Korean J. Math. **22** (2014), No. 1, pp. 133–138
http://dx.doi.org/10.11568/kjm.2014.22.1.133

ON THE SYMMETRY OF ANNULAR BRYANT SURFACE WITH CONSTANT CONTACT ANGLE

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Abstract. We show that a compact immersed annular Bryant surface in $\mathbb{H}^3$ meeting two parallel horospheres in constant contact angles is rotational.

1. introduction

Catenoid is the only nonplanar minimal surface of rotation in $\mathbb{R}^3$. Therefore a catenoid meets each plane perpendicular to the axis of rotation in constant contact angle. Conversely, if a compact embedded minimal or constant mean curvature (cmc) surface in $\mathbb{R}^3$ meets two parallel planes in constant contact angles, then the surface is part of a catenoid or part of a cmc surface of rotation, i.e., a Delaunay surface. This can be proved by using the Alexandrov's moving plane argument [4], [11] to planes perpendicular to the parallel planes. Recently, Pyo showed that a compact immersed minimal annulus meeting two parallel planes in constant contact angles is also part of a catenoid [9]. In the case of cmc surfaces, the result fails to hold: Wente constructed examples of immersed constant mean curvature annuli in a slab or in a ball meeting the boundary planes or the boundary sphere perpendicularly [12].

Received November 12, 2013. Revised March 17, 2014. Accepted March 17, 2014.
2010 Mathematics Subject Classification: 53A10, 53C24.
Key words and phrases: Bryant Surface, Hyperbolic space, Capillarity.
This work was supported by Hankuk University of Foreign Studies Research Fund.

The hyperbolic gauss map of a Bryant surface in $\mathbb{H}^3$ is meromorphic as the gauss map of a minimal surface in $\mathbb{R}^3$ is meromorphic [2]. Morevoer, the cousin correspondence [5] shows a close relation between minimal surfaces in $\mathbb{R}^3$ and Bryant surfaces in $\mathbb{H}^3$: for each simply connected minimal surface in $\mathbb{R}^3$, there exists a differentiable, 2π-periodic family of Bryant surfaces in $\mathbb{H}^3$. The cousin of a plane in $\mathbb{R}^3$ is the associate surfaces of a horosphere in $\mathbb{H}^3$. The cousin of the catenoid is called the catenoid cousin. In this paper, we generalize Pyo's result to Bryant surfaces in $\mathbb{H}^3$.

THEOREM 1. *Let Σ be a compact immersed annular Bryant surface in $\mathbb{H}^3$ meting two parallel horospheres in constant contact angles. Let f be the hyperbolic gauss map of Σ. If f' does not attain* 0 *and* ∞, *then Σ is rotational.*

Two horospheres in $\mathbb{H}^3$ are said to be *parallel* if they have the same ideal boundary point.We note that the gauss map of a minimal surface in a slab in $\mathbb{R}^3$ cannot attain 0 or ∞ [3]. But the hyperbolic gauss map of a catenoid cousin meeting two parallel horospheres can attain 0 or ∞ [10]. In the embedded surface case, one can use the Alexandrov reflection argument to prove that a compact embedded Bryant surface in $\mathbb{H}^3$ meeting two parallel horospheres in constant contact angles is rotational.

We use the Bianchi-Calò method which represents a Bryant surface very simply which is homeomorphic to a region in $\mathbb{C}$ [6].

2. Bianchi-Calò method

We use the upper half space model $(\mathbb{R}^3_+, ds^2_h)$ for $\mathbb{H}^3$: $\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \leq 0\}$ and $ds^2_h = (dx_1^2 + dx_2^2 + dx_3^2)/x_3^2$. In this model, horosphere is either a (euclidean) sphere tangent to the $\{x_3 = 0\}$-plane or a horizontal plane $\{x_3 = \mathit{constant}\}$.

Let $\psi : \Sigma \to \mathbb{H}^3$ be an immersed oriented surface. Let ν be the unit normal vector field on Σ. The hyperbolic gauss map $f : \Sigma \to \partial_\infty \mathbb{H}^3$ relates to $p \in \Sigma$ the end point on the ideal boundary $\partial_\infty \mathbb{H}^3$ of the oriented normal geodesic starting from p in the direction of ν.

Remark 1. *Geometrically, the hyperbolic gauss map f can be interpreted in two ways as follows (the geodesic half sphere and horosphere are assumed to be located in the direction of ν. cf. Figure 1.):*

(a) *$f(p)$ is the euclidean center on $\partial_\infty \mathbb{H}^3 = \mathbb{C}^2 \cup \{\infty\}$ of the geodesic plane tangent to M at p.*

(b) *$f(p)$ is the point on $\partial_\infty \mathbb{H}^3$ of the horosphere tangent to M at p.*

The following Lemma shows the special feature of the Bryant surfaces in $\mathbb{H}^3$ [2].

