

CHARACTERIZATIONS OF GRADED PRÜFER ★-MULTIPLICATION DOMAINS

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ABSTRACT. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain graded by an arbitrary grading torsionless monoid Γ , and \star be a semistar operation on R . In this paper we define and study the graded integral domain analogue of \star -Nagata and Kronecker function rings of R with respect to \star . We say that R is a graded Prüfer \star -multiplication domain if each nonzero finitely generated homogeneous ideal of R is \star_f -invertible. Using \star -Nagata and Kronecker function rings, we give several different equivalent conditions for R to be a graded Prüfer \star -multiplication domain. In particular we give new characterizations for a graded integral domain, to be a PvMD.

1. Introduction

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded (commutative) integral domain graded by an arbitrary grading torsionless monoid Γ , that is Γ is a commutative cancellative monoid (written additively). Let $\langle \Gamma \rangle = \{a - b \mid a, b \in \Gamma\}$, be the quotient group of Γ , which is a torsionfree abelian group.

Let H be the saturated multiplicative set of nonzero homogeneous elements of R . Then $R_H = \bigoplus_{\alpha \in \langle \Gamma \rangle} (R_H)_\alpha$, called the *homogeneous quotient*

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field of R , is a graded integral domain whose nonzero homogeneous elements are units. For a fractional ideal I of R let I_h denote the fractional ideal generated by the set of homogeneous elements of R in I . It is known that if I is a prime ideal, then I_h is also a prime ideal (cf. [29, Page 124]). An integral ideal I of R is said to be homogeneous if $I = \bigoplus_{\alpha \in \Gamma} (I \cap R_\alpha)$; equivalently, if $I = I_h$. A fractional ideal I of R is *homogeneous* if sI is an integral homogeneous ideal of R for some $s \in H$ (thus $I \subseteq R_H$). For $f \in R_H$, let $C_R(f)$ (or simply $C(f)$) denote the fractional ideal of R generated by the homogeneous components of f . For a fractional ideal I of R with $I \subseteq R_H$, let $C(I) = \sum_{f \in I} C(f)$. For more on graded integral domains and their divisibility properties, see [3, 29].

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ and $N_v(H) = \{f \in R \mid C(f)^v = R\}$. (Definitions related to the v -operation will be reviewed in the sequel.) Then $N_v(H)$ is a saturated multiplicative subset of R by [4, Lemma 1.1(2)]. The graded integral domain analogue of the well known Nagata ring is the ring $R_{N_v(H)}$. In [4], Anderson and Chang, studied relationships between the ideal-theoretic properties of $R_{N_v(H)}$ and the homogeneous ideal-theoretic properties of R . For example it is shown that if R has a unit of nonzero degree, $Pic(R_{N_v(H)}) = 0$ and that R is a PvMD if and only if each ideal of $R_{N_v(H)}$ is extended from a homogeneous ideal of R , if and only if $R_{N_v(H)}$ is a Prüfer (or Bézout) domain [4, Theorems 3.3 and 3.4]. Also, they generalized the notion of Kronecker function ring, (for e. a. b. star operations on R) and then showed that this ring is a Bézout domain [4, Theorem 3.5]. For the definition and properties of semistar-Nagata and Kronecker function rings of an integral domain see the interesting survey article [21]. Recall that the *Picard group (or the ideal class group)* of an integral domain D , is $Pic(D) = Inv(D)/Prin(D)$, where $Inv(D)$ is the multiplicative group of invertible fractional ideals of D , and $Prin(D)$ is the subgroup of principal fractional ideal of D .

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be an integral domain, and \star be a semistar operation on R . In Section 2 of this paper we study the homogeneous elements of $QSpec^\star(R)$ denoted by $h-QSpec^\star(R)$. We show that if \star is a finite type semistar operation on R which sends homogeneous fractional ideals to homogeneous ones, and such that $R^\star \subsetneq R_H$, then each homogeneous quasi- \star -ideal of R , is contained in a homogeneous quasi- \star -prime ideal of R . One of key results in this paper is Proposition 2.3, which shows that if $R^\star \subsetneq R_H$, the $\tilde{\star}$ sends homogeneous fractional ideals to homogeneous ones. We also define and study the Nagata ring of R with

respect to \star . The \star -Nagata ring is defined by the quotient ring $R_{N_\star(H)}$, where $N_\star(H) = \{f \in R \mid C(f)^\star = R^\star\}$. Among other things, it is shown that $Pic(R_{N_\star(H)}) = 0$. In Section 3 we define and study the Kronecker function ring of R with respect to \star . The Kronecker function ring, inspired by [20, Theorem 5.1], is defined by $Kr(R, \star) := \{0\} \cup \{f/g \mid 0 \neq f, g \in R, \text{ and there is } 0 \neq h \in R \text{ such that } C(f)C(h) \subseteq (C(g)C(h))^\star\}$. It is shown that if \star sends homogeneous fractional ideals to fractional ones, then $Kr(R, \star)$ is a Bézout domain. In Section 3 we define the notion of graded Prüfer \star -multiplication domains and give several different equivalent conditions to be a graded P \star MD. A graded integral domain R , is called a *graded Prüfer \star -multiplication domain (graded P \star MD)* if every finitely generated homogeneous ideal of R is a \star_f -invertible, i.e., $(II^{-1})^{\star_f} = R^\star$ for each finitely generated homogeneous ideal I of R . Among other results we show that R is a graded P \star MD if and only if $R_{N_\star(H)}$ is a Prüfer domain if and only if $R_{N_\star(H)}$ is a Bézout domain if and only if $R_{N_\star(H)} = Kr(R, \tilde{\star})$ if and only if $Kr(R, \tilde{\star})$ is a flat R -module.

To facilitate the reading of the paper, we review some basic facts on semistar operations. Let D be an integral domain with quotient field K . Let $\overline{\mathcal{F}}(D)$ denote the set of all nonzero D -submodules of K . Let $\mathcal{F}(D)$ be the set of all nonzero *fractional* ideals of D ; i.e., $E \in \mathcal{F}(D)$ if $E \in \overline{\mathcal{F}}(D)$ and there exists a nonzero element $r \in D$ with $rE \subseteq D$. Let $f(D)$ be the set of all nonzero finitely generated fractional ideals of D . Obviously, $f(D) \subseteq \mathcal{F}(D) \subseteq \overline{\mathcal{F}}(D)$. As in [30], a *semistar operation on D* is a map $\star : \overline{\mathcal{F}}(D) \rightarrow \overline{\mathcal{F}}(D)$, $E \mapsto E^\star$, such that, for all $x \in K$, $x \neq 0$, and for all $E, F \in \overline{\mathcal{F}}(D)$, the following three properties hold:

- $\star_1 : (xE)^\star = xE^\star$;
- $\star_2 : E \subseteq F$ implies that $E^\star \subseteq F^\star$;
- $\star_3 : E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

Let \star be a semistar operation on the domain D . For every $E \in \overline{\mathcal{F}}(D)$, put $E^{\star_f} := \cup F^\star$, where the union is taken over all finitely generated $F \in f(D)$ with $F \subseteq E$. It is easy to see that \star_f is a semistar operation on D , and \star_f is called *the semistar operation of finite type associated to \star* . Note that $(\star_f)_f = \star_f$. A semistar operation \star is said to be of *finite type* if $\star = \star_f$; in particular \star_f is of finite type. We say that a nonzero ideal I of D is a *quasi- \star -ideal* of D , if $I^\star \cap D = I$; a *quasi- \star -prime* (ideal of D), if I is a prime quasi- \star -ideal of D ; and a *quasi- \star -maximal* (ideal of D), if I is maximal in the set of all proper quasi- \star -ideals of D . Each quasi- \star -maximal ideal is a prime ideal. It was shown in [16, Lemma 4.20] that

if $D^\star \neq K$, then each proper quasi- \star_f -ideal of D is contained in a quasi- \star_f -maximal ideal of D . We denote by $\text{QMax}^\star(D)$ (resp., $\text{QSpec}^\star(D)$) the set of all quasi- \star -maximal ideals (resp., quasi- \star -prime ideals) of D .

If \star_1 and \star_2 are semistar operations on D , one says that $\star_1 \leq \star_2$ if $E^{\star_1} \subseteq E^{\star_2}$ for each $E \in \overline{\mathcal{F}}(D)$ (cf. [30, page 6]). This is equivalent to saying that $(E^{\star_1})^{\star_2} = E^{\star_2} = (E^{\star_2})^{\star_1}$ for each $E \in \overline{\mathcal{F}}(D)$ (cf. [30, Lemma 16]). Obviously, for each semistar operation \star defined on D , we have $\star_f \leq \star$. Let d_D (or, simply, d) denote the identity (semi)star operation on D . Clearly, $d_D \leq \star$ for all semistar operations \star on D .

It has become standard to say that a semistar operation \star is *stable* if $(E \cap F)^\star = E^\star \cap F^\star$ for all $E, F \in \overline{\mathcal{F}}(D)$. (“Stable” has replaced the earlier usage, “quotient”, in [30, Definition 21].) Given a semistar operation \star on D , it is possible to construct a semistar operation $\tilde{\star}$, which is stable and of finite type defined as follows: for each $E \in \overline{\mathcal{F}}(D)$,

$$E^{\tilde{\star}} := \{x \in K \mid xJ \subseteq E, \text{ for some } J \subseteq R, J \in f(R), J^\star = D^\star\}.$$

It is well known that [16, Corollary 2.7]

$$E^{\tilde{\star}} := \bigcap \{ED_P \mid P \in \text{QMax}^{\star_f}(D)\}, \text{ for each } E \in \overline{\mathcal{F}}(D).$$

The most widely studied (semi)star operations on D have been the identity d , v , $t := v_f$, and $w := \tilde{v}$ operations, where $A^v := (A^{-1})^{-1}$, with $A^{-1} := (R : A) := \{x \in K \mid xA \subseteq D\}$.

