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ON A SEQUENCE OF KANTOROVICH TYPE OPERATORS VIA RIEMANN TYPE q-INTEGRAL

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Abstract. In this work, we construct Kantorovich type generalization of a class of linear positive operators via Riemann type q-integral. We obtain estimations for the rate of convergence by means of modulus of continuity and the elements of Lipschitz class and also investigate weighted approximation properties.

1. Introduction

The classical Meyer-König and Zeller (MKZ) [24] operators are defined by

$$M_n(f;x)=\begin{cases}(1-x)^{n+1}\displaystyle\sum_{k=0}^{\infty}f\left(\frac{k}{n+k+1}\right)\binom{n+k}{k}x^k & ,\ x\in[0,1)\\ f(x) & ,\ x=1.\end{cases}$$

In order to give the monotonicity properties, Cheney and Sharma [8] introduced the slight modification of the MKZ operators

$$M_n^*(f;x)=\begin{cases}(1-x)^{n+1}\displaystyle\sum_{k=0}^{\infty}f\left(\frac{k}{n+k}\right)\binom{n+k}{k}x^k & ,\ x\in[0,1)\\ f(x) & ,\ x=1.\end{cases}$$

In many papers (see, for instance [1, 4, 7, 11, 13, 16, 21]) the authors studied these operators and some new generalizations of them.

Rempulska and Skorupka [27] extended the definition of MKZ operators on an unbounded set as follows:

(1.1)
$$M_n(f,b_n;x)=\begin{cases}\left(1-\frac{x}{b_n}\right)^{n+1}\displaystyle\sum_{k=0}^{\infty}f\left(\frac{k}{n+k}b_n\right)\binom{n+k}{k}\left(\frac{x}{b_n}\right)^k & ,\ x\in[0,b_n)\\ f(x) & ,\ x\geq b_n\end{cases}$$

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where $b_n \geq 1$ is an increasing sequence of real numbers having the properties

$$\lim_{n\to\infty} b_n = \infty, \lim_{n\to\infty} \frac{b_n}{n} = 0.$$

By introducing a generalization of these operators for differentiable functions the authors investigated some approximation properties in polynomial weighted spaces.

In order to give further detail, let us recall some basic concepts and definitions in q-calculus. For any fixed real number $q > 0$ and nonnegative integer r, the q-integer $[r]_q$, the q-factorial $[r]_q!$ and q-binomial coefficients are defined by (see [5])

$$[r]_q = \begin{cases} \dfrac{1-q^r}{1-q}, & \text{if } q \neq 1 \\ r, & \text{if } q = 1, \end{cases}$$

$$[r]_q! = \begin{cases} [1]_q[2]_q \cdots [r]_q, & \text{if } r \geq 1 \\ 1, & \text{if } r = 0, \end{cases}$$

and

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q![n-r]_q!}, \quad n \geq r \geq 0,$$

respectively.

Now suppose that $0 < a < b$, $0 < q < 1$ and f is a real-valued function. The q-Jackson integral of f over the interval $[0, b]$ and a general interval $[a, b]$ are defined by (see [19])

$$\int_0^b f(x) d_q x = (1-q) b \sum_{j=0}^{\infty} f\left(bq^j\right) q^j$$

and

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x,$$

respectively, provided the series converge.

As mentioned in [9], because of the q-Jackson integral of f over an interval $[a, b]$ includes two infinite sums, some problems are encountered in deriving the q-analogues of some well-known integral inequalities which are used to compute order of approximation of linear positive operators containing q-Jackson integral. To solve this problem Marinković et al. (see [23]) defined the Riemann type q-integral as

$$R_q(f; a, b) = \int_a^b f(x) d_q^R x = (1-q)(b-a) \sum_{j=0}^{\infty} f\left(a + (b-a)q^j\right) q^j$$

which contains only points within the interval of integral.
Dalmanoğlu and Doğru [9] showed that Riemann type q-integral is a linear

positive operator and satisfies the Hölder inequality

$$R_q(|fg|;a,b) \leq \left(R_q(|f|^m;a,b)\right)^{\frac{1}{m}} \left(R_q(|g|^n;a,b)\right)^{\frac{1}{n}} \tag{1.2}$$

with $\frac{1}{m}+\frac{1}{n}=1$.

