Korean J. Math. **22** (2014), No. 1, pp. 151–167
http://dx.doi.org/10.11568/kjm.2014.22.1.151

REGULARITY AND SEMIPOTENCY OF HOM

Hamza Hakmi

Abstract. Let M, N be modules over a ring R and $[M,N] = \mathrm{Hom}_R(M,N)$. The concern is study of: (1) Some fundamental properties of $[M,N]$ when $[M,N]$ is regular or semipotent. (2) The substructures of $[M,N]$ such as radical, the singular and co-singular ideals, the total and others has raised new questions for research in this area. New results obtained include necessary and sufficient conditions for $[M,N]$ to be regular or semipotent. New substructures of $[M,N]$ are studied and its relationship with the Tot of $[M,N]$. In this paper we show that, the endomorphism ring of a module M is regular if and only if the module M is semi-injective (projective) and the kernel (image) of every endomorphism is a direct summand.

1. Introduction.

In this paper rings R, are associative with identity unless otherwise indicated. All modules over a ring R are unitary right modules. We write $J(R)$ and $U(R)$ for the Jacobson radical and the group of units of a ring R. A submodule N of a module M is said to be *small* in M, if $N + K \neq M$ for any proper submodule K of M [1]. Also, a submodule Q of a module M is said to be *large* (*essential*) in M if $Q \cap K \neq 0$ for every nonzero submodule K of M [1]. For a submodule N of a module M, we use $N \subseteq^{\oplus} M$ to mean that N is a direct summand of M, and write $N \leq_e M$ and $N \ll M$ to indicate that N is an large, respectively

Received January 18, 2014. Revised March 20, 2014. Accepted March 25, 2014.
2010 Mathematics Subject Classification: Primary: 16E50,16E70. Secondary: 16D40, 16D50.
Key words and phrases: Regular ring, semipotent ring, Radical Jacobson, The Total, (Co)Singular ideal.

small, submodule of M. We use the notation: $E_M = \mathrm{End}_R(M)$ and $[M, N] = \mathrm{Hom}_R(M, N)$. Thus, $[M, N]$ is an (E_M, E_N)-bimodule. Our main concern is about the four substructures of $\mathrm{Hom}_R(M, N)$ and the regularity, semipotency of $\mathrm{Hom}_R(M, N)$ given as follows [9].

• The Jacobson radical.

$$J[M, N] = \{\alpha : \alpha \in [M, N];\ \beta\alpha \in J(E_M) \text{ for all } \beta \in [N, M]\}.$$

$$J[M, N] = \{\alpha : \alpha \in [M, N];\ \alpha\beta \in J(E_N) \text{ for all } \beta \in [N, M]\}.$$

Thus $J[M, M] = J(E_M)$. In particular, $J[R, R] = J(R)$.

• The singular ideal $\triangle[M, N] = \{\alpha : \alpha \in [M, N],\ \ \mathrm{Ker}(\alpha) \leq_e M\}$. In particular, $\Delta(E_M) = \Delta[M, M] = \{\alpha : \alpha \in E_M; \mathrm{Ker}(\alpha) \leq_e M\}$.

• The co-singular ideal $\nabla[M, N] = \{\alpha : \alpha \in [M, N],\ \ \mathrm{Im}(\alpha) \ll M\}$. In particular, $\nabla(E_M) = \nabla[M, M] = \{\alpha : \alpha \in E_M; \mathrm{Im}(\alpha) \ll M\}$

• The total.

$$\mathrm{Tot}[M, N] = \{\alpha : \alpha \in [M, N];\ [N, M]\alpha \ \text{ contains no nonzero idempotents}\}.$$

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The Total is the concept was first introduced by F.Kasch. An excellent reference on the study of the total as will as its connections with the Jacobson radical and the singular and co-singular ideals or other substructures of ring. In section 2, it is proved some basic properties of $[M, N]$ when $[M, N]$ is regular include necessary and sufficient conditions for $[M, N]$ to be regular. In section 3, it is proved that for a module M, E_M is regular if and only if M is semi-projective and $\mathrm{Im}(\alpha) \subseteq^{\oplus} M$ if and only if M is semi-injective and $\mathrm{Ker}(\alpha) \subseteq^{\oplus} M$ for any $\alpha \in E_M$. The semipotentness of $[M, N]$ is studied in section 4, include necessary and sufficient conditions for $[M, N]$ to be semipotent. A new description of $J[M, N]$ is obtained in case $[M, N]$ is semipotent. Also, it is proved that for a semi-projective module P; $J(E_P) = \{\alpha : \alpha \in E_P; \mathrm{Im}(1 - \alpha\beta) = P \text{ for all } \beta \in E_P\}$ and for a semi-injective module Q; $J(E_Q) = \{\alpha : \alpha \in E_Q; \mathrm{Ker}(1 - \alpha\beta) = 0 \text{ for all } \beta \in E_Q\}$. In addition to, it is proved that for a locally projective module P; $\mathrm{Tot}(E_P) = \{\alpha : \alpha \in E_P; \mathrm{Im}(1 - \alpha\beta) = P \text{ for all } \beta \in E_P\}$ and for a locally injective module Q; $\mathrm{Tot}(E_Q) = \{\alpha : \alpha \in E_Q; \mathrm{Ker}(1 - \alpha\beta) = 0 \text{ for all } \beta \in E_Q\}$.

2. Regularity of $[M, N]$.

Let M_R, N_R be modules. An element α of $[M, N]$ is called *regular* [2], if there exists $\beta \in [N, M]$ such that $\alpha = \alpha\beta\alpha$. $[M, N]$ is called *regular* if each $\alpha \in [M, N]$ is regular. We start with the following fundamental lemma which gives information about relationship between any two elements of $[M, N]$.

LEMMA 2.1. *Let M_R, N_R be modules and $\alpha \in [M, N]$, $\beta \in [N, M]$. The following hold:*
(1) $Im(\alpha) + Im(1_N - \alpha\beta) = N$.
(2) $Im(\alpha - \alpha\beta\alpha) = Im(\alpha) \cap Im(1_N - \alpha\beta)$.
(3) $Im(\beta) + Im(1_M - \beta\alpha) = M$.
(4) $Im(\beta - \beta\alpha\beta) = Im(\beta) \cap Im(1_M - \beta\alpha)$.
(5) $Ker(\alpha) \cap Ker(1_M - \beta\alpha) = 0$.
(6) $Ker(\alpha - \alpha\beta\alpha) = Ker(\alpha) + Ker(1_M - \beta\alpha)$.
(7) $Ker(\beta) \cap Ker(1_N - \alpha\beta) = 0$.
(8) $Ker(\beta - \beta\alpha\beta) = Ker(\beta) + Ker(1_N - \alpha\beta)$.

