5) We proceed by induction on $n$. Suppose that $\mathcal{DP}_C\text{-pd}_R(M) \leqslant 1$. Then there exists an exact sequence of $R$-modules $0 \to D_1 \to D_0 \to M \to 0$ with $D_0, D_1 \in \mathcal{DP}_C(R)$. By Lemma 2.1 with $H = 0$, we get the exact sequences of $R$-modules $0 \to C \otimes_R P_1 \to D_0' \to M \to 0$ and $0 \to D_1' \to P_0 \to M \to 0$ with $P_0, P_1 \in \mathcal{P}(R)$ and $D_0', D_1' \in \mathcal{DP}_C(R)$.

Now suppose $n \geqslant 2$. Then there exists an exact sequence of $R$-modules $0 \to D_n \to D_{n-1} \to \cdots \to D_0 \to M \to 0$ where $D_i \in \mathcal{DP}_C(R)$ for $0 \leqslant i \leqslant n$. Set $H = \mathrm{Coker}(D_3 \to D_2)$. By applying Lemma 2.1 to the exact sequence $0 \to H \to D_1 \to D_0 \to M \to 0$, we have an exact sequence $0 \to D_n \to \cdots \to D_2 \to D_1' \to P_0 \to M \to 0$ with $D_1' \in \mathcal{DP}_C(R)$ and $P_0 \in \mathcal{P}(R)$. Put $N = \mathrm{Coker}(D_2 \to D_1')$. Then we have $\mathcal{DP}_C\text{-pd}_R(N) \leqslant n-1$. By the induction hypothesis, there exists an exact sequence of $R$-modules $0 \to Q_n \to \cdots \to Q_{t+1} \to D_t \to Q_{t-1} \to \cdots \to Q_1 \to P_0 \to M \to 0$ such that $P_0 \in \mathcal{P}(R)$, $D_t \in \mathcal{DP}_C(R)$ and $Q_i \in \mathcal{P}(R) \cup \mathcal{P}_C(R)$ for $i \neq t$ and $1 \leqslant t \leqslant n$.

Now we need only to prove (5) for $t = 0$. Set $B = \mathrm{Coker}(D_2 \to D_1)$. By the induction hypothesis, we have an exact sequence $0 \to Q_n \to \cdots \to Q_2 \to D_1' \to B \to 0$ with $Q_i \in \mathcal{P}(R) \cup \mathcal{P}_C(R)$ and $D_1' \in \mathcal{DP}_C(R)$ for $2 \leqslant i \leqslant n$. Set

$G = \mathrm{Coker}(Q_3 \to Q_2)$. Then by applying Lemma 2.1 to the exact sequence $0 \to G \to D'_1 \to D_0 \to M \to 0$, we have an exact sequence $0 \to G \to C \otimes_R P_1 \to D'_0 \to M \to 0$ with $P_1 \in \mathcal{P}(R)$ and $D'_0 \in \mathcal{DP}_C(R)$. Thus we obtain the desired exact sequence $0 \to Q_n \to \cdots \to Q_2 \to Q_1 \to D'_0 \to M \to 0$ with $Q_i \in \mathcal{P}(R) \cup \mathcal{P}_C(R)$ and $D'_0 \in \mathcal{DP}_C(R)$.

The last formulas in the theorem for determination of $\mathcal{DP}_C\text{-pd}_R(M)$ are a direct consequence of the equivalence between (1)-(3). $\square$

The next result shows that the class of $R$-modules of finite $\mathcal{DP}_C$-projective dimension satisfies the two-of-three property.

**Proposition 2.5.** *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of $R$-modules. If two of $\mathcal{DP}_C\text{-pd}_R(M')$, $\mathcal{DP}_C\text{-pd}_R(M)$ and $\mathcal{DP}_C\text{-pd}_R(M'')$ are finite, so is the third.*

*Proof.* Let $P'$ and $P''$ be projective resolutions of $M'$ and $M''$, respectively. For each $n \geqslant 0$, the horseshoe lemma yields a commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M'_n & \to & M_n & \to & M''_n & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P'_{n-1} & \to & P'_{n-1} \oplus P''_{n-1} & \to & P''_{n-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P'_0 & \to & P'_0 \oplus P''_0 & \to & P''_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M' & \to & M & \to & M'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Assume that $\mathcal{DP}_C\text{-pd}_R(M)$, $\mathcal{DP}_C\text{-pd}_R(M'') \leqslant n$. Then Theorem 2.4 implies that $M_n$, $M''_n \in \mathcal{DP}_C(R)$. From the top row of the diagram, we conclude that $M'_n \in \mathcal{DP}_C(R)$ by Theorem 1.12. Hence, the first column of the diagram is a bounded augmented $\mathcal{DP}_C(R)$-resolution of $M'$, so we have $\mathcal{DP}_C\text{-pd}_R(M') \leqslant n$.