The study of approximation by linear positive operators based on q-integers was firstly carried out in 1987 by Lupaş [22] and, ten years later, by Phillips [26] (see, e.g., [25]). Thereafter many authors studied q-generalization of classical linear positive operators (see, for instance [2, 3, 9, 10, 12, 14, 17, 18, 20, 28, 30]). We now mention some works related to generalization of MKZ operators based on q-integers. In 2000, Trif [30] introduced q-MKZ operators and investigated the approximation and monotonicity properties of these operators. But, it was impossible to give an explicit expression for the second moment of q-MKZ operators, Doğru and Duman [12] constructed another kind q-MKZ operators and investigated their statistical approximation properties. Then, Heping [18] derived an explicit formula in terms of q-hypergeometric series for the second moment of the q-MKZ operators defined by Trif, and discussed further approximating properties of these operators. Recently, in [3] Özarslan and Duman presented a new generalization of MKZ operators based on q-integers and obtained a Korovkin type approximation theorem for them. Very recently, Sharma [28] introduced q-MKZ Durrmeyer operators with the help of the q-Jackson integral and obtained their rate of convergence and weighted statistical approximation properties. Gupta and Sharma [17] constructed the q-MKZ Kantorovich operators by q-Jackson integral and investigated statistical approximation properties. In [10], the authors defined the Kantorovich type q-MKZ operators by means of Riemann type q-integral and studied the statistical Korovkin type approximation properties of such operators.

In [14], for $q \in (0,1)$ and every $n \in \mathbb{N}$ we proposed the following q-generalization of the operators $M_n(f, b_n; x)$ given by (1.1),

$$L^*_{n,q}(f;x) = P_{n,q}(x) \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{[n+k]_q} b_n\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k, \quad x \in [0, b_n) \tag{1.3}$$

where

$$P_{n,q}(x) = \prod_{s=0}^{n} \left(1 - q^s \frac{x}{b_n}\right) \tag{1.4}$$

and $b_n \geq 1$ is an increasing sequence of positive numbers such that

$$\lim_{n\to\infty} b_n = \infty.$$

It is clear that from the first condition, the interval $[0, b_n)$ expands infinity as $n \to \infty$. So, we investigated some approximation properties of these operators in weighted spaces of continuous functions on positive semi-axis with the help of weighted Korovkin type theorem proved by Gadjiev in [15]. We also introduced a Stancu type remainder and an application to differential equations related to q-derivatives.

2. Construction of operators

Observe that throughout this work we shall assume $q \in (0,1)$.

In this paper, for every $n \in \mathbb{N}$ we consider the Kantorovich type generalization by means of Riemann type q-integral of the operators defined by (1.3) as follows:

(2.1)
$$L_{n,q}(f;x) = P_{n,q}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k C_{n,k}(q,b_n) \int_{\frac{[k]_q}{[n+k]_q} b_n}^{\frac{[k+1]_q}{[n+k+1]_q} b_n} f(t)\, d_q^R t,\ x \in [0,b_n)$$

with

$$C_{n,k}(q,b_n) = \frac{[n+k]_q [n+k+1]_q}{q^k [n]_q b_n}.$$

In (2.1), $b_n \geq 1$ is an increasing sequence of positive real numbers such that $\lim_{n\to\infty} b_n = \infty$, $P_{n,q}(x)$ defined as in (1.4) and f is a Riemann type q-integrable function over the interval $\left[\frac{[k]_q}{[n+k]_q} b_n, \frac{[k+1]_q}{[n+k+1]_q} b_n\right]$.

By using the definition of Riemann type q-integral, it is easily verified that

$$\int_{\frac{[k]_q}{[n+k]_q} b_n}^{\frac{[k+1]_q}{[n+k+1]_q} b_n} d_q^R t = \frac{1}{C_{n,k}(q,b_n)}, \tag{2.2}$$

$$\int_{\frac{[k]_q}{[n+k]_q} b_n}^{\frac{[k+1]_q}{[n+k+1]_q} b_n} t\, d_q^R t = \frac{1}{C_{n,k}(q,b_n)} \frac{[k]_q}{[n+k]_q} b_n + \frac{1}{[2]_q} \frac{1}{C_{n,k}^2(q,b_n)}, \tag{2.3}$$

$$\begin{aligned} \int_{\frac{[k]_q}{[n+k]_q} b_n}^{\frac{[k+1]_q}{[n+k+1]_q} b_n} t^2 d_q^R t = {} & \frac{1}{C_{n,k}(q,b_n)} \frac{[k]_q^2}{[n+k]_q^2} b_n^2 + \frac{2}{[2]_q} \frac{1}{C_{n,k}^2(q,b_n)} \frac{[k]_q}{[n+k]_q} b_n \\ & + \frac{1}{[3]_q} \frac{1}{C_{n,k}^3(q,b_n)}. \end{aligned} \tag{2.4}$$