Let \star be a semistar operation on an integral domain D . We say that \star is an **e. a. b.** (*endlich arithmetisch brauchbar*) *semistar operation* of D if, for all $E, F, G \in f(D)$, $(EF)^\star \subseteq (EG)^\star$ implies that $F^\star \subseteq G^\star$ ([20, Definition 2.3 and Lemma 2.7]). We can associate to any semistar operation \star on D , an **e. a. b.** semistar operation of finite type \star_a on D , called the **e. a. b.** *semistar operation associated to \star* , defined as follows for each $F \in f(D)$ and for each $E \in \overline{\mathcal{F}}(D)$:

$$F^{\star_a} := \bigcup \{((FH)^\star : H^\star) \mid H \in f(R)\},$$

$$E^{\star_a} := \bigcup \{F^{\star_a} \mid F \subseteq E, F \in f(R)\}$$

[20, Definition 4.4 and Proposition 4.5] (note that $((FH)^\star : H^\star) = ((FH)^\star : H)$). It is known that $\star_f \leq \star_a$ [20, Proposition 4.5(3)]. Obviously $(\star_f)_a = \star_a$. Moreover, when $\star = \star_f$, then \star is **e. a. b.** if and only if $\star = \star_a$ [20, Proposition 4.5(5)].

Let \star be a semistar operation on a domain D . Recall from [17] that, D is called a *Prüfer \star -multiplication domain* (for short, a **P \star MD**) if each

finitely generated ideal of D is \star_f -invertible; i.e., if $(II^{-1})^{\star_f} = D^\star$ for all $I \in f(D)$. When $\star = v$, we recover the classical notion of PvMD; when $\star = d_D$, the identity (semi)star operation, we recover the notion of Prüfer domain.

2. Nagata ring

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, \star be a semistar operation on R , H be the set of nonzero homogeneous elements of R . An overring T of R , with $R \subseteq T \subseteq R_H$ will be called a *homogeneous overring* if $T = \bigoplus_{\alpha \in \langle \Gamma \rangle} (T \cap (R_H)_\alpha)$. Thus T is a graded integral domain with $T_\alpha = T \cap (R_H)_\alpha$.

In this section we study the homogeneous elements of $\text{QSpec}^\star(R)$, denoted by $h\text{-QSpec}^\star(R)$, and the graded integral domain analogue of \star -Nagata ring. Let $h\text{-QMax}^\star(R)$ denote the set of ideals of R which are maximal in the set of all proper homogeneous quasi- \star -ideals of R . The following lemma shows that, if $R^\star \subsetneq R_H$ and $\star = \star_f$ sends homogeneous fractional ideals to homogeneous ones, then $h\text{-QMax}^{\star_f}(R)$ is nonempty and each proper homogeneous quasi- \star_f -ideal is contained in a maximal homogeneous quasi- \star_f -ideal.

LEMMA 2.1. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, \star a finite type semistar operation on R which sends homogeneous fractional ideals to homogeneous ones, and such that $R^\star \subsetneq R_H$. If I is a proper homogeneous quasi- \star -ideal of R , then I is contained in a proper homogeneous quasi- \star -prime ideal.*

Proof. Let $X := \{I \mid I \text{ is a homogeneous quasi-}\star\text{-ideal of } R\}$. Then it is easy to see that X is nonempty. Indeed, in this case R^\star is a homogeneous overring of R , and if $u \in H$ is a nonunit in R^\star , then $uR^\star \cap R$ is a proper homogeneous quasi- \star -ideal of R . Also X is inductive (see proof of [16, Lemma 4.20]). From Zorn's Lemma, we see that every proper homogeneous quasi- \star -ideal of R is contained in some maximal element Q of X .

Now we show that Q is actually prime. Take $f, g \in H \setminus Q$ and suppose that $fg \in Q$. By the maximality of Q we have $(Q, f)^\star = R^\star$ (note that $(Q, f)^\star \cap R$ is a homogeneous quasi- \star -ideal of R and properly contains Q). Since \star is of finite type, we can find a finitely generated ideal $J \subseteq Q$

such that $(J, f)^\star = R^\star$. Then $g \in gR^\star \cap R = g(J, f)^\star \cap R \subseteq Q^\star \cap R = Q$ a contradiction. Thus Q is a prime ideal. \square

The following example shows that we can not drop the condition that, \star sends homogeneous fractional ideals to homogeneous ones, in the above lemma.

EXAMPLE 2.2. Let k be a field and X, Y be indeterminates over k . Let $R = k[X, Y]$, which is a (\mathbb{N}_0) -graded Noetherian integral domain with $\deg X = \deg Y = 1$. Set $M := (X, Y + 1)$ which is a maximal non-homogeneous ideal of R . Let T be a DVR [11], with maximal ideal N , dominating the local ring R_M . If $R_H \subseteq T$, then there exists a prime ideal P of R such that, $P \cap H = \emptyset$ and $N \cap R_H = PR_H$. Thus $M = N \cap R = N \cap R_H \cap R = PR_H \cap R = P$. Hence $M \cap H = \emptyset$, which is a contradiction, since $X \in M \cap H$. So that, $R_H \not\subseteq T$. Let \star be a semistar operation on R defined by $E^\star = ET \cap ER_H$ for each $E \in \overline{\mathcal{F}}(R)$. Then clearly $\star = \star_f$ and $R^\star \subsetneq R_H$. If P is a nonzero prime ideal of R , such that $P \cap H = \emptyset$, then $P^{\star_f} \cap R = PT \cap PR_H \cap R = PT \cap P = P$. Thus P is a quasi- \star_f -prime ideal. On the other hand if P is any nonzero prime ideal of R such that $P \cap H \neq \emptyset$, then $PT = N^k$, for some integer $k \geq 1$. Therefore, if we assume that P is a quasi- \star_f -ideal of R , then we would have $P = PT \cap PR_H \cap R = PT \cap R = N^k \cap R \supseteq M^k$, which implies that $P = M$. Thus $\text{QSpec}^{\star_f}(R) = \{M\} \cup \{P \in \text{Spec}(R) | P \neq 0 \text{ and } P \cap H = \emptyset\}$. Therefore by [16, Lemma 4.1, Remark 4.5], we have $\text{QSpec}^{\tilde{\star}}(R) = \{Q \in \text{Spec}(R) | 0 \neq Q \subseteq M\} \cup \{P \in \text{Spec}(R) | P \neq 0 \text{ and } P \cap H = \emptyset\}$. Hence in the present example we have $h\text{-QSpec}^{\star_f}(R) = h\text{-QMax}^{\star_f}(R) = \emptyset$, and $h\text{-QSpec}^{\tilde{\star}}(R) = h\text{-QMax}^{\tilde{\star}}(R) = \{(X)\}$. Note that in this example $h\text{-QMax}^{\tilde{\star}}(R) \not\subseteq \text{QMax}^{\tilde{\star}}(R) = \text{QMax}^{\star_f}(R)$.

From now on in this paper, we are interested and consider, the semistar operations \star on R , such that $R^\star \subsetneq R_H$ and sends homogeneous fractional ideals to homogeneous ones. For any such semistar operation, if I is a homogeneous ideal of R , we have $I^{\star_f} = R^\star$ if and only if $I \not\subseteq Q$ for each $Q \in h\text{-QMax}^{\star_f}(R)$. Also if P is a quasi- \star -prime ideal of R , then either $P_h = 0$ or P_h is a quasi- \star -prime ideal of R . Indeed, if $P_h \neq 0$, then $P_h \subseteq (P_h)^\star \cap R \subseteq P^\star \cap R = P$, which implies that $P_h = (P_h)^\star \cap R$, since $(P_h)^\star \cap R$ is a homogeneous ideal.

The following proposition is the key result in this paper.

PROPOSITION 2.3. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, and \star be a semistar operation on R such that $R^\star \subsetneq R_H$. Then, $\tilde{\star}$ sends

homogeneous fractional ideals to homogeneous ones. In particular $h\text{-QMax}^\star(R) \neq \emptyset$, and R^\star is a homogeneous overring of R .

Proof. Let E be a homogenous fractional ideal of R . To show that E^\star is homogeneous let $f \in E^\star$. Then $fJ \subseteq E$ for some finitely generated ideal J of R such that $J^\star = R^\star$. Suppose that $J = (g_1, \dots, g_n)$. Using [4, Lemma 1.1(1)], there is an integer $m \geq 1$ such that $C(g_i)^{m+1}C(f) = C(g_i)^mC(fg_i)$ for all $i = 1, \dots, n$. Since E is a homogeneous fractional ideal and $fg_i \in E$, we have $C(fg_i) \subseteq E$. Thus we have $C(g_i)^{m+1}C(f) \subseteq E$. Let $J_0 := C(g_1)^{m+1} + \dots + C(g_n)^{m+1}$. Thus J_0 is a finitely generated homogeneous ideal of R such that $J_0^\star = R^\star$. Since $C(f)J_0 \subseteq E$, $C(f) \subseteq E^\star$. Therefore E^\star is a homogeneous ideal. \square

LEMMA 2.4. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, \star a semistar operation on R which sends homogeneous fractional ideals to homogeneous ones. Then \star_f sends homogeneous fractional ideals to homogeneous ones.

Proof. Let E be a homogenous fractional ideal of R . Let $0 \neq x \in E^{\star_f}$. Then, there exists an $F \in f(R)$ such that $F \subseteq E$ and $x \in F^\star$. Suppose that F is generated by $y_1, \dots, y_n \in R_H$. Let G be a homogeneous fractional ideal of R , generated by homogeneous components of y_1, \dots, y_n . Note that $F \subseteq G \subseteq E$ and $x \in G^\star$. Thus homogeneous components of x belong to $G^\star \subseteq E^{\star_f}$. This shows that E^{\star_f} is homogeneous. \square

Note that the v -operation sends homogeneous fractional ideals to homogeneous ones by [3, Proposition 2.5]. Using the above two results, the t and w -operations also, send homogeneous fractional ideals to homogeneous ones.