Proof. We have $\alpha\beta \in E_N$ and $\beta\alpha \in E_M$.
(1). It is clear that $N = \mathrm{Im}(\alpha\beta) + \mathrm{Im}(1_N - \alpha\beta) \subseteq \mathrm{Im}(\alpha) + \mathrm{Im}(1_N - \alpha\beta) \subseteq N$. Similarly (3) holds.
(2). $\alpha - \alpha\beta\alpha \in [M, N]$. $\mathrm{Im}(\alpha - \alpha\beta\alpha) = \mathrm{Im}((1_N - \alpha\beta)\alpha) \subseteq \mathrm{Im}(1_N - \alpha\beta)$ and $\mathrm{Im}(\alpha - \alpha\beta\alpha) = \mathrm{Im}(\alpha(1_M - \beta\alpha)) \subseteq \mathrm{Im}(\alpha)$. So $\mathrm{Im}(\alpha - \alpha\beta\alpha) \subseteq \mathrm{Im}(\alpha) \cap \mathrm{Im}(1_N - \alpha\beta)$.
Let $x \in \mathrm{Im}(\alpha) \cap \mathrm{Im}(1_N - \alpha\beta)$; $x \in N$ and $x = \alpha(y) = (1_N - \alpha\beta)(z)$ where $y \in M$, $z \in N$. So $x = z - \alpha\beta(z)$, $z = x + \alpha\beta(z) = \alpha(y) + \alpha\beta(z) = \alpha(y + \beta(z))$. Let $y_0 = y + \beta(z) \in M$. Then $z = \alpha(y_0)$ and $x = (1_N - \alpha\beta)(z) = (1_N - \alpha\beta)\alpha(y_0) = (\alpha - \alpha\beta\alpha)(y_0) \in \mathrm{Im}(\alpha - \alpha\beta\alpha)$. Thus, $\mathrm{Im}(\alpha) \cap \mathrm{Im}(1_N - \alpha\beta) \subseteq \mathrm{Im}(\alpha - \alpha\beta\alpha)$. Similarly (4) holds. (5) and (7) are clears.
(6) It is clear that $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\alpha - \alpha\beta\alpha)$ and $\mathrm{Ker}(1_M - \beta\alpha) \subseteq \mathrm{Ker}(\alpha - \alpha\beta\alpha)$, so $\mathrm{Ker}(\alpha) + \mathrm{Ker}(1_M - \beta\alpha) \subseteq \mathrm{Ker}(\alpha - \alpha\beta\alpha)$. Let $x \in \mathrm{Ker}(\alpha - \alpha\beta\alpha)$. Then $x \in M$ and $\alpha(x) = \alpha\beta\alpha(x)$. Since $x = \beta\alpha(x) + (1_M - \beta\alpha)(x)$ and $\beta\alpha(x) \in \mathrm{Ker}(1_M - \beta\alpha)$, $(1_M - \beta\alpha)(x) \in \mathrm{Ker}(\alpha)$, hence $(1_M - \beta\alpha)(\beta\alpha(x)) = \beta\alpha(x) - \beta\alpha\beta\alpha(x) = \beta\alpha(x) - \beta\alpha(x) = 0$, $\alpha(1_M - \beta\alpha)(x) = \alpha(x) - \alpha\beta\alpha(x) = \alpha(x) - \alpha(x) = 0$. So $x \in \mathrm{Ker}(1_M - \beta\alpha) + \mathrm{Ker}(\alpha)$. Thus, $\mathrm{Ker}(\alpha - \alpha\beta\alpha) \subseteq \mathrm{Ker}(\alpha) + \mathrm{Ker}(1_M - \beta\alpha)$. Similarly (8) holds. □

The following Lemma is continuation of Lemma 2.1 [2].

LEMMA 2.2. *Let M_R, N_R be modules and $\alpha \in [M,N]$, $\beta \in [N,M]$. The following hold:*
(1) Im$(1_N - \alpha\beta) = N$ if and only if Im$(1_M - \beta\alpha) = M$.
(2) Ker$(1_N - \alpha\beta) = 0$ if and only if Ker$(1_M - \beta\alpha) = 0$.
(3) $1_N - \alpha\beta \in U(E_N)$ if and only if $1_M - \beta\alpha \in U(E_M)$.

Proof. (1)($\Rightarrow$). Suppose that $\mathrm{Im}(1_N - \alpha\beta) = N$, then $\mathrm{Im}(\beta - \beta\alpha\beta) = \mathrm{Im}(\beta)$. By Lemma 2.1; $\mathrm{Im}(\beta) = \mathrm{Im}(\beta - \beta\alpha\beta) = \mathrm{Im}(\beta) \cap \mathrm{Im}(1_M - \beta\alpha)$, so $\mathrm{Im}(\beta) \subseteq \mathrm{Im}(1_M - \beta\alpha)$. By Lemma 2.1; $M = \mathrm{Im}(\beta) + \mathrm{Im}(1_M - \beta\alpha) = \mathrm{Im}(1_M - \beta\alpha)$. Similarly ($\Leftarrow$) holds.
(2)($\Rightarrow$). Suppose that $\mathrm{Ker}(1_N - \alpha\beta) = 0$. Let $x \in \mathrm{Ker}(1_M - \beta\alpha)$. Then $x \in M$ and $\beta\alpha(x) = x$, so $\alpha\beta\alpha(x) = \alpha(x)$ and $(1_N - \alpha\beta)(\alpha(x)) = 0$. So by assumption; $\alpha(x) \in (1_N - \alpha\beta) = 0$ and $\alpha(x) = 0$, $x \in \mathrm{Ker}(\alpha)$. Thus, $\mathrm{Ker}(1_M - \beta\alpha) \subseteq \mathrm{Ker}(\alpha)$ and by Lemma 2.1; $0 = \mathrm{Ker}(\alpha) \cap \mathrm{Ker}(1_M - \beta\alpha) = \mathrm{Ker}(1_M - \beta\alpha)$. So $\mathrm{Ker}(1_M - \beta\alpha) = 0$. Similarly ($\Leftarrow$) holds.
(3). By (1) and (2). □

Let M_R be a module and $\alpha \in E_M$. R. Ware in [7], proved that, α is regular if and only if $\mathrm{Im}(\alpha)$ and $\mathrm{Ker}(\alpha)$ are direct summands of M. The next Proposition gives information about $\alpha \in [M,N]$, when α is a regular element.

PROPOSITION 2.3. *Let M, N be modules and $\alpha \in [M,N]$. The following are equivalent:*
(1) There exists $\beta \in [M,N]$ such that $\alpha = \alpha\beta\alpha$.
(2) Im$(\alpha) \subseteq^{\oplus} N$ and Ker$(\alpha) \subseteq^{\oplus} M$.
(3) There exists $\beta \in [N,M]$ such that Im$(\alpha) \cap$ Im$(1_N - \alpha\beta) = 0$.
(4) There exists $\beta \in [N,M]$ such that Ker$(\alpha) +$ Ker$(1_M - \beta\alpha) = M$.

Proof. (1) $\Leftrightarrow$ (2). By [3, Characterization 2.2].
(1) $\Leftrightarrow$ (3). $\alpha - \alpha\beta\alpha = 0$ if and only if $\mathrm{Im}(\alpha - \alpha\beta\alpha) = 0$ if and only if $\mathrm{Im}(\alpha) \cap \mathrm{Im}(1_N - \alpha\beta) = 0$, by Lemma 2.1.
(1) $\Leftrightarrow$ (4). $\alpha - \alpha\beta\alpha = 0$ if and only if $\mathrm{Ker}(\alpha - \alpha\beta\alpha) = M$ if and only if $M = \mathrm{Ker}(\alpha) + \mathrm{Ker}(1_M - \beta\alpha)$, by Lemma 2.1. □

Let M_R, N_R be modules and $\alpha \in [M,N]$. The following Theorem describe the submodules $\alpha[N,M]$ and $[M,N]\alpha$ when $[M,N]$ is regular.

THEOREM 2.4. *Let M, N be modules and $\alpha, \beta \in [M,N]$. If $[M,N]$ is regular, then the following hold:*
(1) Im$(\alpha) \subseteq$ Im(β) if and only if $\alpha[N,M] \subseteq \beta[N,M]$.
(2) Im$(\alpha) =$ Im(β) if and only if $\alpha[N,M] = \beta[N,M]$.

(3) $\alpha[N,M] = \{\mu : \mu \in E_N;\ Im(\mu) \subseteq Im(\alpha)\}$.
(4) $Ker(\alpha) \subseteq Ker(\beta)$ *if and only if* $[N,M]\beta \subseteq [N,M]\alpha$.
(5) $Ker(\alpha) = Ker(\beta)$ *if and only if* $[N,M]\beta = [N,M]\alpha$.
(6) $[N,M]\alpha = \{\mu : \mu \in E_M;\ Ker(\alpha) \subseteq Ker(\mu)\}$.