In [14], we proved that

$$\begin{aligned} & L_{n,q}^*(e_0;x) = 1, \\ & L_{n,q}^*(e_1;x) = x, \\ & x^2 \leq L_{n,q}^*(e_2;x) \leq q x^2 + \frac{b_n}{[n]_q} x, \end{aligned}$$

where $e_\upsilon(t) = t^\upsilon, \upsilon = 0,1,2$.

Now we can state the following lemma.

Lemma 1. *The operator $L_{n,q}$ defined by* (2.1) *satisfies*

$$L_{n,q}(e_0;x) = 1, \tag{2.5}$$

$$|L_{n,q}(e_1;x) - x| \le \frac{1}{[2]_q}\frac{b_n}{[n+1]_q}, \tag{2.6}$$

$$\left|L_{n,q}(e_2;x) - x^2\right| \le (1-q)x^2 + \left(\frac{b_n}{[n]_q} + \frac{2}{[2]_q}\frac{b_n}{[n+1]_q}\right)x + \frac{1}{[3]_q}\frac{b_n^2}{[n+1]_q^2} \tag{2.7}$$

for each $x \in [0, b_n)$.

Proof. By means of (2.2), we immediately see that

$$\begin{aligned} L_{n,q}(e_0;x) &= P_{n,q}(x)\sum_{k=0}^{\infty}\begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k \\ &= L^*_{n,q}(e_0;x) \\ &= 1. \end{aligned}$$

Taking into consideration (2.3), one has

$$\begin{aligned} &L_{n,q}(e_1;x) - x \\ &= P_{n,q}(x)\sum_{k=0}^{\infty}\frac{[k]_q}{[n+k]_q}b_n\begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k - x \\ &\quad + \frac{b_n}{[2]_q}P_{n,q}(x)\sum_{k=0}^{\infty}\begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k \frac{q^k[n]_q}{[n+k]_q[n+k+1]_q} \\ &= L^*_{n,q}(e_1;x) - x + \frac{b_n}{[2]_q}P_{n,q}(x)\sum_{k=0}^{\infty}\begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k \frac{q^k[n]_q}{[n+k]_q[n+k+1]_q} \end{aligned}$$

which gives

$$\begin{aligned} &|L_{n,q}(e_1;x) - x| \\ &\le \left|L^*_{n,q}(e_1;x) - x\right| + \frac{b_n}{[2]_q}P_{n,q}(x)\sum_{k=0}^{\infty}\begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k \frac{q^k[n]_q}{[n+k]_q[n+k+1]_q}. \end{aligned}$$

Since $q^k \le 1$, $\frac{[n]_q}{[n+k]_q[n+k+1]_q} \le \frac{1}{[n+1]_q}$ for $0 < q < 1$, $k = 0, 1, \ldots$, $n \in \mathbb{N}$, we may write

$$|L_{n,q}(e_1;x) - x| \le \left|L^*_{n,q}(e_1;x) - x\right| + \frac{1}{[2]_q}\frac{b_n}{[n+1]_q}L^*_{n,q}(e_0;x).$$

Use of the facts $L^*_{n,q}(e_0;x) = 1$ and $L^*_{n,q}(e_1;x) - x = 0$ yields

$$|L_{n,q}(e_1;x) - x| \le \frac{1}{[2]_q}\frac{b_n}{[n+1]_q}.$$

Now we prove (2.7). By means of (2.4), we have

$$\begin{aligned} &L_{n,q}(e_2;x) - x^2 \\ &= P_{n,q}(x)\sum_{k=0}^{\infty}\frac{[k]_q^2}{[n+k]_q^2}b_n^2\begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k - x^2 \end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{[2]_q} b_n P_{n,q}(x) \sum_{k=0}^{\infty} \frac{[k]_q}{[n+k]_q} b_n \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k \frac{q^k [n]_q}{[n+k]_q [n+k+1]_q} \\
&+\frac{b_n^2}{[3]_q} P_{n,q}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k \left(\frac{q^k [n]_q}{[n+k]_q [n+k+1]_q}\right)^2 \\
= {} & L_{n,q}^*(e_2;x) - x^2 \\
&+\frac{2}{[2]_q} b_n P_{n,q}(x) \sum_{k=0}^{\infty} \frac{[k]_q}{[n+k]_q} b_n \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k \frac{q^k [n]_q}{[n+k]_q [n+k+1]_q} \\
&+\frac{b_n^2}{[3]_q} P_{n,q}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k \left(\frac{q^k [n]_q}{[n+k]_q [n+k+1]_q}\right)^2
\end{aligned}
$$