It is well-known that $\text{QMax}^{\star_f}(R) = \text{QMax}^\star(R)$, see [5, Theorem 2.16], for star operation case, and [18, Corollary 3.5(2)], in general semistar operations. Although Example 2.2, shows that it may happen that $h\text{-QMax}^{\star_f}(R) \neq h\text{-QMax}^\star(R)$, we have the following proposition whose proof is almost the same as [4, Theorem 2.16].

PROPOSITION 2.5. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, \star a semistar operation on R such that $R^\star \subsetneq R_H$, which sends homogeneous fractional ideals to homogeneous ones. Then $h\text{-QMax}^{\star_f}(R) = h\text{-QMax}^\star(R)$.

Proof. Assume that $Q \in h\text{-QMax}^{\star_f}(R)$. Then since $\tilde{\star} \leq \star_f$ by [18, Lemma 2.7(1)], we have $Q \subseteq Q^{\tilde{\star}} \cap R \subseteq Q^{\star_f} \cap R = Q$, that is Q is a quasi- $\tilde{\star}$ -ideal. Suppose that $Q \notin h\text{-QMax}^{\tilde{\star}}(R)$. Then Q is properly contained in some $P \in h\text{-QMax}^{\tilde{\star}}(R)$. So since $Q \in h\text{-QMax}^{\star_f}(R)$, using Lemma 2.1, we must have $P^{\star_f} = R^{\star}$. Thus there is some finitely generated ideal $F \subseteq P$ such that $F^{\star} = R^{\star}$. So for any $r \in R$, $rF \subseteq F \subseteq P$. But then, $r \in P^{\tilde{\star}}$, so $R \subseteq P^{\tilde{\star}}$, which implies that $P^{\tilde{\star}} = R^{\tilde{\star}}$, a contradiction. Therefore, we must have $Q \in h\text{-QMax}^{\tilde{\star}}(R)$.

If $Q \in h\text{-QMax}^{\tilde{\star}}(R)$, then $Q = Q^{\tilde{\star}} \cap R \subseteq Q^{\star_f} \cap R \subseteq R$. Suppose that $Q^{\star_f} \cap R = R$, which implies that $Q^{\star_f} = R^{\star}$. Then there is a finitely generated ideal $F \subseteq Q$ such that $F^{\star} = R^{\star}$. Now for any $r \in R$, $rF \subseteq F \subseteq Q$. Therefore $R \subseteq Q^{\tilde{\star}}$, and so $R = Q^{\tilde{\star}} \cap R = Q$, which is a contradiction. So $Q^{\star_f} \cap R \subsetneq R$. Now, since $Q^{\star_f} \cap R$ is a homogeneous quasi- \star_f -ideal, there is a $P \in h\text{-QMax}^{\star_f}(R)$ such that $Q \subseteq Q^{\star_f} \cap R \subseteq P$. From the first half of the proof, we know that $P \in h\text{-QMax}^{\tilde{\star}}(R)$. So we must have $P = Q$. Therefore $Q \in h\text{-QMax}^{\star_f}(R)$. \square

Park in [31, Lemma 3.4], proved that $I^w = \bigcap_{P \in h\text{-QMax}^w(R)} IR_{H \setminus P}$ for each homogeneous ideal I of R .

PROPOSITION 2.6. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, \star a semistar operation on R such that $R^{\star} \subsetneq R_H$. Then $I^{\tilde{\star}} = \bigcap_{P \in h\text{-QMax}^{\tilde{\star}}(R)} IR_{H \setminus P}$ for each homogeneous ideal I of R . Moreover $I^{\tilde{\star}} R_{H \setminus P} = IR_{H \setminus P}$ for all homogeneous ideal I of R and all $P \in h\text{-QMax}^{\tilde{\star}}(R)$.*

Proof. By Proposition 2.3, $I^{\tilde{\star}}$ is a homogeneous ideal. Also note that $\bigcap_{P \in h\text{-QMax}^{\tilde{\star}}(R)} IR_{H \setminus P}$ is a homogeneous ideal of R . Let $f \in I^{\tilde{\star}}$ be homogeneous. Then $fJ \subseteq I$ for some homogeneous finitely generated ideal J of R such that $J^{\star} = R^{\star}$. It is easy to see that $J^{\tilde{\star}} = R^{\tilde{\star}}$. Hence we have $J \not\subseteq P$ for all $P \in h\text{-QMax}^{\tilde{\star}}(R)$. Thus $f \in IR_{H \setminus P}$ for all $P \in h\text{-QMax}^{\tilde{\star}}(R)$. Conversely, let $f \in \bigcap_{P \in h\text{-QMax}^{\tilde{\star}}(R)} IR_{H \setminus P}$ be homogeneous. Then $(I : f)$ is a homogeneous ideal which is not contained in any $P \in h\text{-QMax}^{\tilde{\star}}(R)$. Therefore $(I : f)^{\tilde{\star}} = R^{\tilde{\star}}$. So that there exist a finitely generated ideal $J \subseteq (I : f)$ such that $J^{\star} = R^{\star}$. Thus $fJ \subseteq I$, i.e., $f \in I^{\tilde{\star}}$. The second assertion follows from the first one. \square

Let D be a domain with quotient field K , and let X be an indeterminate over K . For each $f \in K[X]$, we let $c_D(f)$ denote the content of

the polynomial f , i.e., the (fractional) ideal of D generated by the coefficients of f . Let \star be a semistar operation on D . If $N_\star := \{g \in D[X] \mid g \neq 0 \text{ and } c_D(g)^\star = D^\star\}$, then $N_\star = D[X] \setminus \bigcup \{P[X] \mid P \in \text{QMax}^{\star f}(D)\}$ is a saturated multiplicative subset of $D[X]$. The ring of fractions

$$\text{Na}(D, \star) := D[X]_{N_\star}$$

is called the \star -Nagata domain (of D with respect to the semistar operation \star). When $\star = d$, the identity (semi)star operation on D , then $\text{Na}(D, d)$ coincides with the classical Nagata domain $D(X)$ (as in, for instance [28, page 18], [23, Section 33] and [18]).

Let $N_\star(H) = \{f \in R \mid C(f)^\star = R^\star\}$. It is easy to see that $N_\star(H)$ is a saturated multiplicative subset of R . Indeed assume $f, g \in N_\star(H)$. Then $C(f)^{n+1}C(g) = C(f)^nC(fg)$ for some integer $n \geq 1$ by [4, Lemma 1.1(2)], and $C(fg) \subseteq C(f)C(g)$. Thus $fg \in N_\star(H) \Leftrightarrow C(fg)^\star = R^\star \Leftrightarrow C(f)^\star = C(g)^\star = R^\star \Leftrightarrow f, g \in N_\star(H)$. Also it is easy to show that $N_\star(H) = N_{\star_f}(H) = N_{\bar{\star}}(H)$. We define the graded integral domain analogue of \star -Nagata ring, by the quotient ring $R_{N_\star(H)}$. When $\star = v$, $R_{N_\star(H)}$ was studied in [4], denoted by $R_{N(H)}$.

LEMMA 2.7. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, and \star be a semistar operation on R such that $R^\star \subsetneq R_H$, which sends homogeneous fractional ideals to homogeneous ones.

- (1) $N_\star(H) = R \setminus \bigcup_{Q \in h\text{-QMax}^{\star f}(R)} Q$.
- (2) $\text{Max}(R_{N_\star(H)}) = \{QR_{N_\star(H)} \mid Q \in h\text{-QMax}^{\star f}(R)\}$ if and only if R has the property that if I is a nonzero ideal of R with $C(I)^\star = R^\star$, then $I \cap N_\star(H) \neq \emptyset$.

Proof. (1) Let $x \in R$. Then $x \in N_\star(H) \Leftrightarrow C(x)^\star = R^\star \Leftrightarrow C(x) \not\subseteq Q$ for all $Q \in h\text{-QMax}^{\star f}(R) \Leftrightarrow x \notin Q$ for all $Q \in h\text{-QMax}^{\star f}(R) \Leftrightarrow x \in R \setminus \bigcup_{Q \in h\text{-QMax}^{\star f}(R)} Q$.

(2) (\Rightarrow) Let I is a nonzero ideal of R with $C(I)^\star = R^\star$. Then $I \not\subseteq Q$ for all $Q \in h\text{-QMax}^{\star f}(R)$, and hence $IR_{N_\star(H)} = R_{N_\star(H)}$. Thus $I \cap N_\star(H) \neq \emptyset$.

(\Leftarrow) Let I be a nonzero ideal of R such that $I \subseteq \bigcup_{Q \in h\text{-QMax}^{\star f}(R)} Q$. If $C(I)^{\star f} = R^\star$, then, by assumption, there exists an $f \in I$ with $C(f)^\star = R^\star$. But, since $I \subseteq \bigcup_{Q \in h\text{-QMax}^{\star f}(R)} Q$, we have $f \in Q$ for some $Q \in h\text{-QMax}^{\star f}(R)$, a contradiction. Thus $C(I)^\star \subsetneq R^\star$, and hence $I \subseteq Q$ for some $Q \in h\text{-QMax}^{\star f}(R)$. Thus $\{QR_{N_\star(H)} \mid Q \in h\text{-QMax}^{\star f}(R)\}$ is the set of maximal ideals of $R_{N_\star(H)}$ by [23, Proposition 4.8]. \square

We will say that R satisfies property $(\#_\star)$ if, for any nonzero ideal I of R , $C(I)^\star = R^\star$ implies that there exists an $f \in I$ such that $C(f)^\star = R^\star$.

EXAMPLE 2.8. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, and let \star be a semistar operation on R . If R contains a unit of nonzero degree, then R satisfies property $(\#_\star)$ (see [4, Example 1.6] for the case $\star = t$).