Proof. (1)($\Rightarrow$). Suppose that $\mathrm{Im}(\alpha) \subseteq \mathrm{Im}(\beta)$. Since $[M,N]$ is regular there exists $\mu \in [N,M]$ such that $\beta = \beta\mu\beta$. For $e = \beta\mu$; $e^2 = e \in E_N$ and $\mathrm{Im}(e) = \mathrm{Im}(\beta)$, so $\mathrm{Im}(\alpha) \subseteq \mathrm{Im}(e)$. Thus, for all $x \in M$; $e(\alpha(x)) = \alpha(x)$, so $\alpha = e\alpha = \beta\mu\alpha \in \beta E_M$. Therefore, $\alpha[N,M] \subseteq \beta E_M[N,M] \subseteq \beta[N,M]$.
($\Leftarrow$). Suppose that $\alpha[N,M] \subseteq \beta[N,M]$. Since $[M,N]$ is regular; $\alpha = \alpha\lambda\alpha$ for some $\lambda \in [N,M]$. Since $\alpha\lambda \in \alpha[N,M] \subseteq \beta[N,M]$; $\alpha\lambda = \beta\delta$ for some $\delta \in [N,M]$. Thus, $\mathrm{Im}(\alpha) = \mathrm{Im}(\alpha\lambda\alpha) = \mathrm{Im}(\beta\delta\alpha) \subseteq \mathrm{Im}(\beta)$.
(2) and (3) are clear by (1).
(4)($\Rightarrow$). Suppose that $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\beta)$, then $\beta(\mathrm{Ker}(\alpha)) = 0$. Since $[M,N]$ is regular there exists $\mu \in [M,N]$ such that $\alpha = \alpha\mu\alpha$. For $e = \mu\alpha \in E_M$; $e^2 = e$ and $\mathrm{Ker}(\alpha) = \mathrm{Ker}(e)$, so $\beta(\mathrm{Ker}(\alpha)) = \beta(\mathrm{Ker}(e)) = \beta(\mathrm{Im}(1_M - e)) = \mathrm{Im}(\beta(1_M - e)) = 0$. Thus, $\beta(1_M - e) = 0$ and that $\beta = \beta e = \beta\mu\alpha \in (E_N)\alpha$. So $[N,M]\beta \subseteq [N,M](E_N)\alpha \subseteq [N,M]\alpha$.
($\Leftarrow$). Suppose that $[N,M]\beta \subseteq [N,M]\alpha$. Since $[M,N]$ is regular; $\beta = \beta\delta\beta$ for some $\delta \in [N,M]$ and $\delta\beta \in [N,M]\beta \subseteq [N,M]\alpha$. So $\delta\beta = \lambda\alpha$ for some $\lambda \in [N,M]$. Thus, $\beta = \beta\lambda\alpha$ and $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\beta)$.
(5) and (6) are clear by (4). □

The next Corollary is a special case of Theorem 2.4, for $M = N$.

Corollary 2.5. *Let M be a module with E_M is a regular ring and $\alpha, \beta \in E_M$. The following hold:*
(1) $Im(\alpha) \subseteq Im(\beta)$ *if and only if* $\alpha E_M \subseteq \beta E_M$.
(2) $Im(\alpha) = Im(\beta)$ *if and only if* $\alpha E_M = \beta E_M$.
(3) $\alpha E_M = \{\beta : \beta \in E_M;\ Im(\beta) \subseteq Im(\alpha)\}$.
(4) $Ker(\alpha) \subseteq Ker(\beta)$ *if and only if* $(E_M)\beta \subseteq (E_M)\alpha$.
(5) $Ker(\alpha) = Ker(\beta)$ *if and only if* $(E_M)\alpha = (E_M)\beta$.
(6) $(E_M)\alpha = \{\beta : \beta \in E_M;\ Ker(\alpha) \subseteq Ker(\beta)\}$.

3. Semi-injective (projective) modules.

Theorem 3.1 ([8], p.260). *For every module M_R the following are equivalent:*
(1) For every submodule N of M and every epimorphism $\alpha : M \to N$,

homomorphism $\lambda : M \to N$ *there exists* $\beta \in E_M$ *such that* $\alpha\beta = \lambda$.
(2) For every $\alpha \in E_M$*;* $\alpha E_M = Hom_R(M, Im(\alpha))$.
(3) For every $\alpha \in E_M$*;* $\alpha E_M = \{\beta : \beta \in E_M;\ Im(\beta) \subseteq Im(\alpha)\}$.

Proof. (1) $\Rightarrow$ (2). Suppose (1) holds. Let $\alpha \in E_M$ and $\lambda \in \alpha E_M$. Then $\lambda = \alpha\beta$ for some $\beta \in E_M$. So $\mathrm{Im}(\lambda) \subseteq \mathrm{Im}(\alpha)$; $\lambda \in \mathrm{Hom}_R(M, \mathrm{Im}(\alpha))$. Let $\beta \in \mathrm{Hom}_R(M, \mathrm{Im}(\alpha))$. By assumption there exists $\lambda \in E_M$ such that $\alpha\lambda = \beta$, so $\beta \in \alpha E_M$.
(2) $\Rightarrow$ (1). Let N be a submodule of M and $\alpha : M \to N$ is an epimorphism, $\lambda : M \to N$ is a homomorphism. Then $\mathrm{Im}(\lambda) \subseteq N = \mathrm{Im}(\alpha)$, so $\lambda \in \mathrm{Hom}_R(M, \mathrm{Im}(\alpha))$. By assumption $\lambda = \alpha\beta$ for some $\beta \in E_M$.
(2) $\Leftrightarrow$ (3) it is clear. □

A module M_R is called a semi-projective module [8], if it is satisfies the equivalent conditions of Theorem 3.1.

Theorem 3.2 ([8], p.261). *For every module* M_R *the following are equivalent:*
(1) For every factor module N *of* M *and every monomorphism* $\alpha : N \to M$*, homomorphism* $\lambda : N \to M$ *there exists* $\beta \in E_M$ *such that* $\beta\alpha = \lambda$.
(2) For every $\alpha \in E_M$*;* $E_M\alpha = \{\beta : \beta \in E_M;\ Ker(\alpha) \subseteq Ker(\beta)\}$.

Proof. (1) $\Rightarrow$ (2). Suppose (1) holds. Let $\alpha \in E_M$ and $\beta \in E_M\alpha$. Then $\beta = \lambda\alpha$ for some $\lambda \in E_M$, so $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\beta)$.
Let $\beta \in E_M$ such that $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\beta)$. Then the map $\alpha' : M/\mathrm{Ker}(\alpha) \to M$ is defined by $\alpha'(\overline{x}) = \alpha(x)$ for all $\overline{x} \in M/\mathrm{Ker}(\alpha)$, is monomorphism. Also, Since $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\beta)$, the map $\beta' : M/\mathrm{Ker}(\alpha) \to M$ is defined by $\beta'(\overline{x}) = \beta(x)$ for all $\overline{x} \in M/\mathrm{Ker}(\alpha)$, is homomorphism. By assumption, there exists $\lambda \in E_M$ such that $\lambda\alpha' = \beta'$. Thus, for all $x \in M$; $\lambda\alpha(x) = \lambda\alpha'(\overline{x}) = \beta'(\overline{x}) = \beta(x)$, so $\lambda\alpha = \beta$ and $\beta \in E_M\alpha$.
(2) $\Rightarrow$ (1). Let N be a factor module of M and $\alpha : N \to M$ is a monomorphism, $\beta : N \to M$ is a homomorphism. Also, Let $\pi : M \to N$ be a canonical homomorphism of a module M onto factor module N. Then $\alpha\pi, \beta\pi \in E_M$ and $\mathrm{Ker}(\alpha\pi) \subseteq \mathrm{Ker}(\beta\pi)$. By assumption $\beta\pi \in E_M(\alpha\pi)$, so $\beta\pi = \lambda(\alpha\pi)$ for some $\lambda \in E_M$. Let $y \in N$, then $y = \pi(x)$ for some $x \in M$ and $\beta(y) = \beta\pi(x) = \lambda\alpha\pi(x) = \lambda\alpha(y)$. Thus, $\beta = \lambda\alpha$. □

A module M_R is called a semi-injective module [8], if it is satisfies the equivalent conditions of Theorem 3.2.

Theorem 3.3. *For every module* M_R*. The following are equivalent:*
(1) E_M *is a regular ring.*

(2) M is a semi-projective module and Im$(\alpha) \subseteq^{\oplus} M$ for all $\alpha \in E_M$.
(3) M is a semi-injective module and Ker$(\alpha) \subseteq^{\oplus} M$ for all $\alpha \in E_M$.