and so

$$
\begin{aligned}
& \left|L_{n,q}(e_2;x) - x^2\right| \\
\leq {} & \left|L_{n,q}^*(e_2;x) - x^2\right| \\
&+\frac{2}{[2]_q} b_n P_{n,q}(x) \sum_{k=0}^{\infty} \frac{[k]_q}{[n+k]_q} b_n \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k \frac{q^k [n]_q}{[n+k]_q [n+k+1]_q} \\
&+\frac{b_n^2}{[3]_q} P_{n,q}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k \left(\frac{q^k [n]_q}{[n+k]_q [n+k+1]_q}\right)^2 .
\end{aligned}
$$

Again using the inequalities $q^k \leq 1$, $\frac{[n]_q}{[n+k]_q [n+k+1]_q} \leq \frac{1}{[n+1]_q}$ for $0 < q < 1$, $k = 0, 1, \ldots$, $n \in \mathbb{N}$, we get

$$
\begin{aligned}
& \left|L_{n,q}(e_2;x) - x^2\right| \\
\leq {} & \left|L_{n,q}^*(e_2;x) - x^2\right| + \frac{2}{[2]_q} \frac{b_n}{[n+1]_q} L_{n,q}^*(e_1;x) + \frac{1}{[3]_q} \frac{b_n^2}{[n+1]_q^2} L_{n,q}^*(e_0;x).
\end{aligned}
$$

Finally, by means of the facts $L_{n,q}^*(e_0;x) = 1$, $L_{n,q}^*(e_1;x) = x$ and $\left|L_{n,q}^*(e_2;x) - x^2\right| \leq (1-q)x^2 + \frac{b_n}{[n]_q} x$, we arrive at the desired result. $\square$

Lemma 2. *The operator $L_{n,q}$ defined by* (2.1) *satisfies*

$$
L_{n,q}((e_1 - x)^2;x) \leq (1-q)x^2 + \left(\frac{b_n}{[n]_q} + \frac{4}{[2]_q} \frac{b_n}{[n+1]_q}\right) x + \frac{1}{[3]_q} \frac{b_n^2}{[n+1]_q^2} \tag{2.8}
$$

for each $x \in [0, b_n)$.

Proof. From the linearity and positivity of $L_{n,q}$, one has

$$
L_{n,q}((e_1 - x)^2;x) = \left(L_{n,q}(e_2;x) - x^2\right) - 2x \left(L_{n,q}(e_1;x) - x\right)
$$

and

(2.9) $$L_{n,q}((e_1-x)^2;x) \leq \left|L_{n,q}(e_2;x)-x^2\right| + 2x\left|L_{n,q}(e_1;x)-x\right|.$$

Hence using the inequalities (2.6) and (2.7) into (2.9), we obtain the desired result. □

From the inequalities (2.6) and (2.7), it is seen that for a fixed value q with $0<q<1$ the sequence of the operator $L_{n,q}$ does not satisfy the conditions of Korovkin's theorem. In order to guarantee its convergence we shall replace q by a sequence q_n such that $\lim_{n\to\infty} q_n = 1$ and $\lim_{n\to\infty} \frac{b_n}{[n]_{q_n}} = 0$. For instance, if we choose $q_n = e^{-\frac{1}{n}}$ and $b_n = \sqrt{n}$, these conditions are satisfied. Hence, noting that $\frac{1}{[2]_{q_n}} = \frac{1}{1+q_n}$, $\frac{1}{[3]_{q_n}} = \frac{1}{1+q_n+q_n^2}$ and $0 < \frac{b_n}{[n+1]_{q_n}} < \frac{b_n}{[n]_{q_n}}$ we can state the following theorem.