The next result is a generalization of the fact that $I^\sim = I \text{Na}(R, \star) \cap K$, where K is the quotient field of R [18, Proposition 3.4(3)].

LEMMA 2.9. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, and \star be a semistar operation on R such that $R^\star \subsetneq R_H$, with property $(\#_\star)$. Then $I^\sim = IR_{N_\star(H)} \cap R_H$ and $I^\sim R_{N_\star(H)} = IR_{N_\star(H)}$ for each homogeneous ideal I of R . In particular R^\sim is integrally closed if and only if $R_{N_\star(H)}$ is integrally closed.

Proof. If $I^\sim = IR_{N_\star(H)} \cap R_H$, then it is easy to see that $I^\sim R_{N_\star(H)} = IR_{N_\star(H)}$. Hence it suffices to show that $I^\sim = IR_{N_\star(H)} \cap R_H$.

(\subseteq) Let $f \in I^\sim (\subseteq R_H)$, and let J be a finitely generated ideal of R such that $J^\star = R^\star$ and $fJ \subseteq I$. Then $C(J)^\star = R^\star$, and since R satisfies property $(\#_\star)$, there exists an $h \in J$ with $C(h)^\star = R^\star$. Hence $h \in N_\star(H)$ and $fh \in I$. Thus $f \in IR_{N_\star(H)} \cap R_H$.

(\supseteq) Let $f = \frac{g}{h} \in IR_{N_\star(H)} \cap R_H$, where $g \in I$ and $h \in N_\star(H)$. Then $fh = g \in I$, and since $C(h)^{m+1}C(f) = C(h)^mC(fh)$ for some integer $m \geq 1$ by [4, Lemma 1.1(1)], we have $fC(h)^{m+1} \subseteq C(f)C(h)^{m+1} = C(h)^mC(fh) = C(h)^mC(g) \subseteq I$. Also note that $(C(h)^{m+1})^\star = R^\star$, since $C(h)^\star = R^\star$. Thus $f \in I^\sim$.

For the in particular case, assume that $R_{N_\star(H)}$ is integrally closed. Using [3, Proposition 2.1], R_H is a GCD-domain, hence is integrally closed. Therefore $R^\sim = R_{N_\star(H)} \cap R_H$ is integrally closed. Conversely, assume that R^\sim is integrally closed. Then R_Q is integrally closed by [14, Proposition 3.8] for all $Q \in \text{QSpec}^\sim(R)$. Let $QR_{N_\star(H)}$ be a maximal ideal of $R_{N_\star(H)}$ for some $Q \in h\text{-QMax}^\sim(R)$. Then $(R_{N_\star(H)})_{QR_{N_\star(H)}} = R_Q$ is integrally closed. Thus $R_{N_\star(H)}$ is integrally closed. \square

LEMMA 2.10. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, and \star be a semistar operation on R such that $R^\star \subsetneq R_H$, with property $(\#_\star)$. Then for each nonzero finitely generated homogeneous ideal I of R , I is \star_f -invertible if and only if, $IR_{N_\star(H)}$ is invertible.

Proof. Let I be nonzero finitely generated homogeneous ideal of R , such that I is \star_f -invertible. Let $QR_{N_\star(H)} \in \text{Max}(R_{N_\star(H)})$, where $Q \in h\text{-QMax}^{\tilde{\star}}(R)$ by Lemma 2.7(2). Thus by [22, Theorem 2.23], $(IR_{N_\star(H)})_{QR_{N_\star(H)}} = IR_Q$ is invertible (is principal) in R_Q . Hence $IR_{N_\star(H)}$ is invertible by [23, Theorem 7.3]. Conversely, assume that I is finitely generated, and $IR_{N_\star(H)}$ is invertible. By flatness we have $I^{-1}R_{N_\star(H)} = (R : I)R_{N_\star(H)} = (R_{N_\star(H)} : IR_{N_\star(H)}) = (IR_{N_\star(H)})^{-1}$. Therefore, $(II^{-1})R_{N_\star(H)} = (IR_{N_\star(H)})(I^{-1}R_{N_\star(H)}) = (IR_{N_\star(H)})(IR_{N_\star(H)})^{-1} = R_{N_\star(H)}$. Hence $II^{-1} \cap N_\star(H) \neq \emptyset$. Let $f \in II^{-1} \cap N_\star(H)$. So that $R^\star = C(f)^\star \subseteq (II^{-1})^{\star f} \subseteq R^\star$. Thus I is \star_f -invertible. \square

COROLLARY 2.11. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, and \star be a semistar operation on R such that $R^\star \subsetneq R_H$, with property $(\#_\star)$ and $0 \neq f \in R$. Then the following conditions are equivalent:*

- (1) $C(f)$ is \star_f -invertible.
- (2) $C(f)R_{N_\star(H)}$ is invertible.
- (3) $C(f)R_{N_\star(H)} = fR_{N_\star(H)}$.

Proof. Exactly is the same as [4, Corollary 1.9]. \square

Let \mathbb{Z} be the additive group of integers. Clearly, the direct sum $\Gamma \oplus \mathbb{Z}$ of Γ with \mathbb{Z} is a torsionless grading monoid. So if y is an indeterminate over $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$, then $R[y, y^{-1}]$ is a graded integral domain graded by $\Gamma \oplus \mathbb{Z}$. In the following proposition we use a technique for defining semistar operations on integral domains, due to Chang and Fontana [9, Theorem 2.3].

PROPOSITION 2.12. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with quotient field K , let y, X be two indeterminates over R and let \star be a semistar operation on R such that $R^\star \subsetneq R_H$. Set $T := R[y, y^{-1}]$, $K_1 := K(y)$ and take the following subset of $\text{Spec}(T)$:*

$$\Delta^\star := \{Q \in \text{Spec}(T) \mid Q \cap R = (0) \text{ or } Q = (Q \cap R)R[y, y^{-1}] \text{ and } (Q \cap R)^{\star f} \subsetneq R^\star\}.$$

Set $S^\star := T[X] \setminus (\bigcup \{Q[X] \mid Q \in \Delta^\star\})$ and:

$$E^{\star'} := E[X]_{S^\star} \cap K_1, \text{ for all } E \in \overline{\mathcal{F}}(T).$$

- (a) The mapping $\star' : \overline{\mathcal{F}}(T) \rightarrow \overline{\mathcal{F}}(T)$, $E \mapsto E^{\star'}$ is a stable semistar operation of finite type on T , i.e., $\star' = \star'$.
- (b) $(\tilde{\star})' = (\star_f)' = \star'$.
- (c) $(ER[y, y^{-1}])^{\star'} \cap K = E^{\tilde{\star}}$ for all $E \in \overline{\mathcal{F}}(R)$.

- (d) $(ER[y, y^{-1}])^{\star'} = E^{\tilde{\star}}R[y, y^{-1}]$ for all $E \in \overline{\mathcal{F}}(R)$.
- (e) $T^{\star'} \subsetneq T_{H'}$, where H' is the set of nonzero homogeneous elements of T , and \star' sends homogeneous fractional ideals to homogeneous ones.
- (f) $\text{QMax}^{\star'}(T) = \{Q \mid Q \in \text{Spec}(T) \text{ such that } Q \cap R = (0) \text{ and } c_R(Q)^{\star f} = R^{\star}\} \cup \{PR[y, y^{-1}] \mid P \in \text{QMax}^{\star f}(R)\}$.
- (g) $h\text{-QMax}^{\star'}(T) = \{PR[y, y^{-1}] \mid P \in h\text{-QMax}^{\tilde{\star}}(R)\}$.
- (h) $(w_R)' = (t_R)' = (v_R)' = w_T$.

Proof. Set $\nabla^{\star} := \{Q \in \text{Spec}(T) \mid Q \cap R = (0) \text{ and } c_D(Q)^{\star f} = R^{\star} \text{ or } Q = PR[y, y^{-1}] \text{ and } P \in \text{QMax}^{\star f}(D)\}$. Then it is easy to see that the elements of ∇^{\star} are the maximal elements of Δ^{\star} (see proof of [9, Theorem 2.3]). Thus

$$S^{\star} := T[X] \setminus \left(\bigcup \{Q[X] \mid Q \in \Delta^{\star}\} \right) = T[X] \setminus \left(\bigcup \{Q[X] \mid Q \in \nabla^{\star}\} \right).$$

(a) It follows from [9, Theorem 2.1 (a) and (b)], that \star' is a stable semistar operation of finite type on T .

(b) Since $\text{QMax}^{\star f}(D) = \text{QMax}^{\tilde{\star}}(D)$, the conclusion follows easily from the fact that $S^{\tilde{\star}} = S^{\star f} = S^{\star}$.

(c) and (d) Exactly are the same as proof of [9, Theorem 2.3(c) and (d)].

(e) From part (d) we have $T^{\star'} = R^{\tilde{\star}}R[y, y^{-1}] \subsetneq R_H R[y, y^{-1}] = T_{H'}$. The second assertion follows from Proposition 2.3, since $\star' = \star'$ by (a).

(f) Follows from [9, Theorem 2.1(e)] and the remark in the first paragraph in the proof.

(g) Let $M \in h\text{-QMax}^{\star'}(T)$. Since $y, y^{-1} \in T$, clearly we have $M \cap R \neq (0)$. Then by (f), there is $P \in \text{QMax}^{\star f}(R)$ such that $M \subseteq PR[y, y^{-1}]$. If $P \in h\text{-QMax}^{\tilde{\star}}(R)$, then $M = PR[y, y^{-1}]$ and we are done. So suppose that $P \notin h\text{-QMax}^{\tilde{\star}}(R)$. Then note that $P_h \in h\text{-QSpec}^{\tilde{\star}}(R)$ and $M \subseteq P_h R[y, y^{-1}] = (PR[y, y^{-1}])_h$; hence $M = P_h R[y, y^{-1}]$, because M is a homogeneous maximal quasi- \star' -ideal. Note that in this case $P_h \in h\text{-QMax}^{\tilde{\star}}(R)$ by [16, Lemma 4.1, Remark 4.5]. So that $M \in \{PR[y, y^{-1}] \mid P \in h\text{-QMax}^{\tilde{\star}}(R)\}$. The other inclusion is trivial.