Lemma 1. *A surface $\psi : \Sigma \to \mathbb{H}^3$ has mean curvature one if and only if the hyperbolic gauss map $h : \Sigma \to \mathbb{C} \cup \{\infty\}$ is meromorphic.*

Instead of the usual Bryant representation formula, we use the Bianchi-Calò method to represent a Bryant surface which is homeomorphic to a region in $\mathbb{C}$ [6]. Let $f = f(z)$ be a holomorphic map defined in a region $\Omega \subset \mathbb{C}$, and let

$$R_{f(z)} = \frac{1 + |z|^2}{2} \left| f'(z) \right| . \tag{1}$$

Let $S_{f(z)} \subset \mathbb{R}^3_+$ be the sphere which is tangent to $\partial_\infty \mathbb{H}^3$ at $f(z)$ and has euclidean radius $R_{f(z)}$. Note that $S_{f(z)}$ is a two-parameter family of spheres. Clearly, $\partial_\infty \mathbb{H}^3$ is one of the two envelopes of $S_{f(z)}$. The second envelope gives a Bryant surface whose gauss map is f [6].

Bianchi-Calò method: In the above situation, the parametrization

$$x_1 = Re(f) - \frac{|f'|^2 Re(f'z) + \frac{1+|z|^2}{2} Re\left((f')^2 \bar{f}''\right)}{|f'|^2 + Re(f'\bar{f}''\bar{z}) + \frac{|f''|^2(1+|z|^2)}{4}} \tag{2}$$

$$x_2 = Im(f) - \frac{|f'|^2 Im(f'z) + \frac{1+|z|^2}{2} Im\left((f')^2 \bar{f}''\right)}{|f'|^2 + Re(f'\bar{f}''\bar{z}) + \frac{|f''|^2(1+|z|^2)}{4}} \tag{3}$$

$$x_3 = \frac{|f'|^3}{|f'|^2 + Re(f'\bar{f}''\bar{z}) + \frac{|f''|^2(1+|z|^2)}{4}} \tag{4}$$

in terms of f gives a Bryant surface Σ_f in $\mathbb{H}^3$. (Here, we use $'$ to denote d/dz.) Moreover, f is the hyperbolic gauss map of Σ_f in terms of the local complex parameter $z = x + iy$ on Ω.

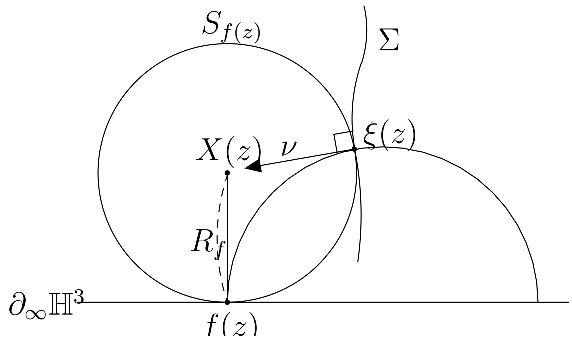

FIGURE 1. Bianchi-Calò method

REMARK 2. *For a given Bryant surface Σ homeomorphic to a region in $\mathbb{C}$ with hyperbolic gauss map f, the radius R_f of* (1) *is just the euclidean radius of the horosphere tangent to $\partial_\infty \mathbb{H}^3$ and Σ. Therefore Σ_f derived from f by the Bianchi-Calò method coincides with Σ.*

We briefly explain Σ_f. Details can be found in [6]. Let $f = f_1 + if_2$, and let

$$X(z) = \big(f_1(z), f_2(z), R_{f(z)}\big)$$

be the surface of centers of $S_{f(z)}$. Since Σ_f is an envelope of $S_{f(z)}$, we have $T_p\Sigma_f = T_pS_{f(z)}$ at each $p \in \Sigma_f$ and for suitable z. Therefore Σ_f is given by

$$\xi(z) = X(z) - R_{f(z)}\nu, \tag{5}$$

where ν is the euclidean unit normal of Σ_f in the direction of $X - \xi$ (cf. Figure 2). Here we have (for simplicity, we let $R = R_f$) [6]

$$\nu = \frac{1}{|\nabla R|^2 + |f'|^2}\left(2\alpha_1, 2\alpha_2, |\nabla R|^2 - |f'|^2\right),$$

where

$$\alpha_1 = R_y f_{2,x} - R_x f_{2,y},\ \alpha_2 = R_x f_{1,y} - R_y f_{2,x}$$

and

$$|\nabla R|^2 + |f'|^2 = \big(|z|^2 + 1\big)\left(|f'|^2 + Re(f'\bar{f}''\bar{z}) + \frac{|f''|^2(1+|z|^2)}{4}\right). \tag{6}$$

Then it is easy to see that

$$x_3 = \xi_3 = R - R\,\nu_3 = \frac{2R|f'|^2}{|\nabla R|^2 + |f'|^2}. \tag{7}$$

3. Proof of the Main result

In the following, the two parallel horospheres under consideration are assumed to be two horizontal planes Π_1 and Π_2 in $\mathbb{R}^3_+$. Let Σ be a compact immersed annular Bryant surface meeting Π_1 and Π_2 in constant contact angles. Let $\xi : A \to \mathbb{H}^3$ be the immersion of Σ, where $A = \{(x, y) \in \mathbb{R}^2 : R_1 \leq r = \sqrt{x^2 + y^2} \leq R_2\}$ is an annulus. Let $f : A \to \partial_\infty \mathbb{H}^3 = \mathbb{C}^2 \cup \{\infty\}$ be the hyperbolic gauss map of Σ. Hereafter we identify the Bryant surface Σ_f derived from f with Σ. Since the hyperbolic gauss map f is assumed to be bounded, f is holomorphic on A. Now we prove Theorem 1.