Proof. (1) $\Rightarrow$ (2). Suppose that E_M is regular. Then $\mathrm{Im}(\alpha) \subseteq^{\oplus} M$ for all $\alpha \in E_M$. On the other hand, by Corollary 2.5(3) and Theorem 3.1, implies that M is semi-projective.
(2) $\Rightarrow$ (1). Let $\alpha \in E_M$, by assumption $\mathrm{Im}(\alpha) \subseteq^{\oplus} M$. Let $\pi : M \to \mathrm{Im}(\alpha)$ the projection. Then $\mathrm{Im}(\alpha) = \mathrm{Im}(\pi)$, by Theorem 4.1, $\pi \in \alpha E_M$, so $\pi = \alpha\beta$ for some $\beta \in E_M$. On the other hand, for every $x \in M$; $\alpha(x) \in \mathrm{Im}(\alpha)$, so $\pi(\alpha(x)) = \alpha(x)$. Thus, $\pi\alpha = \alpha$ and that $\alpha\beta\alpha = \alpha$. So E_M is regular.
(1) $\Rightarrow$ (3). Suppose that E_M is regular. Then $\ker(\alpha) \subseteq^{\oplus} M$ for all $\alpha \in E_M$. On the other hand, by Corollary 2.5(6) and Theorem 3.2, implies that M is semi-injective.
(3) $\Rightarrow$ (1). Let $\alpha \in E_M$, by assumption $\mathrm{Ker}(\alpha) \subseteq^{\oplus} M$. Then $M = \mathrm{Ker}(\alpha) \oplus K$ for some submodule K of M. Let $\pi : M \to K$ be the projection. Then $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\pi)$ and $\alpha(\mathrm{Ker}(\pi)) = \alpha(\mathrm{Im}(1-\pi)) = 0$, so $\alpha = \alpha\pi$. Since M is semi-injective and $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\pi)$; $\pi \in E_M\alpha$. Thus, $\pi = \beta\alpha$ for some $\beta \in E_M$, so $\alpha = \alpha\beta\alpha$. □

A module M_R is called semi-simple [1], if every submodule of M is a direct summand of M. A ring R is semi-simple if R_R is semi-simple.

Corollary 3.4. *For any ring R the following are equivalent:*
(1) A ring R is semi-simple.
(2) M is semi-simple for every $M \in \mathrm{mod}-R$.
(3) E_M is a regular ring for every $M \in \mathrm{mod}-R$.
(4) E_F is a regular ring for every free module $F \in \mathrm{mod}-R$.
(5) For every $M \in \mathrm{mod}-R$, M is semi-injective and Ker$(\alpha) \subseteq^{\oplus} M$ for all $\alpha \in E_M$.
(6) For every $M \in \mathrm{mod}-R$, M is semi-projective and Im$(\alpha) \subseteq^{\oplus} M$ for all $\alpha \in E_M$.

Proof. (1) $\Leftrightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are clear. (4) $\Rightarrow$ (1) by [6, Theorem 1]. (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) by Theorem 3.3. □

4. Semipotency of $[M, N]$.

An element a of a ring R is called *partially invertible* or *pi* for short, if a is a divisor of an idempotent [2]. The next Proposition gives information about $\alpha \in E_M$, when α is a divisor of an idempotent.

Proposition 4.1. *Let M, N be modules and $\alpha \in [M, N]$. The following are equivalent:*
(1) There exists $\beta \in [N, M]$ such that $\beta = \beta\alpha\beta$.
(2) There exists $\delta \in [N, M]$ such that Im$(\alpha\delta)$, Ker$(\alpha\delta)$ are direct summands of N.
(3) There exists $\gamma \in [N, M]$ such that Im$(\gamma\alpha)$, Ker$(\gamma\alpha)$ are direct summands of M.
(4) There exists $\beta \in [N, M]$ such that Im$(\beta) \cap$ Im$(1_M - \beta\alpha) = 0$.
(5) There exists $\beta \in [N, M]$ such that Ker$(\beta) +$ Ker$(1_N - \alpha\beta) = N$.

Proof. (1) $\Rightarrow$ (2). If $\beta = \beta\alpha\beta$ for some $\beta \in [N, M]$; $(\alpha\beta)^2 = \alpha\beta \in E_N$, so $\mathrm{Im}(\alpha\beta)$ and $\mathrm{Ker}(\alpha\beta)$ are direct summand of N.
(2) $\Rightarrow$ (1). If $\mathrm{Im}(\alpha\delta)$ and $\mathrm{Ker}(\alpha\delta)$ are direct summand of N for some $\delta \in [N, M]$; by Lemma 2.3 there exists $\mu \in E_N$ such that $(\alpha\delta)\mu(\alpha\delta) = \alpha\delta$. Then for $\beta = \delta\mu\alpha\delta\mu \in [N, M]$; $\beta\alpha\beta = \beta$. Similarly (1) $\Leftrightarrow$ (3) holds.
(1) $\Rightarrow$ (4). Suppose that $\beta\alpha\beta = \beta$ for some $\beta \in [N, M]$. Then $\mathrm{Im}(\beta - \beta\alpha\beta) = 0$, by Lemma 2.1 $\mathrm{Im}(\alpha) \cap \mathrm{Im}(1_M - \beta\alpha) = 0$.
(4) $\Rightarrow$ (1). If $\mathrm{Im}(\alpha) \cap \mathrm{Im}(1_M - \beta\alpha) = 0$ for some $\beta \in [N, M]$, then by Lemma 2.1 $\mathrm{Im}(\beta - \beta\alpha\beta) = 0$, so $\beta\alpha\beta = \beta$.
(1) $\Leftrightarrow$ (5). For some $\beta \in [N, M]$; $\beta\alpha\beta = \beta$ if and only if $\mathrm{Ker}(\beta - \beta\alpha\beta) = M$ if and only if $\mathrm{Ker}(\alpha) + \mathrm{Ker}(1_N - \alpha\beta) = N$ by Lemma 2.1. □

Let M, N be modules. Recall that $[M, N]$ is *semipotent* by Zhou [9, Theorem 2.2], if $\mathrm{Tot}[M, N] = J[M, N]$.

Corollary 4.2. *Let M_R, N_R be modules. The following are equivalent:*
(1) $[M, N]$ is semipotent.
(2) For every $\alpha \in [M, N] \setminus J[M, N]$ there exists $\beta \in [N, M]$ such that $\beta = \beta\alpha\beta$.
(3) For every $\alpha \in [M, N] \setminus J[M, N]$ there exists $\beta \in [N, M]$ such that Im$(\alpha\beta)$, Ker$(\alpha\beta)$ are direct summands of N.
(4) For every $\alpha \in [M, N] \setminus J[M, N]$ there exists $\beta \in [N, M]$ such that Im$(\beta\alpha)$, Ker$(\beta\alpha)$ are direct summands of M.
(5) For every $\alpha \in [M, N] \setminus J[M, N]$ there exists $\beta \in [N, M]$ such that Im$(\beta) \cap$ Im$(1_M - \beta\alpha) = 0$.
(6) For every $\alpha \in [M, N] \setminus J[M, N]$ there exists $\beta \in [N, M]$ such that Ker$(\beta) +$ Ker$(1_N - \alpha\beta) = N$.

Proof. By Proposition 4.1. □

Let M, N be modules. Write:

$$\nabla_1[M,N] = \{\alpha : \alpha \in [M,N];\ \operatorname{Im}(1_N - \alpha\beta) = N \text{ for all } \ \beta \in [N,M]\}.$$

$$\nabla_2[M,N] = \{\alpha : \alpha \in [M,N];\ \operatorname{Im}(1_M - \beta\alpha) = M \text{ for all } \ \beta \in [N,M]\}.$$

It is clear that $\nabla_1[M,N]$ and $\nabla_2[M,N]$ are non empty subsets in $[M,N]$, $(0 \in \nabla_1[M,N],\ 0 \in \nabla_2[M,N])$. By using Lemma 2.2(1), it is easy to see that $\nabla_1[M,N] = \nabla_2[M,N]$. Therefore we use the notation:

$$\widehat{\nabla}[M,N] = \{\alpha : \alpha \in [M,N];\ \operatorname{Im}(1_N - \alpha\beta) = N \text{ for all } \ \beta \in [N,M]\}.$$

$$= \{\alpha : \alpha \in [M,N];\ \operatorname{Im}(1_M - \beta\alpha) = M \text{ for all } \ \beta \in [N,M]\}.$$

$\widehat{\nabla}[M,N]$ is a semi-ideal in mod$-R$, which means hat it is closed under arbitrary multiplication from either side, by the following Lemma.

LEMMA 4.3. *For arbitrary $M, N, X, Y \in$ mod$-R$, the following hold:*
(1) $\widehat{\nabla}[M,N][X,M] \subseteq \widehat{\nabla}[X,N]$.
(2) $[N,Y]\widehat{\nabla}[M,N] \subseteq \widehat{\nabla}[M,Y]$.
(3) $[N,Y]\widehat{\nabla}[M,N][X,M] \subseteq \widehat{\nabla}[X,Y]$.