(h) Suppose that $\star_f = t$. Note that if $M \in \text{QMax}^{\star'}(T)$, and $M \cap R \neq (0)$, then, $M = (M \cap R)[y, y^{-1}]$ and $M \cap R \in \text{QMax}^t(R)$ (cf. [24, Proposition 1.1]). Moreover, if $Q \in \text{Spec}(T)$ is such that $Q \cap R = (0)$, then Q is a quasi- t -maximal ideal of T if and only if $c_R(Q)^t = R$. Indeed, if Q is a quasi- t -maximal ideal of T , and $c_R(Q)^t \subsetneq R$, then there exists

a quasi- t -maximal ideal P of R such that $c_R(Q)^t \subseteq P$. Hence $Q \subseteq P[y, y^{-1}]$, and therefore $Q = P[y, y^{-1}]$. Consequently $(0) = Q \cap R = P[y, y^{-1}] \cap R = P$ which is a contradiction. Conversely assume that $c_R(Q)^t = R$. Suppose Q is not a quasi- t -maximal ideal of T , and let M be a quasi- t -maximal ideal of T which contains Q . Since the containment is proper, we have $M \cap R \neq (0)$. Thus $M = (M \cap R)[y, y^{-1}]$ and $M \cap R \in \text{QMax}^t(R)$ (cf. [24, Proposition 1.1]). Since $Q \subseteq M$, $c_R(Q)$ is contained in the quasi- t -ideal $M \cap R$, so that $c_R(Q)^t \neq R$ which is a contradiction. Thus we showed that $\text{QMax}^t(T) = \{Q \mid Q \in \text{Spec}(T) \text{ such that } Q \cap R = (0) \text{ and } c_R(Q)^{\star f} = R^{\star}\} \cup \{PR[y, y^{-1}] \mid P \in \text{QMax}^{\star f}(R)\} = \text{QMax}^{\star f}(T)$, where the second equality is by (f). Thus using (a) and (b), we obtain $(w_R)' = (t_R)' = (v_R)' = w_T$. \square

It is known that $\text{Pic}(D(X)) = 0$ [1, Theorem 2]. More generally, if \star is a star operation on D , then $\text{Pic}(\text{Na}(D, \star)) = 0$, [26, Theorem 2.14]. Also in the graded case it is shown in [4, Theorem 3.3], that $\text{Pic}(R_{N_v(H)}) = 0$, where $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a graded integral domain containing a unit of nonzero degree. We next show in general that $\text{Pic}(R_{N_\star(H)}) = 0$.

THEOREM 2.13. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree, and \star be a semistar operation on R such that $R^\star \subsetneq R_H$. Then $\text{Pic}(R_{N_\star(H)}) = 0$.*

Proof. Let y be an indeterminate over R , and $T = R[y, y^{-1}]$. Using Proposition 2.12(e) and (g) and Lemma 2.7, we deduce that $\text{Max}(T_{N_{\star'}(H)}) = \{QT_{N_{\star'}(H)} \mid Q \in h\text{-QMax}^{\star f}(R)\}$. Next since $\text{Max}((R_{N_\star(H)})(y)) = \{P(y) \mid P \text{ is a maximal ideal of } R_{N_\star(H)}\}$, [23, Proposition 33.1], we have $\text{Max}((R_{N_\star(H)})(y)) = \{(QR_{N_\star(H)})(y) \mid Q \in h\text{-QMax}^{\star f}(R)\}$. Thus by a computation similar to the proof of [4, Lemma 3.2], we obtain the equality $T_{N_{\star'}(H)} = (R_{N_\star(H)})(y)$. The rest of the proof is exactly the same as proof of [4, Theorem 3.3], using Proposition 2.12. \square

Let D be a domain and T an overring of D . Let \star and \star' be semistar operations on D and T , respectively. One says that T is (\star, \star') -linked to D (or that T is a (\star, \star') -linked overring of D) if

$$F^\star = D^\star \Rightarrow (FT)^{\star'} = T^{\star'}$$

for each nonzero finitely generated ideal F of D . (The preceding definition generalizes the notion of “ t -linked overring” which was introduced

in [13].) It is shown in [15, Theorem 3.8], that T is a (\star, \star') -linked overring of D if and only if $\text{Na}(D, \star) \subseteq \text{Na}(T, \star')$. We need a graded analogue of linkedness.

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, and T be a homogeneous overring of R . Let \star and \star' be semistar operations on R and T , respectively. We say that T is *homogeneously (\star, \star') -linked overring of R* if

$$F^\star = D^\star \Rightarrow (FT)^{\star'} = T^{\star'}$$

for each nonzero homogeneous finitely generated ideal F of R . We say that T is *homogeneously t -linked overring of R* if T is homogeneously (t, t) -linked overring of R . Also it can be seen that T is homogeneously (\star, \star') -linked overring of R if and only if T is homogeneously $(\tilde{\star}, \tilde{\star}')$ -linked overring of R (cf. [15, Theorem 3.8]).

EXAMPLE 2.14. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, and let \star be a semistar operation on R such that $R^\star \subsetneq R_H$. Let $P \in h\text{-QSpec}^\star(R)$. Then, $R_{H \setminus P}$ is a homogeneously (\star, \star') -linked overring of R , for all semistar operation \star' on $R_{H \setminus P}$. Indeed assume that F is a nonzero finitely generated homogeneous ideal of R such that $F^\star = R^\star$. Then we have $F^\star = R^\star$. Thus using Proposition 2.6, we have $FR_{H \setminus P} = F^\star R_{H \setminus P} = R^\star R_{H \setminus P} = R_{H \setminus P}$.

LEMMA 2.15. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree, and let T be a homogeneous overring of R . Let \star (resp. \star') be a semistar operation on R (resp. on T). Then, T is a homogeneously (\star, \star') -linked overring of R if and only if $R_{N_\star(H)} \subseteq T_{N_{\star'}(H)}$.

Proof. Let $f \in R$ such that $C_R(f)^\star = R^\star$. Then by assumption $C_T(f)^{\star'} = (C_R(f)T)^{\star'} = R^{\star'}$. Hence $R_{N_\star(H)} \subseteq T_{N_{\star'}(H)}$. Conversely let F be a nonzero homogeneous finitely generated ideal of R such that $F^\star = R^\star$. Since R has a unit of nonzero degree we can choose an element $f \in R$ such that $C_R(f) = F$. From the fact that $C_R(f)^\star = R^\star$, we have that f is a unit in $R_{N_\star(H)}$ and so by assumption, f is a unit in $T_{N_{\star'}(H)}$. This implies that $C_T(f)^{\star'} = (C_R(f)T)^{\star'} = T^{\star'}$, i.e., $(FT)^{\star'} = T^{\star'}$. \square

3. Kronecker function ring

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, \ast an e.a.b. star operation on R . The graded analogue of the well known Kronecker

function ring (see [23, Theorem 32.7]) of R with respect to \star is defined by

$$\text{Kr}(R, \star) := \left\{ \frac{f}{g} \mid f, g \in R, g \neq 0, \text{ and } C(f) \subseteq C(g)^\star \right\}$$

in [4]. The following lemma is proved in [4, Theorems 2.9 and 3.5], for an e.a.b. star operation \star . We need to state it for e.a.b. semistar operations. Since the proof is exactly the same as star operation case, we omit the proof.

LEMMA 3.1. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, \star an e.a.b. semistar operation on R , and*

$$\text{Kr}(R, \star) := \left\{ \frac{f}{g} \mid f, g \in R, g \neq 0, \text{ and } C(f) \subseteq C(g)^\star \right\}.$$

Then

- (1) $\text{Kr}(R, \star)$ is an integral domain.

In addition, if R has a unit of nonzero degree, then,

- (2) $\text{Kr}(R, \star)$ is a Bézout domain.
- (3) $I \text{Kr}(R, \star) \cap R_H = I^\star$ for every nonzero finitely generated homogeneous ideal I of R .

Inspired by the work of Fontana and Loper in [20], we can generalize this definition of $\text{Kr}(R, \star)$ to all semistar operations on R which send homogeneous fractional ideals, to homogeneous ones, provided that R has a unit of nonzero degree. Before doing that we need a lemma.

LEMMA 3.2. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, \star a semistar operation on R which sends homogeneous fractional ideals to homogeneous ones. Suppose that $a \in R$ is homogeneous and $B, F \in f(R)$, with B homogeneous and $F \subseteq R_H$, such that $aF \subseteq (BF)^\star$. Then there exists a homogeneous $T \in f(R)$ such that $aT \subseteq (BT)^\star$.*

Proof. Suppose that F is generated by $y_1, \dots, y_n \in R_H$. Let $y_i = \sum t_{ij}$ be the decomposition of y_i to homogeneous elements for $i = 1, \dots, n$. Then $ay_i \in (BF)^\star = (\sum y_i B)^\star \subseteq (\sum t_{ij} B)^\star$. Since $(\sum t_{ij} B)^\star$ is homogeneous we have $at_{ij} \in (\sum t_{ij} B)^\star$. Let T be the fractional ideal of R , generated by all homogeneous elements t_{ij} . So that $aT \subseteq (BT)^\star$ and $T \in f(R)$ is homogeneous. \square

THEOREM 3.3. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree, \star a semistar operation on R which sends homogeneous fractional ideals to homogeneous ones, and*

$$\text{Kr}(R, \star) := \left\{ \frac{f}{g} \mid \begin{array}{l} f, g \in R, g \neq 0, \text{ and there is } 0 \neq h \in R \\ \text{such that } C(f)C(h) \subseteq (C(g)C(h))^\star \end{array} \right\}.$$

Then

- (1) $\text{Kr}(R, \star) = \text{Kr}(R, \star_a)$.
- (2) $\text{Kr}(R, \star)$ is a Bézout domain.
- (3) $I \text{Kr}(R, \star) \cap R_H = I^{\star_a}$ for every nonzero finitely generated homogeneous ideal I of R .
- (4) If $f, g \in R$ are nonzero such that $C(f + g)^\star = (C(f) + C(g))^\star$, then $(f, g) \text{Kr}(R, \star) = (f + g) \text{Kr}(R, \star)$. In particular, $f \text{Kr}(R, \star) = C(f) \text{Kr}(R, \star)$ for all $f \in R$.

