



# Stability of Explicit Symplectic Partitioned Runge-Kutta Methods

Toshiyuki Koto<sup>1\*</sup> and Eunjee Song<sup>2</sup>, *Member, KIICE*

<sup>1</sup>Department of Information Systems and Mathematical Sciences, Nanzan University, Seto 489-0863, Japan

<sup>2</sup>Department of Computer Science, Namseoul University, Cheonan 331-707, Korea

## Abstract

A numerical method for solving Hamiltonian equations is said to be symplectic if it preserves the symplectic structure associated with the equations. Various symplectic methods are widely used in many fields of science and technology. A symplectic method preserves an approximate Hamiltonian perturbed from the original Hamiltonian. It theoretically supports the effectiveness of symplectic methods for long-term integration. Although it is also related to long-term integration, numerical stability of symplectic methods have received little attention. In this paper, we consider explicit symplectic methods defined for Hamiltonian equations with Hamiltonians of the special form, and study their numerical stability using the harmonic oscillator as a test equation. We propose a new stability criterion and clarify the stability of some existing methods that are visually based on the criterion. We also derive a new method that is better than the existing methods with respect to a Courant-Friedrichs-Lewy condition for hyperbolic equations; this new method is tested through a numerical experiment with a nonlinear wave equation.

**Index Terms:** Courant-Friedrichs-Lewy condition, Hamiltonian equations, Nonlinear wave equations, Numerical stability, Symplectic methods

## I. INTRODUCTION

A symplectic integration method is a numerical method for solving Hamiltonian equations,

$$\frac{dq}{dt} = \nabla_p H, \quad \frac{dp}{dt} = -\nabla_q H, \quad (1)$$

a special class of differential equations related to classical mechanics and symplectic geometry. Various symplectic methods are designed and widely used in celestial mechanics, molecular dynamics, electromagnetic field analysis, etc., particularly for the longterm integration of Hamiltonian

equations.

The time evolution of Hamiltonian equations preserves a special differential 2-form  $dp \wedge dq$  called the symplectic form. A numerical method is said to be symplectic if it also preserves the symplectic form. Since the concept of symplectic integration methods was proposed in the mid-1980s [1], many mathematical researches have been carried out [2-4]. In particular, it has been revealed that a symplectic method preserves an approximate Hamiltonian perturbed from the original Hamiltonian [5, 6]. It theoretically supports the effectiveness of symplectic methods for long-term integration.

On the other hand, the numerical stability of symplectic

Received 09 October 2013, Revised 11 November 2013, Accepted 02 December 2013

\*Corresponding Author Toshiyuki Koto (E-mail: [koto@ms.nanzan-u.ac.jp](mailto:koto@ms.nanzan-u.ac.jp), Tel:81-561-89-2000 ext. 2408)

Department of Information Systems and Mathematical Sciences, Nanzan University, Seto 489-0863, Japan.

**Open Access** <http://dx.doi.org/10.6109/jicce.2014.12.1.039>

print ISSN: 2234-8255 online ISSN: 2234-8883

© This is an Open Access article distributed under the terms of the Creative Commons Attribution Non-Commercial License (<http://creativecommons.org/licenses/by-nc/3.0/>) which permits unrestricted non-commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

Copyright © The Korea Institute of Information and Communication Engineering

methods has received little attention, although it is also related to long-term integration; only a few papers [7, 8] deal with this subject. It is certain that many outstanding symplectic methods are implicit and possess originally superior stability. However, in a large-scale computation, e.g., in the solution of partial differential equations, explicit methods are still effective tools. A study of their stability has significance for practical computation because the stability of numerical methods is closely related to step size restrictions, such as a Courant-Friedrichs-Lewy (CFL) condition for hyperbolic equations.

In this paper, we study the stability of an explicit symplectic method by using the harmonic oscillator as a test equation, following [8]. An outline of this paper is as follows: In Section II, we describe the fundamental concept and notation concerning explicit symplectic methods and their numerical stability. In Section III, we propose a new stability criterion for the symplectic methods and discuss the stability of the basic methods on the basis of this criterion. In Section IV, we continue to analyze more advanced methods and derive a new method, which is tested through a numerical experiment with the sine-Gordon equation, a nonlinear wave equation in Section V.

## II. PRELIMINARIES

### A. Explicit Symplectic Methods

We consider a Hamiltonian of the special form

$$H(t, p, q) = T(p) + U(t, q), \quad (2)$$

and the initial value problem

$$\begin{cases} \frac{dq}{dt} = f(p), & \frac{dp}{dt} = g(t, q) \quad (t \geq 0) \\ q(0) = q_0, & p(0) = p_0, \end{cases} \quad (3)$$

for the corresponding Hamiltonian equation, where

$$f(p) = \nabla_p T(p), \quad g(t, q) = -\nabla_q U(t, q). \quad (4)$$

In mechanics,  $T$  and  $U$  represent kinetic energy and potential energy, respectively.