Proof. (1). Let $\alpha \in \widehat{\nabla}[M,N]$ and $\lambda \in [X,M]$. Then $\alpha\lambda \in [X,N]$. For all $\beta \in [N,X]$; $\operatorname{Im}(1_N - (\alpha\lambda)\beta) = \operatorname{Im}(1_N - \alpha(\lambda\beta)) = N$, hence $\lambda\beta \in [N,M]$. Thus, $\alpha\lambda \in \widehat{\nabla}[X,N]$. (2) is analogous.
(3) by (1) and (2). □

Let M, N be modules. Write

$$\triangle_1[M,N] = \{\alpha : \alpha \in [M,N];\ \operatorname{Ker}(1_N - \alpha\beta) = 0 \text{ for all } \ \beta \in [N,M]\}.$$

$$\triangle_2[M,N] = \{\alpha : \alpha \in [M,N];\ \operatorname{Ker}(1_M - \beta\alpha) = 0 \text{ for all } \ \beta \in [N,M]\}.$$

It is clear that $\triangle_1[M,N]$ and $\triangle_2[M,N]$ are non empty subsets in $[M,N]$, $(0 \in \triangle_1[M,N],\ 0 \in \triangle_2[M,N])$. By using Lemma 2.2(2), it is easy to see that $\triangle_1[M,N] = \triangle_2[M,N]$. Therefore we use the notation:

$$\widehat{\triangle}[M,N] = \{\alpha : \alpha \in [M,N];\ \operatorname{Ker}(1_N - \alpha\beta) = 0 \text{ for all } \ \beta \in [N,M]\}.$$

$$= \{\alpha : \alpha \in [M,N];\ \operatorname{Ker}(1_M - \beta\alpha) = 0 \text{ for all } \ \beta \in [N,M]\}.$$

$\widehat{\triangle}[M,N]$ is a semi-ideal in mod$-R$, which means hat it is closed under arbitrary multiplication from either side, by the following Lemma.

LEMMA 4.4. *For arbitrary $M, N, X, Y \in mod-R$, the following hold:*
(1) $\widehat{\triangle}[M, N][X, M] \subseteq \widehat{\triangle}[X, N]$.
(2) $[N, Y]\widehat{\triangle}[M, N] \subseteq \widehat{\triangle}[M, Y]$.
(3) $[N, Y]\widehat{\triangle}[M, N][X, M] \subseteq \widehat{\triangle}[X, Y]$.

Proof. (1). Let $\alpha \in \widehat{\triangle}[M, N]$ and $\lambda \in [X, M]$. Then $\alpha\lambda \in [X, N]$. For all $\beta \in [N, X]$; $\mathrm{Ker}(1_N - (\alpha\lambda)\beta) = \mathrm{Ker}(1_N - \alpha(\lambda\beta)) = 0$, hence $\lambda\beta \in [N, M]$. Thus, $\alpha\lambda \in \widehat{\triangle}[X, N]$. (2) is analogous.
(3) by (1) and (2). □

COROLLARY 4.5. *Let M, N be modules. The following hold:*
(1) $\nabla[M, N] \subseteq \widehat{\nabla}[M, N]$.
(2) $\triangle[M, N] \subseteq \widehat{\triangle}[M, N]$.

Proof. It is clear by Lemma 2.1. □

LEMMA 4.6. *Let M, N be modules. The following hold:*
(1) $J[M, N] \subseteq \widehat{\nabla}[M, N] \cap \widehat{\triangle}[M, N]$.
(2) $\widehat{\nabla}[M, N] \cup \widehat{\triangle}[M, N] \subseteq Tot[M, N]$.
(3) $J[M, N] \subseteq Tot[M, N]$.

Proof. (1). Let $\alpha \in J[M, N]$. Then for all $\beta \in [N, M]$; $1_N - \alpha\beta \in U(E_N)$ and $1_M - \beta\alpha \in U(E_M)$, so there exists $g \in E_N$, $\lambda \in E_M$ such that $(1_N - \alpha\beta)g = 1_N$ and $\lambda(1_M - \beta\alpha) = 1_M$. Therefore $\mathrm{Im}(1_N - \alpha\beta) = N$ and $\mathrm{Ker}(1_M - \beta\alpha) = 0$. Thus $\alpha \in \widehat{\nabla}[M, N]$ and $\alpha \in \widehat{\triangle}[M, N]$. So $J[M, N] \subseteq \widehat{\nabla}[M, N] \cap \widehat{\triangle}[M, N]$.
(2). Let $\alpha \in \widehat{\nabla}[M, N]$. Then for all $\beta \in [N, M]$; $\mathrm{Im}(1_N - \alpha\beta) = N$. If $\alpha \notin \mathrm{Tot}[M, N]$, there exists $\delta \in [N, M]$ such that $0 \neq (\alpha\delta)^2 = \alpha\delta \in E_N$. So $\mathrm{Ker}(\alpha\delta) = \mathrm{Im}(1_N - \alpha\delta) = N$, thus $\alpha\delta = 0$ a contradiction. So $\alpha \in \mathrm{Tot}[M, N]$.
Let $\alpha \in \widehat{\triangle}[M, N]$. Then for all $\beta \in [N, M]$; $\mathrm{Ker}(1_N - \alpha\beta) = 0$. If $\alpha \notin \mathrm{Tot}[M, N]$, there exists $\delta \in [N, M]$ such that $0 \neq (\alpha\delta)^2 = \alpha\delta \in E_N$. So $\mathrm{Im}(\alpha\delta) = \mathrm{Ker}(1_N - \alpha\delta) = 0$, thus $\alpha\delta = 0$ a contradiction. So $\alpha \in \mathrm{Tot}[M, N]$.
(3) by (1) and (2). □

The following Proposition describe the Jacobson radical of $[M, N]$ when $[M, N]$ is semipotent.

PROPOSITION 4.7. *Let M, N be modules with $[M, N]$ is semipotent. Then the following hold:*

(1) $J[M,N] = \widehat{\nabla}[M,N]$.
(2) $J[M,N] = \widehat{\triangle}[M,N]$.
(3) $\widehat{\nabla}[M,N] = \widehat{\triangle}[M,N]$.

Proof. Suppose that $[M,N]$ is semipotent.
(1). By Lemma 4.6, we have $J[M,N] \subseteq \widehat{\nabla}[M,N]$. Let $\alpha \in \widehat{\nabla}[M,N]$. Then for all $\beta \in [N,M]$; $\mathrm{Im}(1_N - \alpha\beta) = N$. Suppose that $\alpha \notin J[M,N]$, then there exists $\delta \in [N,M]$ such that $\delta\alpha\delta = \delta \neq 0$, so $0 \neq (\alpha\delta)^2 = \alpha\delta \in E_N$ and $\mathrm{Ker}(\alpha\delta) = \mathrm{Im}(1_N - \alpha\delta) = N$, so $\alpha\delta = 0$ a contradiction. Thus, $\alpha \in J[M,N]$.
(2). By Lemma 4.6, we have $J[M,N] \subseteq \widehat{\triangle}[M,N]$. Let $\alpha \in \widehat{\triangle}[M,N]$. Then for all $\beta \in [N,M]$; $\mathrm{Ker}(1_N - \alpha\beta) = 0$. Suppose that $\alpha \notin J[M,N]$, then there exists $\delta \in [N,M]$ such that $\delta\alpha\delta = \delta \neq 0$, so $0 \neq (\alpha\delta)^2 = \alpha\delta \in E_N$ and $\mathrm{Im}(\alpha\delta) = \mathrm{Ker}(1_N - \alpha\delta) = 0$, so $\alpha\delta = 0$ a contradiction. Thus, $\alpha \in J[M,N]$.
(3) by (1) and (2). □

Corollary 4.8. *Let M, N be modules with $[M,N]$ is semipotent and $\alpha \in [N,M]$. Then the following hold:*
(1) $\alpha \in J[N,M]$ if and only if $\mathrm{Im}(1_N - \alpha\beta) = N$ for all $\beta \in [N,M]$ if and only if $\mathrm{Ker}(1_N - \alpha\beta) = 0$ for all $\beta \in [N,M]$.
(2) $\alpha \in J[N,M]$ if and only if $\mathrm{Im}(1_M - \beta\alpha) = M$ for all $\beta \in [N,M]$ if and only if $\mathrm{Ker}(1_M - \beta\alpha) = 0$ for all $\beta \in [N,M]$.