Proof. It is clear from the definition that $\text{Kr}(R, \star) = \text{Kr}(R, \star_f)$. Thus using Lemma 2.4, we can assume, without loss of generality, that \star is a semistar operation of finite type.

Parts (2) and (3) are direct consequences of (1) using Lemma 3.1. For the proof of (1) we have two cases:

Case 1: Assume that \star is an **e.a.b.** semistar operation of finite type. In this case, for $f, g, h \in R \setminus \{0\}$ we have

$$C(f)C(h) \subseteq (C(g)C(h))^\star \Leftrightarrow C(f) \subseteq C(g)^\star.$$

Therefore $\text{Kr}(R, \star)$ -as defined in this theorem- coincides with $\text{Kr}(R, \star)$ of an **e.a.b.** semistar operation \star , as defined in Lemma 3.1. Also in this case $\star = \star_a$ by [20, Proposition 4.5(5)]. Hence in this case (1) is true.

Case 2: General case. Let \star be a semistar operation of finite type on R . By definition it is easy to see that, given two semistar operations on R with $\star_1 \leq \star_2$, then $\text{Kr}(R, \star_1) \subseteq \text{Kr}(R, \star_2)$. Using [20, Proposition 4.5(3)] we have $\star \leq \star_a$. Therefore $\text{Kr}(R, \star) \subseteq \text{Kr}(R, \star_a)$. Conversely let $f/g \in \text{Kr}(R, \star_a)$. Then, by Case 1, $C(f) \subseteq C(g)^{\star_a}$. Set $A := C(f)$ and $B := C(g)$. Then $A \subseteq B^{\star_a} = \bigcup \{ (BH)^\star : H \in f(R) \}$. Suppose that A is generated by homogeneous elements $x_1, \dots, x_n \in R$. Then there is $H_i \in f(R)$, such that $x_i H_i \subseteq (BH_i)^\star$ for $i = 1, \dots, n$. Choose $0 \neq r_i \in R$ such that $F_i = r_i H_i \subseteq R$. Thus $x_i F_i \subseteq (BF_i)^\star$. Therefore Lemma 3.2 gives a homogeneous $T_i \in f(R)$ such that $x_i T_i \subseteq (BT_i)^\star$. Now set $T := T_1 T_2 \cdots T_n$ which is a finitely generated homogeneous

fractional ideal of R such that $AT \subseteq (BT)^\star$. Now since R has a unit of nonzero degree, we can find an element $h \in R$ such that $C(h) = T$. Then $C(f)C(h) \subseteq (C(g)C(h))^\star$. This means that $f/g \in \text{Kr}(R, \star)$ to complete the proof of (1).

The proof of (4) is exactly the same as [4, Theorem 2.9(3)]. □

4. Graded P \star MDs

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, \star be a semistar operation on R , H be the set of nonzero homogeneous elements of R , and $N_\star(H) = \{f \in R \mid C(f)^\star = R^\star\}$. In this section we define the notion of graded Prüfer \star -multiplication domain (graded P \star MD for short) and give several characterization of it.

We say that a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ with a semistar operation \star , is a *graded Prüfer \star -multiplication domain (graded P \star MD)* if every nonzero finitely generated homogeneous ideal of R is a \star_f -invertible, i.e., $(II^{-1})^{\star_f} = R^\star$ for every nonzero finitely generated homogeneous ideal I of R . It is easy to see that a graded P \star MD is the same as a graded P \star_f MD by definition, and is the same as a graded P $\tilde{\star}$ MD by [22, Proposition 2.18]. When $\star = v$ we recover the classical notion of a *graded Prüfer v -multiplication domain (graded PvMD)* [2]. It is known that R is a graded PvMD if and only if R is a PvMD [2, Theorem 6.4].

Also when $\star = d$, a graded PdMD is called a *graded Prüfer domain* [4]. It is clear that every graded Prüfer domain is a graded PvMD and hence a PvMD. In particular every graded Prüfer domain is an integrally closed domain. Although R is a graded PvMD if and only if R is a PvMD, Anderson and Chang in [4, Example 3.6] provided an example of a graded Prüfer domain which is not Prüfer. It is known that if A, B, C are ideals of an integral domain D , then $(A+B)(A+C)(B+C) = (A+B+C)(AB+AC+BC)$. Thus $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a graded Prüfer domain if and only if every nonzero ideal of R generated by two homogeneous elements is invertible. We use this result in this section without comments.

The following proposition is inspired by [23, Theorem 24.3].

PROPOSITION 4.1. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then the following conditions are equivalent:*

- (1) R is a graded Prüfer domain.

- (2) Each finitely generated nonzero homogeneous ideal of R is a cancellation ideal.
- (3) If A, B, C are finitely generated homogeneous ideals of R such that $AB = AC$ and A is nonzero, then $B = C$.
- (4) R is integrally closed and there is a positive integer $n > 1$ such that $(a, b)^n = (a^n, b^n)$ for each $a, b \in H$.
- (5) R is integrally closed and there exists an integer $n > 1$ such that $a^{n-1}b \in (a^n, b^n)$ for each $a, b \in H$.

Proof. The implications (1) \Rightarrow (2) \Rightarrow (3) and (4) \Rightarrow (5) are clear.

(3) \Rightarrow (4) By the same argument as in the proof of part (2) \Rightarrow (3), in [23, Proposition 24.1], we have that R is integrally closed in R_H . Therefore by [3, Proposition 5.4], R is integrally closed. Now if $a, b \in H$ we have $(a, b)^3 = (a, b)(a^2, b^2)$. Thus by (3) we obtain that $(a, b)^2 = (a^2, b^2)$.

(5) \Rightarrow (1) If (5) holds then [23, Proposition 24.2], implies that each nonzero homogeneous ideal generated by two homogeneous elements is invertible. Therefore R is a graded Prüfer domain. \square

The ungraded version of the following theorem is due to Gilmer (see [23, Corollary 28.5]).

THEOREM 4.2. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree. Then R is a graded Prüfer domain if and only if $C(f)C(g) = C(fg)$ for all $f, g \in R_H$.*

Proof. (\Rightarrow) Let $f, g \in R_H$. Then by [4, Lemma 1.1(1)], there exists some positive integer n such that $C(f)^{n+1}C(g) = C(f)^nC(fg)$. Now since R is a graded Prüfer domain, the homogeneous fractional ideal $C(f)^n$ is invertible. Thus $C(f)C(g) = C(fg)$ for all $f, g \in R_H$.

(\Leftarrow) Let $\alpha \in H$ be a unit of nonzero degree. Assume that $C(f)C(g) = C(fg)$ for all $f, g \in R_H$. Hence R is integrally closed by [2, Theorem 3.7]. Now let $a, b \in H$ be arbitrary. We can choose a positive integer n such that $\deg(a) \neq \deg(\alpha^n b)$. So that $C(a + \alpha^n b) = (a, b)$. Hence, since $(a + \alpha^n b)(a - \alpha^n b) = a^2 - (\alpha^n b)^2$, we have $(a, b)(a, -b) = (a^2, -b^2)$. Consequently $(a, b)^2 = (a^2, b^2)$. Thus by Proposition 4.1, we see that R is a graded Prüfer domain. \square

LEMMA 4.3. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain and P be a homogeneous prime ideal. Then, the following statements are equivalent:*

- (1) $R_{H \setminus P}$ is a graded Prüfer domain
- (2) R_P is a valuation domain.
- (3) For each nonzero homogeneous $u \in R_H$, u or u^{-1} is in $R_{H \setminus P}$.

Proof. (1) \Rightarrow (2) Suppose that $R_{H \setminus P}$ is a graded Prüfer domain. In particular $R_{H \setminus P}$ is a (graded) PvMD and each nonzero homogeneous ideal of $R_{H \setminus P}$ is a t -ideal. So that $h\text{-QMax}^t(R_{H \setminus P}) = \{PR_{H \setminus P}\}$. Thus by [10, Lemma 2.7], we see that $(R_{H \setminus P})_{PR_{H \setminus P}} = R_P$ is a valuation domain.

(2) \Rightarrow (3) Let $0 \neq u \in R_H$. Thus by the hypothesis u or u^{-1} is in R_P . Thus u or u^{-1} is in $R_{H \setminus P}$.

(3) \Rightarrow (1) Let I, J be two nonzero homogeneous ideals of $R_{H \setminus P}$ and assume that $I \not\subseteq J$. So there is a homogeneous element $a \in I \setminus J$. For each $b \in J$, we have $\frac{a}{b} \notin R_{H \setminus P}$, since otherwise we have $a = (\frac{a}{b})b \in J$. Thus by the hypothesis $\frac{b}{a} \in R_{H \setminus P}$. Hence $b = (\frac{b}{a})a \in I$. Thus we showed that $J \subseteq I$, and so every two homogeneous ideal are comparable.