In general, symplectic methods are implicit; i.e., it is necessary to solve nonlinear equations for the implementation of these methods. For problem (3), there are explicit symplectic methods by virtue of the special form (2). A well-known instance is a symplectic partitioned Runge-Kutta method, whose general form is as follows (see, e.g., [2, 4]):

$$\begin{aligned} Q_0 &= q_n, \quad P_1 = p_n \\ \text{for } i &:= 1 \text{ to } s \text{ do} \\ \quad \{ & Q_i = Q_{i-1} + \Delta t \, b_i f(P_i) \\ & P_{i+1} = P_i + \Delta t \, \hat{b}_i g(t_n + c_i \Delta t, Q_i) \\ Q_{n+1} &= Q_s, \quad p_{n+1} = P_{s+1} \end{aligned} \quad (5)$$

Here,  $\Delta t > 0$  is the time step size,  $t_n = n\Delta t$  ( $n = 0, 1, \dots$ ), and  $q_n$  and  $p_n$  are approximate values for  $q(t_n)$  and  $p(t_n)$ , respectively. Further,  $b_1, b_2, \dots, b_s$ ,  $c_i = \sum_{j=1}^i b_j$ , and  $\hat{b}_1, \hat{b}_2, \dots, \hat{b}_s$  are parameters of the method, and  $Q_i$  and  $P_i$  are intermediate variables for computation. The parameters of the method, determined from order conditions [2, 4], are often written as

$$b : b_1, b_2, \dots, b_s, \quad \hat{b} : \hat{b}_1, \hat{b}_2, \dots, \hat{b}_s.$$

### B. Test Equation for Stability Analysis

To study the stability of the symplectic method (5), we adopt the harmonic oscillator

$$\frac{dq}{dt} = \omega p, \quad \frac{dp}{dt} = -\omega q, \quad \omega \geq 0, \quad (6)$$

as a test equation ([8]; see also [7] for another test equation). This is a Hamiltonian equation with the Hamiltonian  $H(p, q) = (\omega/2)(p^2 + q^2)$ ,  $\omega \geq 0$ . We also adopt the scaled step size

$$\theta = \omega \Delta t, \quad (7)$$

as a basic parameter for the stability analysis. Upon the restriction of the frequency  $\omega \geq 0$ , the range of the parameter is  $\theta \geq 0$ .

It should be noted that exact solutions to (6) satisfy

$$\begin{aligned} \begin{bmatrix} q(t_{n+1}) \\ p(t_{n+1}) \end{bmatrix} &= M(\theta) \begin{bmatrix} q(t_n) \\ p(t_n) \end{bmatrix}, \\ M(\theta) &= \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}. \end{aligned} \quad (8)$$

The matrix  $M(\theta)$  is an orthogonal matrix, and its eigenvalues are  $e^{\pm i\theta}$ , both of which have unit modulus.

In the case  $f(p) = \omega p$  and  $g(t, q) = -\omega q$ , the equations for the intermediate variables in (5) becomes

$$\begin{cases} Q_i = Q_{i-1} + b_i \theta P_i \\ P_{i+1} = P_i - \hat{b}_i \theta Q_i \end{cases} \quad (9)$$

The substitution of the first equation into the second equation gives  $P_{i+1} = -\hat{b}_i \theta Q_{i-1} + (1 - \hat{b}_i b_i \theta^2) P_i$ . Hence, (9) is rewritten as

$$\begin{bmatrix} Q_i \\ P_{i+1} \end{bmatrix} = M_i(\theta) \begin{bmatrix} Q_{i-1} \\ P_i \end{bmatrix}, \quad M_i(\theta) = \begin{bmatrix} 1 & b_i \theta \\ -\hat{b}_i \theta & 1 - \hat{b}_i b_i \theta^2 \end{bmatrix}, \quad (10)$$

and application of method (5) to test equation (6) yields an analogue to (8),

$$\begin{bmatrix} q_{n+1} \\ p_{n+1} \end{bmatrix} = M_*(\theta) \begin{bmatrix} q_n \\ p_n \end{bmatrix}, M_*(\theta) = M_s(\theta) \cdots M_1(\theta). \quad (11)$$