Theorem 1. *Let q_n be a sequence such that $\lim_{n\to\infty} q_n = 1$ for $0<q_n<1$. If $\lim_{n\to\infty}\frac{b_n}{[n]_{q_n}} = 0$, then for $f\in C[0,\infty)$, the sequence of the linear positive operator L_{n,q_n} defined by (2.1) converges uniformly to $f(x)$ on each closed finite interval $[0,a]$ where a is a fixed positive real number.*

3. Rate of convergence

In this part, we compute the rate of convergence by means of the modulus of continuity and the elements of Lipschitz class.

By $C_B[0,\infty)$, we denote the class of real-valued functions f which are bounded and continuous on $[0,\infty)$.

Theorem 2. *Let q_n be a sequence such that $\lim_{n\to\infty} q_n = 1$ for $0<q_n<1$. If $\lim_{n\to\infty}\frac{b_n}{[n]_{q_n}} = 0$, then for $f\in C_B[0,\infty)$ and $x\in[0,b_n)$ we have*

$$|L_{n,q_n}(f;x)-f(x)| \leq 2\omega\left(f,\delta_{n,q_n}(x)\right),$$

where $\omega(f,\delta)$ is the usual modulus of continuity of f and

$$\delta_{n,q_n}(x) = \sqrt{(1-q_n)x^2 + \left(\frac{b_n}{[n]_{q_n}} + \frac{4}{[2]_{q_n}}\frac{b_n}{[n+1]_{q_n}}\right)x + \frac{1}{[3]_{q_n}}\frac{b_n^2}{[n+1]_{q_n}^2}}.$$

Proof. By the definition of Riemann type q-integral, it is easily seen that

(3.1) $$|R_q(f;a,b)| \leq R_q(|f|;a,b)$$

and

(3.2) $$R_q(f;a,b) \leq R_q(g;a,b) \quad \text{for} \quad f\leq g.$$

Then using (3.1), (3.2) and the following inequality (see [6])

$$|f(t)-f(x)| \leq \left(1+\frac{(t-x)^2}{\delta^2}\right)\omega(f,\delta), \quad \delta>0$$

respectively, we may write

$$|L_{n,q}(f;x)-f(x)|$$

$$
\begin{aligned}
&= P_{n,q}(x)\sum_{k=0}^{\infty}\begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k C_{n,k}(q,b_n)\left|\int_{\frac{[k]_q}{[n+k]_q}b_n}^{\frac{[k+1]_q}{[n+k+1]_q}b_n}(f(t)-f(x))\,d_q^R t\right| \\
&\leq P_{n,q}(x)\sum_{k=0}^{\infty}\begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k C_{n,k}(q,b_n)\int_{\frac{[k]_q}{[n+k]_q}b_n}^{\frac{[k+1]_q}{[n+k+1]_q}b_n}|f(t)-f(x)|\,d_q^R t \\
&\leq P_{n,q}(x)\sum_{k=0}^{\infty}\begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k C_{n,k}(q,b_n) \\
&\quad\times\int_{\frac{[k]_q}{[n+k]_q}b_n}^{\frac{[k+1]_q}{[n+k+1]_q}b_n}\left(1+\frac{(t-x)^2}{\delta^2}\right)\omega(f,\delta)d_q^R t \\
&= \left(L_{n,q}(e_0;x)+\frac{1}{\delta^2}L_{n,q}((e_1-x)^2;x)\right)\omega(f,\delta).
\end{aligned}
$$

Now with the help of (2.5) and (2.8), one gets

$$
\begin{aligned}
&|L_{n,q}(f;x)-f(x)| \\
&\leq \left\{1+\frac{1}{\delta^2}\left((1-q)x^2+\left(\frac{b_n}{[n]_q}+\frac{4}{[2]_q}\frac{b_n}{[n+1]_q}\right)x+\frac{1}{[3]_q}\frac{b_n^2}{[n+1]_q^2}\right)\right\}\omega(f,\delta).
\end{aligned}
$$