Now let (a, b) be an ideal generated by two homogeneous elements of $R_{H \setminus P}$. Now by the first paragraph $(a, b) = (a)$ or $(a, b) = (b)$. Thus (a, b) is invertible. Hence $R_{H \setminus P}$ is a graded Prüfer domain. \square

THEOREM 4.4. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, and \star be a semistar operation on R such that $R^\star \subsetneq R_H$. Then, the following statements are equivalent:*

- (1) R is a graded $P\star$ MD.
- (2) $R_{H \setminus P}$ is a graded Prüfer domain for each $P \in h\text{-QSpec}^{\tilde{\star}}(R)$.
- (3) $R_{H \setminus P}$ is a graded Prüfer domain for each $P \in h\text{-QMax}^{\tilde{\star}}(R)$.
- (4) R_P is a valuation domain for each $P \in h\text{-QSpec}^{\tilde{\star}}(R)$.
- (5) R_P is a valuation domain for each $P \in h\text{-QMax}^{\tilde{\star}}(R)$.

Proof. (2) \Rightarrow (3) is trivial, and, (2) \Leftrightarrow (4) and (3) \Leftrightarrow (5), follow from Lemma 4.3.

(1) \Rightarrow (2) Let I be a nonzero finitely generated homogeneous ideal of R . Then I is $\tilde{\star}$ -invertible. Therefore, for each $P \in h\text{-QSpec}^{\tilde{\star}}(R)$, since $II^{-1} \not\subseteq P$, we have $R_{H \setminus P} = (II^{-1})R_{H \setminus P} = IR_{H \setminus P}I^{-1}R_{H \setminus P} = (IR_{H \setminus P})(IR_{H \setminus P})^{-1}$. So that $IR_{H \setminus P}$ is invertible. Thus $R_{H \setminus P}$ is a graded Prüfer domain for each $P \in h\text{-QSpec}^{\tilde{\star}}(R)$.

(3) \Rightarrow (1) Let I be a nonzero finitely generated homogeneous ideal of R . Suppose that I is not $\tilde{\star}$ -invertible. Hence there exists $P \in h\text{-QMax}^{\tilde{\star}}(R)$ such that $II^{-1} \subseteq P$. Thus $R_{H \setminus P} = (IR_{H \setminus P})(IR_{H \setminus P})^{-1} = II^{-1}R_{H \setminus P} \subseteq PR_{H \setminus P}$, which is a contradiction. So that $II^{-1} \not\subseteq P$ for

each $P \in h\text{-QMax}^{\tilde{\star}}(R)$. Therefore $(II^{-1})^{\tilde{\star}} = R^{\tilde{\star}}$, that is I is $\tilde{\star}$ -invertible, and hence R is a graded $P\star\text{MD}$. \square

The ungraded version of the following theorem is due to Chang in the star operation case [8, Theorem 3.7], and is due to Anderson, Fontana, and Zafrullah in the case of semistar operations [6, Theorem 1.1].

THEOREM 4.5. *Let $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ be a graded integral domain with a unit of nonzero degree, and \star be a semistar operation on R such that $R^{\star} \subsetneq R_H$. Then R is a graded $P\star\text{MD}$ if and only if $(C(f)C(g))^{\tilde{\star}} = C(fg)^{\tilde{\star}}$ for all $f, g \in R_H$.*

Proof. (\Rightarrow) Let $f, g \in R_H$. Choose a positive integer n such that $C(f)^{n+1}C(g) = C(f)^nC(fg)$ by [4, Lemma 1.1(1)]. Thus $(C(f)^{n+1}C(g))^{\tilde{\star}} = (C(f)^nC(fg))^{\tilde{\star}}$. Since R is a graded $P\star\text{MD}$, the homogeneous fractional ideal $C(f)^n$ is $\tilde{\star}$ -invertible. Thus $(C(f)C(g))^{\tilde{\star}} = C(fg)^{\tilde{\star}}$ for all $f, g \in R_H$.

(\Leftarrow) Assume that $(C(f)C(g))^{\tilde{\star}} = C(fg)^{\tilde{\star}}$ for all $f, g \in R_H$. Let $P \in h\text{-QMax}^{\tilde{\star}}(R)$. Then using Proposition 2.6, we have $C(f)R_{H \setminus P}C(g)R_{H \setminus P} = C(f)C(g)R_{H \setminus P} = (C(f)C(g))^{\tilde{\star}}R_{H \setminus P} = C(fg)^{\tilde{\star}}R_{H \setminus P} = C(fg)R_{H \setminus P}$. Since $R_{H \setminus P}$ has a unit of nonzero degree, Theorem 4.2 shows that $R_{H \setminus P}$ is a graded Prüfer domain. Now Theorem 4.4, implies that R is a graded $P\star\text{MD}$. \square

We now recall the notion of \star -valuation overring (a notion due essentially to P. Jaffard [25, page 46]). For a domain D and a semistar operation \star on D , we say that a valuation overring V of D is a \star -valuation overring of D provided $F^{\star} \subseteq FV$, for each $F \in f(D)$.

REMARK 4.6. (1) *Let \star be a semistar operation on a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$. Recall that for each $F \in f(R)$ we have*

$$F^{\star a} = \bigcap \{FV \mid V \text{ is a } \star\text{-valuation overring of } R\},$$

by [19, Propositions 3.3 and 3.4 and Theorem 3.5].

(2) *We have $N_{\star}(H) = N_{\tilde{\star}_a}(H)$. Indeed, since $\tilde{\star} \leq \tilde{\star}_a$ by [20, Proposition 4.5], we have $N_{\star}(H) = N_{\tilde{\star}}(H) \subseteq N_{\tilde{\star}_a}(H)$. Now if $f \in R \setminus N_{\star}(H)$ then, $C(f)^{\tilde{\star}} \subsetneq R^{\tilde{\star}}$. Thus there is a homogeneous quasi- $\tilde{\star}$ -prime ideal P of R such that $C(f) \subseteq P$. Let V be a valuation domain dominating R_P with maximal ideal M [23, Corollary 19.7]. Therefore V is a $\tilde{\star}$ -valuation overring of R by [18, Theorem 3.9], and $C(f)V \subseteq M$; so $C(f)^{(\tilde{\star})a} \subsetneq R^{(\tilde{\star})a}$ and $f \notin N_{\tilde{\star}_a}(H)$. Thus we obtain that $N_{\star}(H) = N_{\tilde{\star}_a}(H)$.*

In the following theorem we generalize a characterization of PvMDs proved by Arnold and Brewer [7, Theorem 3]. It also generalizes [8, Theorem 3.7], [4, Theorems 3.4 and 3.5], and [17, Theorem 3.1].

THEOREM 4.7. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree, and \star be a semistar operation on R such that $R^\star \subsetneq R_H$. Then, the following statements are equivalent:*

- (1) R is a graded $P\star$ MD.
- (2) Every ideal of $R_{N_\star(H)}$ is extended from a homogeneous ideal of R .
- (3) Every principal ideal of $R_{N_\star(H)}$ is extended from a homogeneous ideal of R .
- (4) $R_{N_\star(H)}$ is a Prüfer domain.
- (5) $R_{N_\star(H)}$ is a Bézout domain.
- (6) $R_{N_\star(H)} = \text{Kr}(R, \tilde{\star})$.
- (7) $\text{Kr}(R, \tilde{\star})$ is a quotient ring of R .
- (8) $\text{Kr}(R, \tilde{\star})$ is a flat R -module.
- (9) $I^\tilde{\star} = I^{\tilde{\star}^a}$ for each nonzero homogeneous finitely generated ideal of R .

In particular if R is a graded $P\star$ MD, then $R^{\tilde{\star}}$ is integrally closed.

Proof. By Proposition 2.3 and Theorem 3.3, we have $\text{Kr}(R, \tilde{\star})$ is well-defined and is a Bézout domain.

(1) \Rightarrow (2) Let $0 \neq f \in R$. Then $C(f)$ is $\tilde{\star}$ -invertible, because R is a graded $P\star$ MD, and thus $fR_{N_\star(H)} = C(f)R_{N_\star(H)}$ by Corollary 2.11. Hence if A is an ideal of $R_{N_\star(H)}$, then $A = IR_{N_\star(H)}$ for some ideal I of R , and thus $A = (\sum_{f \in I} C(f))R_{N_\star(H)}$.

(2) \Rightarrow (3) Clear.

(3) \Rightarrow (1) Is the same as part (3) \Rightarrow (1) in [4, Theorem 3.4].

(1) \Rightarrow (4) Let A be a nonzero finitely generated ideal of $R_{N_\star(H)}$. Then by Corollary 2.11, $A = IR_{N_\star(H)}$ for some nonzero finitely generated homogeneous ideal I of R . Since R is a graded $P\star$ MD, I is $\tilde{\star}$ -invertible, and thus $A = IR_{N_\star(H)}$ is invertible by Lemma 2.10.

(4) \Rightarrow (5) Follows from Theorem 2.13.