It is clear that  $\det M_i(\theta) = 1$ . Hence,  $\det M_*(\theta) = 1$  holds for any method of the form (5). The characteristic equation of  $M_*(\theta)$  is written as

$$\lambda^2 - \{\text{tr } M_*(\theta)\}\lambda + 1 = 0, \quad (12)$$

and the eigenvalues are given by

$$\lambda = \frac{\text{tr } M_*(\theta) \pm \sqrt{\{\text{tr } M_*(\theta)\}^2 - 4}}{2}, \quad (13)$$

where  $\text{tr } M_*(\theta)$  denotes the trace of the matrix  $M_*(\theta)$ . If  $|\text{tr } M_*(\theta)| < 2$ , the eigenvalues are complex numbers with  $|\lambda| = 1$ . If  $\text{tr } M_*(\theta) = 2$ , then  $\lambda = 1$ , and if  $\text{tr } M_*(\theta) = -2$ , then  $\lambda = -1$ . If  $|\text{tr } M_*(\theta)| > 2$ , the eigenvalues are real, and one of them satisfies  $|\lambda| > 1$ . The set  $\{\theta \geq 0 : |\text{tr } M_*(\theta)| \leq 2\}$  is a union of closed intervals. The connected component of the set that contains the origin is called the *stability interval* of method (5), which has been used for comparing the stability of numerical methods [8].

### III. STABILITY CRITERION

If  $|\text{tr } M_*(\theta)| < 2$ ,  $M_*(\theta)$  has complex conjugate eigenvalues  $\lambda, \bar{\lambda}$  which satisfy  $|\lambda| = |\bar{\lambda}| = 1$  and  $\lambda \neq \bar{\lambda}$ . Hence,  $M_*(\theta)$  is represented in the form

$$M_*(\theta) = T \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} T^{-1}, \quad (14)$$

with some nonsingular matrix  $T$ . Since

$$M_*(\theta)^n = T \begin{bmatrix} \lambda^n & 0 \\ 0 & \bar{\lambda}^n \end{bmatrix} T^{-1}, \quad (15)$$

and  $|\lambda| = |\bar{\lambda}| = 1$ , we have  $\|M_*(\theta)^n\| \leq \|T\| \|T^{-1}\|$  for any integer  $n \geq 0$ , where  $\|\cdot\|$  denotes the matrix norm induced from the Euclidean norm. The upper bound  $\|T\| \|T^{-1}\|$  is represented as follows.

**Theorem 1.** Let  $a, b, c, d$  be real numbers. Assume that  $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  satisfies  $\det M = 1$  and  $|\text{tr } M| < 2$ . Then, we have

$$\|M^n\| \leq \phi, \quad (16)$$

for any integer  $n \geq 0$ , where

$$\phi = \frac{|b - c| + \sqrt{(b - c)^2 - \{4 - (a + d)^2\}}}{\sqrt{4 - (a + d)^2}}. \quad (17)$$

The proof of the theorem is obtained by a simple but tiresome computation. We omit the proof (cf. the proof of Theorem 3.1 in [9]). As shown below,  $\phi$  in Theorem 1 is used as a criterion for the stability of the numerical methods.

In the case  $s = 1$  and  $b_1 = \hat{b}_1 = 1$ , (5) is reduced to

$$\begin{cases} q_{n+1} = q_n + \Delta t f(p_n) \\ p_{n+1} = p_n + \Delta t g(t_{n+1}, q_{n+1}) \end{cases}. \quad (18)$$

This is called the symplectic Euler method and is of the order 1 in accuracy. In the case of the symplectic Euler method, we have

$$M_*(\theta) = M_1(\theta) = \begin{bmatrix} 1 & \theta \\ -\theta & 1 - \theta^2 \end{bmatrix}. \quad (19)$$

Since  $\text{tr } M(\theta) = 2 - \theta^2$ , the stability interval of the method is  $[0, 2]$ . For  $0 < \theta < 2$ ,  $\phi$  in Theorem 1 is computed as

$$\begin{aligned} \phi &= \frac{2\theta + \sqrt{(2\theta)^2 - \{4 - (2 - \theta^2)^2\}}}{\sqrt{4 - (2 - \theta^2)^2}} \\ &= \frac{2\theta + \theta^2}{\theta\sqrt{4 - \theta^2}} = \frac{2 + \theta}{\sqrt{4 - \theta^2}}. \end{aligned} \quad (20)$$

In the case  $s = 2$ , method (5) is rewritten as

$$\begin{cases} Q_1 = q_n + \Delta t b_1 f(p_n) \\ P_2 = p_n + \Delta t \hat{b}_1 g(t_n + c_1 \Delta t, Q_1) \\ q_{n+1} = Q_1 + \Delta t b_2 f(P_2) \\ p_{n+1} = p_n + \Delta t \hat{b}_2 g(t_n + c_2 \Delta t, q_{n+1}) \end{cases}, \quad (21)$$

which is of the order 2 if the parameters satisfy

$$b_1 + b_2 = \hat{b}_1 + \hat{b}_2 = 1, \quad b_2 \hat{b}_1 = \frac{1}{2}. \quad (22)$$

In particular, the parameter values

$$b_1 = 0, \quad b_2 = 1, \quad \hat{b}_1 = \hat{b}_2 = \frac{1}{2}, \quad (23)$$

satisfy the condition, and the corresponding method is known as the Störmer-Verlet method [4, 8].