Proof. By Proposition 4.7. □

Theorem 4.9. *(1) For a module N the following conditions are equivalent:*
(i) $\widehat{\nabla}[M,N] \subseteq J[M,N]$ for all $M \in \mathrm{mod}-R$.
(ii) $\widehat{\nabla}(E_N) \subseteq J(E_N)$.
(iii) For every $\alpha \in E_N$ with $1-\alpha \in \widehat{\nabla}(E_N)$ is one-to-one.
(2) For a module M the following conditions are equivalent:
(i) $\widehat{\triangle}[M,N] \subseteq J[M,N]$ for all $N \in \mathrm{mod}-R$.
(ii) $\widehat{\triangle}(E_M) \subseteq J(E_M)$.
(iii) For every $\alpha \in E_M$ with $1-\alpha \in \widehat{\triangle}(E_M)$ is onto.

Proof. $(1)(i) \Rightarrow (ii)$. It is clear.
$(ii) \Rightarrow (iii)$. Let $\alpha \in E_N$ with $1-\alpha \in \widehat{\nabla}(E_N)$. Then $\mathrm{Im}(1-(1-\alpha)\beta) = N$ for all $\beta \in E_N$. On the other hand, $1-\alpha \in J(E_N)$ by assumption. So, $\alpha = 1-(1-\alpha) \in U(E_N)$ and α is one-to-one.

$(iii) \Rightarrow (i)$. Let $\alpha \in \widehat{\nabla}[M,N]$. Then $\mathrm{Im}(1_N - \alpha\beta) = N$ for all $\beta \in [N,M]$. Thus, for every $\lambda \in E_N$; $\mathrm{Im}(1_N - (\alpha\beta)\lambda) = \mathrm{Im}(1_N - \alpha(\beta\lambda)) = N$, hence $\alpha \in \widehat{\nabla}[M,N]$ and $\beta\lambda \in [N,M]$. So $\alpha\beta = (1_N - (1_N - \alpha\beta)) \in \widehat{\nabla}(E_N)$, by assumption $1_N - \alpha\beta$ is one to one. Thus, $1_N - \alpha\beta \in U(E_N)$, $\alpha\beta \in J(E_N)$ and $\alpha \in J[M,N]$.
(2)$(i) \Rightarrow (ii)$. It is clear.
$(ii) \Rightarrow (iii)$. Let $\alpha \in E_M$ with $1 - \alpha \in \widehat{\triangle}(E_M)$. Then $\mathrm{Ker}(1_M - \beta(1_M - \alpha)) = 0$ for all $\beta \in E_M$. On the other hand, $1 - \alpha \in J(E_M)$ by assumption. So, $\alpha = 1 - (1 - \alpha) \in U(E_M)$ and α is one-to-one.
$(iii) \Rightarrow (i)$. Let $\alpha \in \widehat{\triangle}[M,N]$. Then $\mathrm{Ker}(1_M - \beta\alpha) = 0$ for all $\beta \in [N,M]$. Thus, for every $\lambda \in E_M$; $\mathrm{Ker}(1_M - \lambda(\beta\alpha)) = \mathrm{Ker}(1_M - (\lambda\beta)\alpha) = 0$, hence $\alpha \in \widehat{\triangle}[M,N]$ and $\lambda\beta \in [N,M]$. So $\beta\alpha = (1_M - (1_M - \beta\alpha)) \in \widehat{\triangle}(E_M)$ is onto by assumption. Thus, $1_M - \beta\alpha \in U(E_M)$, $\beta\alpha \in J(E_M)$ and $\alpha \in J[M,N]$. □

Theorem 4.10. *Let M, N be modules. The following conditions are equivalent:*
(1) $\mathrm{Tot}[M,N] = \nabla[M,N]$.
(2) For all $\alpha \in [M,N]$ with $\mathrm{Im}(\alpha)$ is not small in N there exists $\beta \in [N,M]$ such that $\mathrm{Im}(\beta) \cap \mathrm{Im}(1_M - \beta\alpha) = 0$.
(3) For all $\alpha \in [M,N]$ with $\mathrm{Im}(\alpha)$ is not small in N there exists $\beta \in [N,M]$ such that $\mathrm{Ker}(\beta) + \mathrm{Ker}(1_N - \alpha\beta) = N$.

Proof. (1) $\Rightarrow$ (2). Suppose (1) holds. Let $\alpha \in [M,N]$ with $\mathrm{Im}(\alpha)$ is not small in N. Then $\alpha \notin \nabla[M,N]$, by assumption there exists $\lambda \in [M,N]$ such that $0 \neq (\lambda\alpha)^2 = \lambda\alpha \in E_M$. For $\beta = \lambda\alpha\lambda$; $\beta\alpha\beta = \beta$. By Lemma 2.1; $0 = \mathrm{Im}(\beta - \beta\alpha\beta) = \mathrm{Im}(\beta) \cap \mathrm{Im}(1_M - \beta\alpha)$.
(2) $\Rightarrow$ (3). Suppose (2) holds. Let $\alpha \in [M,N]$ with $\mathrm{Im}(\alpha)$ is not small in N. By assumption there exists $\beta \in [M,N]$ such that $\mathrm{Im}(\beta) \cap \mathrm{Im}(1_M - \beta\alpha) = 0$. By Lemma 2.1(4); $\beta - \beta\alpha\beta = 0$, so $N = \mathrm{Ker}(\beta - \beta\alpha\beta) = \mathrm{Ker}(\beta) + \mathrm{Ker}(1_N - \alpha\beta)$ by Lemma 2.1(8), giving (3).
(3) $\Rightarrow$ (1). It is clear that $\nabla[M,N] \subseteq \mathrm{Tot}[M,N]$. Let $\alpha \in \mathrm{Tot}[M,N]$. Suppose that $\alpha \notin \nabla[M,N]$, then $\mathrm{Im}(\alpha)$ is not small in N, by assumption there exists $\beta \in [N,M]$ such that $N = \mathrm{Ker}(\beta) + \mathrm{Ker}(1_N - \alpha\beta)$. By Lemma 4.1; $\beta = \beta\alpha\beta$, so $0 \neq (\alpha\beta)^2 = \alpha\beta \in E_N$ a contradiction, hence $\alpha \in \mathrm{Tot}[M,N]$. Thus, $\alpha \in \nabla[M,N]$. □

Theorem 4.11. *Let M, N be modules. The following are equivalent:*
(1) $\mathrm{Tot}[M,N] = \triangle[M,N]$.
(2) For all $\alpha \in [M,N]$ with $\mathrm{Ker}(\alpha)$ is not large in M there exists $\beta \in$

$[N, M]$ such that $Im(\beta) \cap Im(1_M - \beta\alpha) = 0$.
(3) For all $\alpha \in [M, N]$ with $Ker(\alpha)$ is not large in M there exists $\beta \in [N, M]$ such that $Ker(\beta) + Ker(1_N - \alpha\beta) = N$.