(5) \Rightarrow (6) Clearly $R_{N_\star(H)} \subseteq \text{Kr}(R, \tilde{\star})$. Since $R_{N_\star(H)}$ is a Bézout domain, then $\text{Kr}(R, \tilde{\star})$ is a quotient ring of $R_{N_\star(H)}$, by [23, Proposition 27.3]. If $Q \in h\text{-QMax}^{\tilde{\star}}(R)$, then $Q\text{Kr}(R, \tilde{\star}) \subsetneq \text{Kr}(R, \tilde{\star})$. Otherwise $Q\text{Kr}(R, \tilde{\star}) = \text{Kr}(R, \tilde{\star})$, and hence there is an element $f \in Q$, such that $f\text{Kr}(R, \tilde{\star}) = \text{Kr}(R, \tilde{\star})$. Thus $\frac{1}{f} \in \text{Kr}(R, \tilde{\star})$. Therefore $R = C(1) \subseteq C(f)^{(\tilde{\star})^a} \subseteq R^{(\tilde{\star})^a}$, so that $C(f)^{(\tilde{\star})^a} = R^{(\tilde{\star})^a}$. Hence $f \in N_{(\tilde{\star})^a}(H) = N_\star(H)$

by Remark 4.6(2). This means that $Q^{\tilde{\star}} = R^{\tilde{\star}}$, a contradiction. Thus $Q \text{Kr}(R, \tilde{\star}) \subsetneq \text{Kr}(R, \tilde{\star})$, and so there is a maximal ideal M of $\text{Kr}(R, \tilde{\star})$ such that $Q \text{Kr}(R, \tilde{\star}) \subseteq M$. Hence $M \cap R_{N_{\star}(H)} = QR_{N_{\star}(H)}$, by Lemma 2.7. Consequently $R_Q \subseteq \text{Kr}(R, \tilde{\star})_M$, and since R_Q is a valuation domain, we have $R_Q = \text{Kr}(R, \tilde{\star})_M$. Therefore $R_{N_{\star}(H)} = \bigcap_{Q \in h\text{-QMax}^{\tilde{\star}}(R)} R_Q \supseteq \bigcap_{M \in \text{Max}(\text{Kr}(R, \tilde{\star}))} \text{Kr}(R, \tilde{\star})_M$. Hence $R_{N_{\star}(H)} = \text{Kr}(R, \tilde{\star})$.

(6) \Rightarrow (7) and (7) \Rightarrow (8) are clear.

(8) \Rightarrow (6) Recall that an overring T of an integral domain S is a flat S -module if and only if $T_M = S_{M \cap S}$ for all $M \in \text{Max}(T)$ by [32, Theorem 2].

Let A be an ideal of R such that $A \text{Kr}(R, \tilde{\star}) = \text{Kr}(R, \tilde{\star})$. Then there exists an element $f \in A$ such that $f \text{Kr}(R, \tilde{\star}) = \text{Kr}(R, \tilde{\star})$ using Theorem 3.3; so $\frac{1}{f} \in \text{Kr}(R, \tilde{\star}) = \text{Kr}(R, \tilde{\star}_a)$. Thus $R = C(1) \subseteq C(f)^{\tilde{\star}_a} \subseteq R^{\tilde{\star}_a}$, and so $C(f)^{\tilde{\star}_a} = R^{\tilde{\star}_a}$. Hence $C(f)^{\tilde{\star}} = R^{\tilde{\star}}$. Therefore $f \in A \cap N_{\star}(H) \neq \emptyset$. Hence, if P_0 is a homogeneous maximal quasi- $\tilde{\star}$ -ideal of R , then $P_0 \text{Kr}(R, \tilde{\star}) \subsetneq \text{Kr}(R, \tilde{\star})$, and since $P_0 R_{N_{\star}(H)}$ is a maximal ideal of $R_{N_{\star}(H)}$, there is a maximal ideal M_0 of $\text{Kr}(R, \tilde{\star})$ such that $M_0 \cap R = (M_0 \cap R_{N_{\star}(H)}) \cap R = P_0 R_{N_{\star}(H)} \cap R = P_0$. Thus by (8), $\text{Kr}(R, w)_{M_0} = R_{P_0} = (R_{N(H)})_{P_0 R_{N(H)}}$.

Let M_1 be a maximal ideal of $\text{Kr}(R, \tilde{\star})$, and let P_1 be a homogeneous maximal quasi- $\tilde{\star}$ -ideal of R such that $M_1 \cap R_{N_{\star}(H)} \subseteq P_1 R_{N_{\star}(H)}$. By the above paragraph, there is a maximal ideal M_2 of $\text{Kr}(R, \tilde{\star})$ such that $\text{Kr}(R, \tilde{\star})_{M_2} = (R_{N_{\star}(H)})_{P_1 R_{N_{\star}(H)}}$. Note that $\text{Kr}(R, \tilde{\star})_{M_2} \subseteq \text{Kr}(R, \tilde{\star})_{M_1}$, M_1 and M_2 are maximal ideals, and $\text{Kr}(R, \tilde{\star})$ is a Prüfer domain; hence $M_1 = M_2$ (cf. [23, Theorem 17.6(c)]) and $\text{Kr}(R, \tilde{\star})_{M_1} = (R_{N_{\star}(H)})_{P_1 R_{N_{\star}(H)}}$. Thus

$$\begin{aligned} \text{Kr}(R, \tilde{\star}) &= \bigcap_{M \in \text{Max}(\text{Kr}(R, \tilde{\star}))} \text{Kr}(R, \tilde{\star})_M = \bigcap_{P \in h\text{-QMax}^{\tilde{\star}}(R)} (R_{N_{\star}(H)})_{P R_{N_{\star}(H)}} \\ &= R_{N_{\star}(H)}. \end{aligned}$$

(6) \Rightarrow (9) Assume that $R_{N_{\star}(H)} = \text{Kr}(R, \tilde{\star})$. Let I be a nonzero homogeneous finitely generated ideal of R . Then by Lemma 2.9 and Theorem 3.3(3), we have $I^{\tilde{\star}} = I R_{N_{\star}(H)} \cap R_H = I \text{Kr}(R, \tilde{\star}) \cap R_H = I^{\tilde{\star}_a}$.

(9) \Rightarrow (1) Let a and b be two nonzero homogeneous elements of R . Then $((a, b)^3)^{\tilde{\star}_a} = ((a, b)(a^2, b^2))^{\tilde{\star}_a}$ which implies that $((a, b)^2)^{\tilde{\star}_a} = (a^2, b^2)^{\tilde{\star}_a}$. Hence $((a, b)^2)^{\tilde{\star}} = (a^2, b^2)^{\tilde{\star}}$ and so $(a, b)^2 R_{H \setminus P} = (a^2, b^2) R_{H \setminus P}$ for each homogeneous maximal quasi- $\tilde{\star}$ -ideal P of R . On the other hand $R^{\tilde{\star}} = R^{\tilde{\star}_a}$ by (9). Hence $R^{\tilde{\star}}$ is integrally closed. Thus $R^{\tilde{\star}} R_{H \setminus P} = R_{H \setminus P}$ is

integrally closed. Therefore by Proposition 4.1, $R_{H \setminus P}$ is a graded Prüfer domain for each homogeneous maximal quasi- \star_f -ideal of R . Thus R is a graded P \star MD by Theorem 4.4. \square

The following theorem is a graded version of a characterization of Prüfer domains proved by Davis [12, Theorem 1]. It also generalizes [13, Theorem 2.10], in the t -operation, and [15, Theorem 5.3], in the case of semistar operations.

THEOREM 4.8. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree, and \star be a semistar operation on R such that $R^\star \subsetneq R_H$. Then, the following statements are equivalent:*

- (1) R is a graded P \star MD.
- (2) Each homogeneously (\star, t) -linked overring of R is a PvMD.
- (3) Each homogeneously (\star, d) -linked overring of R is a graded Prüfer domain.
- (4) Each homogeneously (\star, t) -linked overring of R , is integrally closed.
- (5) Each homogeneously (\star, d) -linked overring of R , is integrally closed.

Proof. (1) \Rightarrow (2) Let T be a homogeneously (\star, t) -linked overring of R . Thus by Lemma 2.15, we have $R_{N_\star(H)} \subseteq T_{N_v(H)}$. Since R is a graded P \star MD, by Theorem 4.7, we have $R_{N_\star(H)}$ is a Prüfer domain. Thus by [23, Theorem 26.1], we have $T_{N_v(H)}$ is a Prüfer domain. Hence, again by Theorem 4.7, we have T is a graded PvMD. Therefore using [2, Theorem 6.4], T is a PvMD.

(2) \Rightarrow (4) \Rightarrow (5) and (3) \Rightarrow (5) are clear.

(5) \Rightarrow (1) Let $P \in h\text{-QMax}^\star(R)$. For a nonzero homogeneous $u \in R_H$, let $T = R[u^2, u^3]_{H \setminus P}$. Then $R_{H \setminus P}$ and T are homogeneously (\star, d) -linked overring of R by Example 2.14. So that $R_{H \setminus P}$ and T are integrally closed. Hence $u \in T$, and since $T = R_{H \setminus P}[u^2, u^3]$, there exists a polynomial $\gamma \in R_{H \setminus P}[X]$ such that $\gamma(u) = 0$ and one of the coefficients of γ is a unit in $R_{H \setminus P}$. So u or u^{-1} is in $R_{H \setminus P}$ by [27, Theorem 67]. Therefore by Lemma 4.3, $R_{H \setminus P}$ is a graded Prüfer domain. Thus R is a graded P \star MD by Theorem 4.4.

(1) \Rightarrow (3) Is the same argument as in part (1) \Rightarrow (2). \square

The next result gives new characterizations of PvMDs for graded integral domains, which is the special cases of Theorems 4.4, 4.5, 4.7, and 4.8, for $\star = v$.

COROLLARY 4.9. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree. Then, the following statements are equivalent:

- (1) R is a (graded) PvMD.
- (2) $R_{H \setminus P}$ is a graded Prüfer domain for each $P \in h\text{-QMax}^t(R)$.
- (3) R_P is a valuation domain for each $P \in h\text{-QMax}^t(R)$.
- (4) Every ideal of $R_{N_v(H)}$ is extended from a homogeneous ideal of R .
- (5) $R_{N_v(H)}$ is a Prüfer domain.
- (6) $R_{N_v(H)}$ is a Bézout domain.
- (7) $R_{N_v(H)} = \text{Kr}(R, w)$.
- (8) $\text{Kr}(R, w)$ is a quotient ring of R .
- (9) $\text{Kr}(R, w)$ is a flat R -module.
- (10) Each homogeneously t -linked overring of R is a PvMD.
- (11) Each homogeneously t -linked overring of R , is integrally closed.
- (12) $(C(f)C(g))^w = C(fg)^w$ for all $f, g \in R_H$.
- (13) $I^w = I^{w_a}$ for each nonzero homogeneous finitely generated ideal of R .

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