A similar argument shows that, if $\mathcal{DP}_C\text{-pd}_R(M')$, $\mathcal{DP}_C\text{-pd}_R(M'') \leqslant n$, then $\mathcal{DP}_C\text{-pd}_R(M) \leqslant n$.

Assume that $\mathcal{DP}_C\text{-pd}_R(M')$, $\mathcal{DP}_C\text{-pd}_R(M) \leqslant n$. Again, it follows that $M'_n$, $M_n \in \mathcal{DP}_C(R)$. Thus, the top row of the diagram shows that $\mathcal{DP}_C\text{-pd}_R(M''_n) \leqslant 1$. Furthermore, by combing the top row and the rightmost column of this diagram, we obtain an exact sequence

$$0 \to M'_n \to M_n \to P''_{n-1} \to \cdots \to M'' \to 0.$$

This is an augmented $\mathcal{DP}_C(R)$-resolution of $M''$ of length $n+1$, so we conclude that $\mathcal{DP}_C$-$\mathrm{pd}_R(M'') \leqslant n+1$. □

When $C = R$, it was proved in [5, Lem. 2.4(2)] that if $M$ is a Ding projective $R$-module of finite flat dimension, then $M$ is projective. Naturally, it makes sense to give the relation between $\mathcal{P}_C$-projective dimensions and $\mathcal{DP}_C$-projective dimensions.

**Proposition 2.6.** *If $M$ is an $R$-module of finite $\mathcal{F}_C$-projective dimension, then*

$$\mathcal{DP}_C\text{-}\mathrm{pd}_R(M) = \mathcal{P}_C\text{-}\mathrm{pd}_R(M).$$

*In particular, there is an equality of classes $\mathcal{DP}_C \bigcap \overline{\mathcal{F}_C(R)} = \mathcal{P}_C(R)$.*

*Proof.* Using Theorem 2.4, it suffices to show that if $M$ is $D_C$-projective with $\mathcal{F}_C$-$\mathrm{pd}_R(M) < \infty$, then $M$ is $C$-projective. To this end, consider an exact sequence of the form

$$0 \to K \to C \otimes_R P \to M \to 0,$$

where $P \in \mathcal{P}(R)$ and $\mathcal{F}_C$-$\mathrm{pd}_R(K) < \infty$. By Theorem 2.4, $\mathrm{Ext}_R^{\geqslant 1}(M,K) = 0$, so the above sequence splits, forcing $M$ to be a summand of $C \otimes_R P$. Since the class $\mathcal{P}_C(R)$ is closed under summands by [13, Prop. 5.5], $M \in \mathcal{P}_C(R)$, as desired. □

We complete this article with the following application of Lemma 2.1. We denote $\mathcal{DP}_C^2(R)$ the class of $R$-modules $M$ for which there exists an exact sequence of $D_C$-projective $R$-modules

$$X = \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \xrightarrow{\partial_0^X} X_{-1} \xrightarrow{\partial_{-1}^X} \cdots$$

such that the complex $\mathrm{Hom}_R(X,F)$ is exact for each flat $R$-module $F$ and $M \cong \mathrm{Coker}\partial_1^X$.

The following result shows that an iteration of the procedure used to define the $D_C$-projective $R$-modules yields exactly the $D_C$-projective $R$-modules.

**Theorem 2.7.** $\mathcal{DP}_C^2(R) = \mathcal{DP}_C(R)$.

*Proof.* One checks readily that there is a containment $\mathcal{DP}_C(R) \subseteq \mathcal{DP}_C^2(R)$. Now, we prove the converse containment.

Let $M \in \mathcal{DP}_C^2(R)$. By definition we have an exact sequence of $D_C$-projective $R$-modules

$$X = \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \xrightarrow{\partial_0^X} X_{-1} \xrightarrow{\partial_{-1}^X} \cdots$$

such that the complex $\mathrm{Hom}_R(X, C \otimes_R F)$ is exact for each flat $R$-module $F$ and $M \cong \mathrm{Coker}\partial_1^X$. So $\mathrm{Ext}_R^i(M, C \otimes_R F) = 0$ for any flat $R$-module $F$ and any $i \geqslant 1$.