Proof. (1) $\Rightarrow$ (2). Suppose (1) holds. Let $\alpha \in [M, N]$ with $\mathrm{Ker}(\alpha)$ is not large in M. Then $\alpha \notin \triangle[M, N]$, by assumption there exists $\lambda \in [M, N]$ such that $0 \neq (\lambda\alpha)^2 = \lambda\alpha \in E_M$. For $\beta = \lambda\alpha\lambda$; $\beta\alpha\beta = \beta$. By Lemma 4.1; $\mathrm{Im}(\beta) \cap \mathrm{Im}(1_M - \beta\alpha) = 0$.
(2) $\Rightarrow$ (3). Suppose (2) holds. Let $\alpha \in [M, N]$ with $\mathrm{Ker}(\alpha)$ is not large in M. By assumption there exists $\beta \in [M, N]$ such that $\mathrm{Im}(\beta) \cap \mathrm{Im}(1_M - \beta\alpha) = 0$. By Lemma 2.1; $\mathrm{Im}(\beta - \beta\alpha\beta) = 0$, so $\mathrm{Ker}(\beta - \beta\alpha\beta) = N = \mathrm{Ker}(\beta) + \mathrm{Ker}(1_N - \alpha\beta)$.
(3) $\Rightarrow$ (1). It is clear that $\triangle[M, N] \subseteq \mathrm{Tot}[M, N]$. Let $\alpha \in \mathrm{Tot}[M, N]$. Suppose that $\alpha \notin \triangle[M, N]$, then $\mathrm{Ker}(\alpha)$ is not large in M, by assumption there exists $\beta \in [N, M]$ such that $N = \mathrm{Ker}(\beta) + \mathrm{Ker}(1_N - \alpha\beta)$. By Lemma 4.1; $\mathrm{Ker}(\beta - \beta\alpha\beta) = N$, so $\beta = \beta\alpha\beta$ and $0 \neq (\alpha\beta)^2 = \alpha\beta \in E_N$ a contradiction, hence $\alpha \in \mathrm{Tot}[M, N]$. Thus, $\alpha \in \triangle[M, N]$. $\square$

THEOREM 4.12. *Let N be a module. The following are equivalent:*
(1) $Tot(E_N) = \widehat{\nabla}(E_N)$.
(2) $Tot[M, N] = \widehat{\nabla}[M, N]$ for all $M \in mod - R$.
(3) $Tot[N, W] = \widehat{\nabla}[N, W]$ for all $W \in mod - R$.

Proof. (1) $\Rightarrow$ (2). It is clear that $\widehat{\nabla}[M, N] \subseteq \mathrm{Tot}[M, N]$ by Lemma 4.6.
Let $\alpha \in \mathrm{Tot}[M, N]$. If $\alpha \notin \widehat{\nabla}[M, N]$, there exists $\beta \in [N, M]$ such that $\mathrm{Im}(1_N - \alpha\beta) \neq N$, so $\alpha\beta \notin \widehat{\nabla}(E_N) = \mathrm{Tot}(E_N)$. Thus, there exists $\lambda \in E_N$ such that $0 \neq ((\alpha\beta)\lambda)^2 = (\alpha\beta)\lambda \in E_N$, so $0 \neq (\alpha(\beta\lambda))^2 = \alpha(\beta\lambda)$ and $\beta\lambda \in [N, M]$ a contradiction, hence $\alpha \in \mathrm{Tot}[M, N]$. Thus $\alpha \in \widehat{\nabla}[M, N]$.
(2) $\Rightarrow$ (1). It is clear. Similarly equivalent (1) $\Leftrightarrow$ (3) holds. $\square$

THEOREM 4.13. *Let N be a module. The following are equivalent:*
(1) $Tot(E_N) = \widehat{\triangle}(E_N)$.
(2) $Tot[M, N] = \widehat{\triangle}[M, N]$ for all $M \in mod - R$.
(3) $Tot[N, W] = \widehat{\triangle}[N, W]$ for all $W \in mod - R$.

Proof. (1) $\Rightarrow$ (2). It is clear that $\widehat{\triangle}[M, N] \subseteq \mathrm{Tot}[M, N]$ by Lemma 4.6.
Let $\alpha \in \mathrm{Tot}[M, N]$. Suppose that $\alpha \notin \widehat{\triangle}[M, N]$. Then there exists

$\beta \in [N, M]$ such that $\mathrm{Ker}(1_N - \alpha\beta) \neq 0$. So $\alpha\beta \notin \widehat{\triangle}(E_M) = \mathrm{Tot}(E_N)$. Thus, there exists $\delta \in E_N$ such that $0 \neq ((\alpha\beta)\delta)^2 = (\alpha\beta)\delta \in E_N$. So $0 \neq (\alpha(\beta\delta))^2 = \alpha(\beta\delta) \in E_N$ and $\beta\delta \in [N, M]$ a contradiction, hence $\alpha \in \mathrm{Tot}[M, N]$.
(2) $\Rightarrow$ (1). It is clear. Similarly equivalent (1) $\Leftrightarrow$ (3) holds. $\square$

Write
$\widehat{\triangle}\Phi(R) = \{M : M \in \mathrm{mod} - R;\ \mathrm{Tot}[M, N] = \widehat{\triangle}[M, N]$ for all $N \in \mathrm{mod} - R\}$.
$\widehat{\triangle}\Gamma(R) = \{N : N \in \mathrm{mod} - R;\ \mathrm{Tot}[M, N] = \widehat{\triangle}[M, N]$ for all $M \in \mathrm{mod} - R\}$.
$\widehat{\triangle}(R) = \{M : M \in \mathrm{mod} - R;\ \mathrm{Tot}(E_M) = \widehat{\triangle}(E_M)\}$.
$\widehat{\nabla}\Phi(R) = \{M : M \in \mathrm{mod} - R;\ \mathrm{Tot}[M, N] = \widehat{\nabla}[M, N]$ for all $N \in \mathrm{mod} - R\}$.
$\widehat{\nabla}\Gamma(R) = \{N : N \in \mathrm{mod} - R;\ \mathrm{Tot}[M, N] = \widehat{\nabla}[M, N]$ for all $M \in \mathrm{mod} - R\}$.
$\widehat{\nabla}(R) = \{M : M \in \mathrm{mod} - R;\ \mathrm{Tot}(E_M) = \widehat{\nabla}(E_M)\}$.

Theorem 4.14. *For any ring R the following hold:*
(1) $\widehat{\triangle}\Phi(R) = \widehat{\triangle}\Gamma(R) = \widehat{\triangle}(R)$.
(2) $\widehat{\nabla}\Phi(R) = \widehat{\nabla}\Gamma(R) = \widehat{\nabla}(R)$.

Proof. (1). Let $M \in \widehat{\triangle}\Phi(R)$. Then $\mathrm{Tot}[M, N] = \widehat{\triangle}[M, N]$ for all $N \in \mathrm{mod} - R$. By Lemma 4.12; $\mathrm{Tot}(E_M) = \widehat{\triangle}(E_M)$ and $\mathrm{Tot}[W, M] = \widehat{\triangle}[W, M]$ for all $W \in \mathrm{mod} - R$, so $M \in \widehat{\triangle}\Gamma(R)$. Thus, $\widehat{\triangle}\Phi(R) \subseteq \widehat{\triangle}\Gamma(R)$.
Let $N \in \widehat{\triangle}\Gamma(R)$. Then $\mathrm{Tot}[M, N] = \widehat{\triangle}[M, N]$ for all $M \in \mathrm{mod} - R$. By Lemma 3.12(2); $\mathrm{Tot}(E_N) = \widehat{\triangle}(E_N)$ and $\mathrm{Tot}[N, V] = \widehat{\triangle}[N, V]$ for all $V \in \mathrm{mod} - R$, so $N \in \widehat{\triangle}\Phi(R)$. Thus, $\widehat{\triangle}\Gamma(R) \subseteq \widehat{\triangle}\Phi(R)$.
Let $M \in \widehat{\triangle}\Phi(R)$, then $\mathrm{Tot}[M, N] = \widehat{\triangle}[M, N]$ for all $M \in \mathrm{mod} - R$, so $\mathrm{Tot}(E_M) = \widehat{\triangle}(E_M)$ and that $\widehat{\triangle}\Phi(R) \subseteq \widehat{\triangle}(R)$.
If $M \in \widehat{\triangle}(R)$, then $\mathrm{Tot}(E_M) = \widehat{\triangle}(E_M)$ by Lemma 4.13; $\mathrm{Tot}[M, N] = \widehat{\triangle}[M, N]$ for all $N \in \mathrm{mod} - R$, so $M \in \widehat{\Phi}(R)$ and that $\widehat{\triangle}(R) \subseteq \widehat{\triangle}\Phi(R)$.
Similarly (2) holds. $\square$

F. Kasch in [3] studied conditions on modules Q and P, which imply that $\mathrm{Tot}[Q, N] = \triangle[Q, N] = J[Q, N]$ and $\mathrm{Tot}[M, P] = \nabla[M, P] = J[M, P]$ for all $N, M \in \mathrm{mod} - R$. He showed that these equalities hold if Q is injective, respectively P is semiperfect and projective.

A module Q is called *locally injective* [3] if, for every submodule $A \subseteq Q$, which is not large in Q, there exists an injective submodule $0 \neq B \subseteq Q$, with $A \cap B = 0$.