Consequently, choosing

$$
\delta=\delta_{n,q}(x)=\sqrt{(1-q)x^2+\left(\frac{b_n}{[n]_q}+\frac{4}{[2]_q}\frac{b_n}{[n+1]_q}\right)x+\frac{1}{[3]_q}\frac{b_n^2}{[n+1]_q^2}}
$$

and then replacing q by q_n we complete the proof. □

Note that for each $x\in[0,b_n)$, it is not guarantee that $\delta_{n,q_n}(x)\to 0$ as $n\to\infty$. Really, for $x=\frac{b_n}{4}$ we have

$$
\delta_{n,q_n}\left(\frac{b_n}{4}\right)=\sqrt{(1-q_n)\frac{b_n^2}{16}+\frac{1}{4}\frac{b_n^2}{[n]_{q_n}}+\frac{1}{[2]_{q_n}}\frac{b_n^2}{[n+1]_{q_n}}+\frac{1}{[3]_q}\frac{b_n^2}{[n+1]_q^2}}
$$

which may not converge to 0 as $n\to\infty$. A similar situation occurs in q-Chlodovsky operators (see [20]). Therefore we introduce the following rate of convergence theorems for the operator L_{n,q_n} on any closed finite interval $[0,a]$.

Theorem 3. *Let q_n be a sequence such that $\lim_{n\to\infty}q_n=1$ for $0<q_n<1$. If $\lim_{n\to\infty}\frac{b_n}{[n]_{q_n}}=0$, then for $f\in C[0,\infty)$ and $x\in[0,a]$ we have*

$$
|L_{n,q_n}(f;x)-f(x)|\leq 2\omega\left(f,\delta_{n,q_n}\right),
$$

where $\omega(f,\delta)$ is the usual modulus of continuity of f on $[0,a]$ and

$$
\delta_{n,q_n}=\sqrt{(1-q_n)a^2+\left(\frac{b_n}{[n]_{q_n}}+\frac{4}{[2]_{q_n}}\frac{b_n}{[n+1]_{q_n}}\right)a+\frac{1}{[3]_{q_n}}\frac{b_n^2}{[n+1]_{q_n}^2}}.
$$

It can be proved in a similar way that of the proof of Theorem 2.

Theorem 4. *Let q_n be a sequence such that* $\lim_{n\to\infty} q_n = 1$ *for* $0 < q_n < 1$. *If* $\lim_{n\to\infty} \frac{b_n}{[n]_{q_n}} = 0$, *then for* $f \in C[0,\infty)$ *which has continuous first derivative on* $[0,a]$ *and* $x \in [0,a]$, *we have*

$$|L_{n,q_n}(f;x) - f(x)| \le ||f'||_{C[0,a]} \frac{1}{[2]_{q_n}} \frac{b_n}{[n+1]_{q_n}} + 2\delta_{n,q_n}\omega(f', \delta_{n,q_n}),$$

where $\omega(f',\delta)$ *is the usual modulus of continuity of* f' *on* $[0,a]$, $||\cdot||_{C[0,a]}$ *is the standard supremum norm on the space* $C[0,a]$ *and* δ_{n,q_n} *defined as in Theorem 3.*

Proof. Using the fact $L_{n,q_n}(e_0;x) = 1$, from the well-known result of Shisha and Mond [29], it follows that

$$\begin{aligned}
&|L_{n,q_n}(f;x) - f(x)| \\
&\le |f(x)||L_{n,q_n}(e_0;x) - 1| + |f'(x)||L_{n,q_n}(e_1;x) - x| \\
&\quad + \sqrt{L_{n,q_n}((e_1-x)^2;x)} \left\{ \sqrt{L_{n,q_n}(e_0;x)} + \frac{1}{\delta}\sqrt{L_{n,q_n}((e_1-x)^2;x)} \right\} \omega(f',\delta) \\
&\le ||f'||_{C[0,a]} |L_{n,q_n}(e_1;x) - x| \\
&\quad + \sqrt{L_{n,q_n}((e_1-x)^2;x)} \left\{ 1 + \frac{1}{\delta}\sqrt{L_{n,q_n}((e_1-x)^2;x)} \right\} \omega(f',\delta).
\end{aligned}$$

By means of (2.6) and (2.8), this shows that for $x \in [0,a]$

$$|L_{n,q_n}(f;x) - f(x)| \le ||f'||_{C[0,a]} \frac{1}{[2]_{q_n}} \frac{b_n}{[n+1]_{q_n}} + 2\delta_{n,q_n}\omega(f', \delta_{n,q_n}).$$