Set $M_i = \mathrm{Coker}\partial_{i+1}^X$ for any $i \in \mathbb{Z}$. By Lemma 2.1(1), there exists exact sequences of $R$-modules

$$0 \to M \to C \otimes_R P_{-1} \to N_{-1} \to 0,$$

and

$$0 \to M_{-1} \to N_{-1} \to D \to 0$$

such that $P_{-1} \in \mathcal{P}(R)$ and $D \in \mathcal{DP}_C(R)$ with the former one is $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact. Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & M_{-1} & \to & N_{-1} & \to & D & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & X_{-2} & \to & D' & \to & D & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & M_{-2} & = & M_{-2} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

Since $X_{-2}$, $D \in \mathcal{DP}_C(R)$, the middle row with Theorem 1.12 yields $D' \in \mathcal{DP}_C(R)$. Note that the leftmost column is $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact, then $\mathrm{Ext}^1_R(M_{-2}, C \otimes_R F) = 0$ for any flat $R$-module $F$. Hence, the middle column is also $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact, and so we have a $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact exact sequence of $R$-modules

$$0 \to N_{-1} \to D' \to X_{-3} \to X_{-4} \to \cdots .$$

Then by the same argument, we obtain $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact exact sequences of $R$-modules

$$0 \to N_{-1} \to C \otimes_R P_{-2} \to N_{-2} \to 0,$$

and

$$0 \to N_{-2} \to D'' \to X_{-4} \to X_{-5} \to \cdots .$$

By iterating the above argument, we have a $\mathrm{Hom}_R(-, \mathcal{F}_C(R))$-exact exact sequence of $R$-modules

$$0 \to M \to C \otimes_R P_{-1} \to C \otimes_R P_{-2} \to \cdots ,$$

in which each $P_i \in \mathcal{P}(R)$. So $M \in \mathcal{DP}_C(R)$ by Proposition 1.4, and the desired conclusion $\mathcal{DP}^2_C(R) \subseteq \mathcal{DP}_C(R)$ is obtained. □

**Acknowledgements.** The authors would like to thank the referees for helpful suggestions and corrections that have improved this article.

## References

[1] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer, Berlin, 1992.

[2] M. Auslander and M. Bridger, *Stable Module Theory*, Memoirs of the American Mathematical Society, No. 94 American Mathematical Society, Providence, R.I. 1969.

[3] L. W. Christensen, *Gorenstein Dimensions*, Springer, Berlin, 2000.

[4] L. W. Christensen, A. Frankild, and H. Holm, *On Gorenstein projective, injective and flat dimensions—a functorial description with applications*, J. Algebra **302** (2006), no. 1, 231–279.
[5] N. Ding, Y. Li, and L. Mao, *Strongly Gorenstein flat modules*, J. Aust. Math. Soc. **86** (2009), no. 3, 323–338.
[6] N. Ding and L. Mao, *Gorenstein FP-injective and Gorenstein flat modules*, J. Algebra Appl. **7** (2008), no. 4, 491–506.
[7] E. E. Enochs and O. M. G. Jenda, *Relative Homological Algebra*, Walter de Gruyter, Berlin, 2000.
[8] H.-B. Foxby, *Gorenstein modules and related modules*, Math. Scand. **31** (1972), 267–284.
[9] Y. Geng and N. Ding, *$W$-Gorenstein modules*, J. Algebra **325** (2011), 132–146.
[10] J. Gillespie, *Model structures on modules over Ding-Chen rings*, Homology, Homotopy Appl. **12** (2010), no. 1, 61–73.
[11] E. S. Golod, *G-dimension and generalized perfect ideals*, Algebraic geometry and its applications, Trudy Mat. Inst. Steklov. **165** (1984), 62–66.
[12] H. Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra **189** (2004), no. 1-3, 167–193.
[13] H. Holm and D. White, *Foxby equivalence over associative rings*, J. Math. Kyoto Univ. **47** (2007), no. 4, 781–808.
[14] H. Holm and Jørgensen, *Semi-dualizing modules and related Gorenstein homological dimensions*, J. Pure Appl. Algebra **205** (2006), no. 2, 423–445.
[15] N. Mahdou and M. Tamekkante, *Strongly Gorenstein flat modules and dimensions*, Chin. Ann. Math. Ser. B **32** (2011), no. 4, 533–548.
[16] K. Pinzon, *Absolutely pure covers*, Comm. Algebra **36** (2008), no. 6, 2186–2194.
[17] J. J. Rotman, *An Introductions to Homological Algebra*, Academic Press, New York, 1979.
[18] W. V. Vasconcelos, *Divisor Theory in Module Categories*, North-Holland Publishing Co., Amsterdam, 1974.
[19] D. White, *Gorenstein projective dimension with respect to a semidualizing module*, J. Commut. Algebra **2** (2010), no. 1, 111–137.

Chunxia Zhang
Department of Mathematics
Northwest Normal University
Lanzhou, 730070, P. R. China
*E-mail address*: zhangcx@nwnu.edu.cn

Limin Wang
Department of Mathematics
Northwest Normal University
Lanzhou, 730070, P. R. China
*E-mail address*: wanglm@nwnu.edu.cn

Zhongkui Liu
Department of Mathematics
Northwest Normal University
Lanzhou, 730070, P. R. China
*E-mail address*: liuzk@nwnu.edu.cn