A module P is called *locally projective* [3] if, for every submodule $B \subseteq P$, which is not small in P, there exists a projective direct summand $0 \neq A \subseteq^{\oplus} P$, with $A \subseteq B$.

It was proved by Kasch [3], that $\mathrm{Tot}[Q,N] = \triangle[Q,N]$ for all $N \in \mathrm{mod}-R$ if and only if Q is a locally injective module and that $\mathrm{Tot}[M,P] = \nabla[M,P]$ for all $M \in \mathrm{mod}-R$ if and only if P is a locally projective module.

The following questions were raised by Kasch in [3].

(1) If Q is locally injective, then it is true that $\mathrm{Tot}[Q,N] = \triangle[Q,N] = J[Q,N]$ for all $N \in \mathrm{mod}-R$?.

(2) If P is locally projective, then it is true that $\mathrm{Tot}[M,P] = \nabla[M,P] = J[M,P]$ for all $M \in \mathrm{mod}-R$?.

Zhou in [9], proved that the answer to question (1) is "Yes" if a ring R is left Noetherian. But in general, the answer to the question is "No" by [9, Example 4.2].

During our study of answer to questions it is obtained the following results:

Corollary 4.15. *The following hold:*
(1) If Q is a locally injective module, then $Tot[Q,N] = \triangle[Q,N] = \widehat{\triangle}[Q,N]$ for all $N \in mod-R$.
(2) If P is a locally projective module, then $Tot[M,P] = \nabla[M,P] = \widehat{\nabla}[M,P]$ for all $M \in mod-R$.

Proof. (1). If Q is locally injective, then $\triangle[Q,N] \subseteq \widehat{\triangle}[Q,N] \subseteq \mathrm{Tot}[Q,N] = \triangle[Q,N]$ for all $N \in \mathrm{mod}-R$ by Lemma 4.6.
(2). If P is locally projective, then $\nabla[M,P] \subseteq \widehat{\nabla}[M,P] \subseteq \mathrm{Tot}[M,P] = \nabla[M,P]$ by Lemma 4.6. □

Corollary 4.16. *The following hold:*
(1) If Q is a locally injective module and $\alpha \in E_Q$, then $Ker(\alpha) \leq_e Q$ if and only if $Ker(1-\alpha\beta) = 0$ for all $\beta \in E_Q$ if and only if $Ker(1-\beta\alpha) = 0$ for all $\beta \in E_Q$.
(2) If P is a locally projective module and $\alpha \in E_P$, then $Im(\alpha) \ll P$ if and only if $Im(1-\alpha\beta) = P$ for all $\beta \in E_P$ if and only if $Im(1-\beta\alpha) = P$ for all $\beta \in E_P$.

Proof. By corollary 4.15. □

PROPOSITION 4.17. *(1) Let N be a semi-projective module. The following hold:*
(i) $J(E_N) = \widehat{\nabla}(E_N)$.
(ii) $J[M,N] = \widehat{\nabla}[M,N]$ *for all* $M \in mod-R$.
(2) Let N be a semi-injective module. The following hold:
(i) $J(E_N) = \widehat{\triangle}(E_N)$.
(ii) $J[N,W] = \widehat{\triangle}[N,W]$ *for all* $W \in mod-R$.

Proof. (1)(i). It is clear by Lemma 4.6, that $J(E_N) \subseteq \widehat{\nabla}(E_N)$. Let $\alpha \in \widehat{\nabla}(E_N)$. Then $\mathrm{Im}(1-\alpha\beta) = N$ for all $\beta \in E_N$. Since N semi-projective; $(1-\alpha\beta)g = 1$ for some $g \in E_N$, so $\alpha \in J(E_N)$.
(ii) it is clear.
(2)(i). It is clear by Lemma 4.6, that $J(E_N) \subseteq \widehat{\triangle}(E_N)$. Let $\alpha \in \widehat{\triangle}(E_N)$. Then $\mathrm{Ker}(1-\alpha\beta) = 0$ for all $\beta \in E_N$. Since N semi-injective and $\mathrm{Ker}(1-\alpha\beta) \subseteq \mathrm{Ker}(\beta)$; $\beta = \lambda(1-\alpha\beta)$ for some $\lambda \in E_N$. Also, since $1 = \alpha\beta + (1-\alpha\beta) = \alpha\lambda(1-\alpha\beta) + (1-\alpha\beta) = (1+\alpha\lambda)(1-\alpha\beta)$; so $1-\alpha\beta \in U(E_N)$ and that $\alpha \in J(E_N)$.
(ii) it is clear. □

COROLLARY 4.18. *The following hold:*
(1) If N be a semi-projective module, then $\nabla(E_N) \subseteq J(E_N)$.
(2) If N be a semi-injective module, then $\triangle(E_N) \subseteq J(E_N)$.

PROPOSITION 4.19. *(1) Let N be a semi-projective module. The following are equivalent:*
(i) For every $\alpha \in E_N$ *there exists* $\beta \in E_N$ *such that* $\beta\alpha\beta = \beta$.
(ii) For every $\alpha \in E_N$ *there exists* $\beta \in E_N$ *such that* $\mathrm{Im}(\alpha\beta) \subseteq^{\oplus} N$.
(2) Let N be a semi-injective module. The following are equivalent:
(i) For every $\alpha \in E_N$ *there exists* $\beta \in E_N$ *such that* $\beta\alpha\beta = \beta$.
(ii) For every $\alpha \in E_N$ *there exists* $\beta \in E_N$ *such that* $\mathrm{Ker}(\beta\alpha) \subseteq^{\oplus} N$.

Proof. (1)(i) $\Rightarrow$ (ii). It is clear by Proposition 4.1.
(ii) $\Rightarrow$ (i). Let $\alpha \in E_N$. Then $\mathrm{Im}(\alpha\beta) \subseteq^{\oplus} N$ for some $\beta \in E_N$. Let $\pi : N \to \mathrm{Im}(\alpha\beta)$ the projection. Since N is semi-projective; $(\alpha\beta)\lambda = \pi$ for some $\lambda \in E_N$. For $\mu = \lambda\alpha\beta\lambda$; $(\beta\mu)\alpha(\beta\mu) = \beta\mu$.
(2)(i) $\Rightarrow$ (ii). It is clear by Proposition 4.1.
(ii) $\Rightarrow$ (i). Let $\alpha \in E_N$. Then $\mathrm{Ker}(\beta\alpha) \subseteq^{\oplus} N$ for some $\beta \in E_N$. Let π the projection on the complementary summand of $\mathrm{Ker}(\beta\alpha)$. Then

$\mathrm{Ker}(\beta\alpha) = \mathrm{Ker}(\pi)$. By Theorem 3.2; $\pi \in E_N(\beta\alpha) \subseteq E_N\alpha$ hence N is semi-injective. So $\pi = \lambda\alpha$ for some $\lambda \in E_N$. For $\mu = \lambda\alpha\lambda$; $\mu\alpha\mu = \mu$. $\square$

Acknowledgments

The author is very grateful to the referees for their valuable comments and suggestions.

References

[1] F. Kasch, *Modules and Rings*, Academic Press. 1982.

[2] F. Kasch and A. Mader, *Rings, Modules, and the Total, Front. Math.*, Birkhauser Verlag, Basel, 2004.

[3] F. Kasch, *Regular Substructures of Hom*, Appl. Categ. Structures **16** (2008), 159–166.

[4] F. Kasch, *Locally injective and locally projective modules*, Rocky Mountain J. Math. **32** (4) (2002), 1493–1504.

[5] W. K. Nicholson, *$I-$ Rings*, Trans. Amer. Math. Soc. **207** (1975), 361–373.

[6] G.M. Tsukerman, *Rings of endomorphisms of free modules*, Sibirsk. Mat. Zh. **7** (7) (1966), 1161 - 1167.

[7] R. Ware, *Endomorphism Rings of Projective Modules*, Trans. Amer. Math. Soc. **155** (1971), p. 233 - 256.

[8] R. Wisbaure, *Foundations of Module and Ring Theory*, Gordon and Breach, Philadelphia, 1991.

[9] Y. Zhou, *On (Semi)regularity and the total of rings and modules*, J. Algebra. **322** (2009), 562–578.

Hamza Hakmi
Department of Mathematics
Damascus University
Damascus, Syria
E-mail: hhakmi-64@hotmail.com