Thus the proof is completed. □

Theorem 5. *Let q_n be a sequence such that* $\lim_{n\to\infty} q_n = 1$ *for* $0 < q_n < 1$. *If* $\lim_{n\to\infty} \frac{b_n}{[n]_{q_n}} = 0$, *then for* $f \in Lip_M(\alpha)$ *with* $0 < \alpha \le 1$, $M > 0$ *on* $[0,a]$ *and* $x \in [0,a]$ *we have*

$$|L_{n,q_n}(f;x) - f(x)| \le M\delta_{n,q_n}^{\alpha},$$

where δ_{n,q_n} *defined as in Theorem 3.*

Proof. In a similar way that of the proof of Theorem 2, we have

$$\begin{aligned}
&|L_{n,q}(f;x) - f(x)| \\
&\le P_{n,q}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k C_{n,k}(q,b_n) \int_{\frac{[k]_q}{[n+k]_q}b_n}^{\frac{[k+1]_q}{[n+k+1]_q}b_n} |f(t) - f(x)| \, d_q^R t \qquad (3.3) \\
&\le M P_{n,q}(x) \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k C_{n,k}(q,b_n) \int_{\frac{[k]_q}{[n+k]_q}b_n}^{\frac{[k+1]_q}{[n+k+1]_q}b_n} |t - x|^{\alpha} \, d_q^R t.
\end{aligned}$$

Now using the Hölder inequality for Riemann type q-integral given by (1.2) with $m=\frac{2}{\alpha}$ and $n=\frac{2}{2-\alpha}$ and (2.2), one gets

$$\begin{aligned}&\int_{\frac{[k]_q}{[n+k]_q}b_n}^{\frac{[k+1]_q}{[n+k+1]_q}b_n}|t-x|^{\alpha}\,d_q^R t\\ &\leq\left(\int_{\frac{[k]_q}{[n+k]_q}b_n}^{\frac{[k+1]_q}{[n+k+1]_q}b_n}(t-x)^2\,d_q^R t\right)^{\frac{\alpha}{2}}\left(\int_{\frac{[k]_q}{[n+k]_q}b_n}^{\frac{[k+1]_q}{[n+k+1]_q}b_n}d_q^R t\right)^{\frac{2-\alpha}{2}}\\ &=\left(\int_{\frac{[k]_q}{[n+k]_q}b_n}^{\frac{[k+1]_q}{[n+k+1]_q}b_n}(t-x)^2\,d_q^R t\right)^{\frac{\alpha}{2}}\left(\frac{1}{C_{n,k}(q,b_n)}\right)^{\frac{2-\alpha}{2}}.\end{aligned}$$

Substitution of this result into (3.3) leads to

$$\begin{aligned}&|L_{n,q}(f;x)-f(x)|\\ &\leq MP_{n,q}(x)\sum_{k=0}^{\infty}\begin{bmatrix}n+k\\k\end{bmatrix}_q\left(\frac{x}{b_n}\right)^k\left(C_{n,k}(q,b_n)\int_{\frac{[k]_q}{[n+k]_q}b_n}^{\frac{[k+1]_q}{[n+k+1]_q}b_n}(t-x)^2\,d_q^R t\right)^{\frac{\alpha}{2}}\\ &=M\sum_{k=0}^{\infty}\left(P_{n,q}(x)\begin{bmatrix}n+k\\k\end{bmatrix}_q\left(\frac{x}{b_n}\right)^k C_{n,k}(q,b_n)\int_{\frac{[k]_q}{[n+k]_q}b_n}^{\frac{[k+1]_q}{[n+k+1]_q}b_n}(t-x)^2\,d_q^R t\right)^{\frac{\alpha}{2}}\\ &\quad\times\left(P_{n,q}(x)\begin{bmatrix}n+k\\k\end{bmatrix}_q\left(\frac{x}{b_n}\right)^k\right)^{\frac{2-\alpha}{2}}.\end{aligned}$$

Applying the Hölder inequality for sums, the last inequality takes the form

$$\begin{aligned}&|L_{n,q}(f;x)-f(x)|\\ &\leq M\left(P_{n,q}(x)\sum_{k=0}^{\infty}\begin{bmatrix}n+k\\k\end{bmatrix}_q\left(\frac{x}{b_n}\right)^k C_{n,k}(q,b_n)\int_{\frac{[k]_q}{[n+k]_q}b_n}^{\frac{[k+1]_q}{[n+k+1]_q}b_n}(t-x)^2\,d_q^R t\right)^{\frac{\alpha}{2}}\\ &\quad\times\left(P_{n,q}(x)\sum_{k=0}^{\infty}\begin{bmatrix}n+k\\k\end{bmatrix}_q\left(\frac{x}{b_n}\right)^k\right)^{\frac{2-\alpha}{2}}\\ &=M\left(L_{n,q}((e_1-x)^2;x)\right)^{\frac{\alpha}{2}}\left(L_{n,q}^*(e_0;x)\right)^{\frac{2-\alpha}{2}}.\end{aligned}$$