Proof of Theorem 1. The constant contact angle condition implies that the third component $\nu_3 = (|\nabla R|^2 - |f'|^2)/(|\nabla R|^2 + |f'|^2)$ of ν is constant on each component of ∂A. Therefore $|f'|^2/|\nabla R|^2$ is constant on each component of ∂A.

Since $\partial \Sigma_f$ lies on horizontal planes, $x_3 = 2R|f'|^2/(|\nabla R|^2 + |f'|^2)$ is also constant on each component of ∂A. From the constancy of $|f'|^2/|\nabla R|^2$ on ∂A, it follows that $2R = 2|f'|^2/(1 + |z|^2)$ is constant on each component of ∂A. Since ∂A consists of two concentric circles centered at the origin, $|f'|$ is also constant on each component of ∂A. Since $|f'|$ is assumed not to attain 0 and ∞, $\log |f'|$ is a bounded harmonic function on A. Since $|f'|$ is constant on each component of ∂A, we have $\log |f'| = a \log |z| + b$ for some real constants a and b. Hence we have $f'(z) = e^b z^a = Bz^a$. Since f is a single-valued holomorphic function on A, we have $f'(z) = Bz^n$ for some integer n.

From (4), we see that

$$
\begin{aligned}
x_3 &= \frac{|Bz^n|^3}{|Bz^n|^2 + B^2 Re(z^n \cdot n \cdot \bar{z}^{n-1} \cdot \bar{z}) + \frac{|nBz^{n-1}|^2(1+|z|^2)}{4}}. \\
&= \frac{|B||z|^{n+2}}{(n+1)|z|^2 + \frac{n^2}{4}(1 + |z|^2)}
\end{aligned}
$$

Hence x_3 is constant on each circle $C_r = \{z : |z| = r\}$, for $R_1 \leq r \leq R_2$. It follows that the x_3-level curves of Σ_f are images of C_r.

From (1), it follows that R is also constant on each circle $C_r = \{z : |z| = r\}$. We may assume that $f(z) = \frac{B}{n+1} z^{n+1}$. Hence the image of C_r under f is a circle on $\partial_\infty \mathbb{H}^3$. Since Σ_f is one of the envelopes of $S_{f(z)}$, we conclude that $\xi(C_r)$ is a circle on on a horizontal plane. It is clear that $\xi(C_r)$ are coaxial. Hence Σ_f is rotational. $\square$

Finally, we raise the following question.

Let Σ *be a compact immersed annular Bryant surface in* $\mathbb{H}^3$ *meeting two parallel horospheres in constant contact angles. Is* Σ *rotational, even if the derivative of the hyperbolic gauss map attain* 0 *or* ∞?

References

[1] A. D. Alexandrov, *Uniqueness theorems for surfaces in the large V*, Amer. Math. Soc. Transl. **21** (1962), 412–416.

[2] R. Bryant, *Surfaces with constant mean curvature one in hyperbolic space*, Astérisque 154–155: 321–347.

[3] Y. Fang, *Lectures on minimal surfaces in* $\mathbb{R}^3$, Center for Mathematics and Its applications, Australian National University, (1996).

[4] H. Hopf, *Differential Geometry in the large*, Springer, Berlin, (1989).

[5] H. B. Lawson, *Complete Minimal Surfaces in* $\mathbb{S}^3$, Ann. of Math. 2nd Ser. **92** (3) (1970), 335–374.

[6] L. Lima and P. Roitman, *Constant mean curvature one surfaces in hyperbolic space using Bianchi-Calò method*, Annals of the Braz. Acad. of Sci. **74** (2002) (1), 19–24; arXiv:math/0110021.

[7] J. McCuan, *Symmetry via spherical reflection and spanning drops in a wedge*, Pacific J. Math. **180** (1997) (2), 291–323.

[8] J. C. C. Nitsche, *Stationary partitioning of convex bodies*, Arch. Rat. Mech. Anal. **89** (1985), 1–19.

[9] J. Pyo, *Minimal annuli with constant contact angle along the planar boundaries*, Geom. Dedicata **146** (1) (2010), 159–164.

[10] W. Rossman and K. Sato, *Constant mean curvature surfaces with two ends in hyperbolic space*, Experiment. Math. Volume **7**, Issue 2 (1998), 101–119.

[11] J. Serrin, *A symmetry problem in potential theory*, Arch. Rat. Mech. and Anal. **43** (1971), 304–318.

[12] H. Wente, *Tubular capillary surfaces in a convex body.* Advances in geometric analysis and continuum mechanics (Stanford, CA, 1993), 288–298, International Press, Cambridge, MA, (1995).

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