Therefore, if we use the fact $L_{n,q}^*(e_0;x)=1$ and the inequality (2.8) and replace q by q_n then the proof is completed. □

4. Weighted approximation

In this part, we introduce approximation properties of the operator L_{n,q_n} in weighted spaces of continuous functions on positive semi-axis with the help of weighted Korovkin type theorem proved by Gadjiev in [15]. For this purpose,

let us recall the notations and result of [15].
$B_\rho[0,\infty)$: The space of all functions satisfying the condition

$$|f(x)| \le M_f \rho(x),$$

where $x \in [0,\infty)$, M_f is a positive constant depending only on f and $\rho(x) = 1 + x^2$.
$C_\rho[0,\infty)$: The space of all continuous functions in the space $B_\rho[0,\infty)$.
$C_\rho^0[0,\infty)$: The subspace of all functions $f \in C_\rho[0,\infty)$ for which

$$\lim_{x\to\infty} \frac{|f(x)|}{\rho(x)} < \infty.$$

The space $B_\rho[0,\infty)$ is a linear normed space with the following norm:

$$||f||_\rho = \sup_{x\in[0,\infty)} \frac{|f(x)|}{\rho(x)}.$$

Theorem A ([15]). *Let A_n be a sequence of positive linear operators acting from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ satisfying the conditions*

$$\lim_{n\to\infty} ||A_n(e_\nu;x) - x^\nu||_\rho = 0, \quad \nu = 0,1,2.$$

Then for any function $f \in C_\rho^0[0,\infty)$ we have

$$\lim_{n\to\infty} ||A_n(f;x) - f(x)||_\rho = 0,$$

where $\rho(x) = 1 + x^2$.

Theorem 6. *Let q_n be a sequence such that $\lim_{n\to\infty} q_n = 1$ for $0 < q_n < 1$. If $\lim_{n\to\infty} \frac{b_n}{[n]_{q_n}} = 0$, then for each $f \in C_\rho^0[0,\infty)$ we have*

$$\lim_{n\to\infty} \sup_{x\in[0,b_n)} \frac{|L_{n,q_n}(f;x) - f(x)|}{1+x^2} = 0.$$

Proof. It is easily verified that L_{n,q_n} acts from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ for all $n \in \mathbb{N}$. So if we apply Theorem A to the operators $A_n(f;x)$ defined by

$$A_n(f;x) = \begin{cases} L_{n,q_n}(f;x), & \text{if } x \in [0,b_n) \\ f(x), & \text{if } x \ge b_n, \end{cases}$$

then to complete the proof it is enough to show that the conditions

$$\lim_{n\to\infty} \sup_{x\in[0,b_n)} \frac{|L_{n,q_n}(e_\nu;x) - x^\nu|}{1+x^2} = 0, \quad \nu = 0,1,2 \tag{4.1}$$

hold.
Since $L_{n,q_n}(e_0;x) = 1$, (4.1) is valid for $\nu = 0$.
From (2.6), we have

$$\sup_{x\in[0,b_n)} \frac{|L_{n,q_n}(e_1;x) - x|}{1+x^2} \le \frac{1}{[2]_{q_n}} \frac{b_n}{[n+1]_{q_n}}.$$

This implies that

$$\lim_{n\to\infty}\sup_{x\in[0,b_n)}\frac{|L_{n,q_n}(e_1;x)-x|}{1+x^2}=0.$$

By using (2.7), one gets

$$\sup_{x\in[0,b_n)}\frac{|L_{n,q_n}(e_2;x)-x^2|}{1+x^2}\leq 1-q_n+\frac{b_n}{[n]_{q_n}}+\frac{2}{[2]_{q_n}}\frac{b_n}{[n+1]_{q_n}}+\frac{1}{[3]_{q_n}}\frac{b_n^2}{[n+1]_{q_n}^2}$$

which gives

$$\lim_{n\to\infty}\sup_{x\in[0,b_n)}\frac{|L_{n,q_n}(e_2;x)-x^2|}{1+x^2}=0.$$

Thus the proof is completed. □

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