For this method, we have

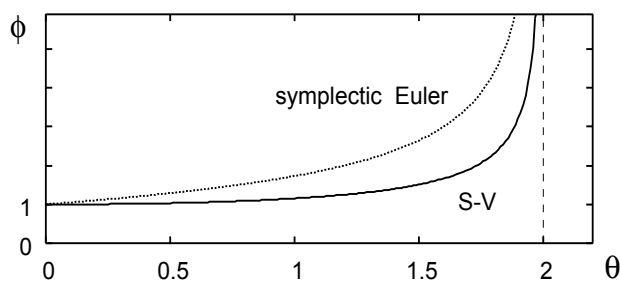
$$M_*(\theta) = M_2(\theta)M_1(\theta) = \begin{bmatrix} 1 - \frac{\theta^2}{2} & \theta \\ -\theta + \frac{\theta^3}{4} & 1 - \frac{\theta^2}{2} \end{bmatrix}. \quad (24)$$

Since  $\text{tr } M_*(\theta) = 2 - \theta^2$ , the stability interval of the Störmer-Verlet method is  $[0, 2]$ , which is the same as that of the symplectic Euler method. However, since  $(2\theta - \theta^3/4)^2 - \{4 - (2 - \theta^2)^2\} = \theta^6/16$ , we have, for  $0 < \theta < 2$ ,

$$\begin{aligned} \phi &= \frac{|2\theta - \theta^3/4| + \sqrt{\theta^6/16}}{\sqrt{4 - (2 - \theta^2)^2}} \\ &= \frac{2\theta - \theta^3/4 + \theta^3/4}{\theta\sqrt{4 - \theta^2}} = \frac{2}{\sqrt{4 - \theta^2}}. \end{aligned} \quad (25)$$

Fig. 1 shows the functions  $\phi$  for the two methods. Function (25) for the Störmer-Verlet method is closer to the line  $\phi = 1$  than (20) for the symplectic Euler method. The matrix  $M(\theta)$  in (8) is an orthogonal matrix and satisfies  $\|M(\theta)^n\| = 1$  for any  $\theta \geq 0$  and any integer  $n \geq 0$ . Since (25) reflects this property more appropriately than (20), we can consider the Störmer-Verlet method has a better stability property than the symplectic Euler method although the two methods have the same stability intervals.

Table 1 presents  $\phi$  and  $\phi_{100} = \max_{0 \leq n \leq 100} \|M_*(\theta)^n\|$ , computed numerically, for several values of  $\theta$ . This shows that  $\phi$  gives an appropriate approximation to  $\sup_{n \geq 0} \|M_*(\theta)^n\|$  except  $\theta = 1$ .



**Fig. 1.** Functions  $\phi$  for the symplectic Euler and Störmer-Verlet methods.

**Table 1.** Comparison between  $\phi$  and  $\phi = \max_{0 \leq n \leq 100} \|M_*(\theta)^n\|$

$\theta$	0.3	1.0	1.5	1.8	1.9
<b>The symplectic Euler method</b>					
$\phi$	1.163160	1.732051	2.645751	4.358899	6.244998
$\phi_{100}$	1.163137	1.618034	2.645681	4.358459	6.244907
<b>The Störmer-Verlet method</b>					
$\phi$	1.011443	1.154701	1.511858	2.294157	3.202563
$\phi_{100}$	1.011442	1.132782	1.511837	2.293982	3.202523

## IV. STABILITY OF METHODS OF ORDER 3 AND ORDER 4

Method (5) for  $s = 3$  corresponding to the parameter values

$$b : 7/24, 3/4, -1/24, \hat{b} : 2/3, -2/3, 1, \quad (26)$$

is called Ruth's method, which is of the order 3 in accuracy. For Ruth's method, we have

$$M_*(\theta) = \begin{bmatrix} 1 - \frac{\theta^2}{2} + \frac{\theta^4}{72} & \theta - \frac{\theta^3}{6} + \frac{7\theta^5}{1728} \\ -\theta + \frac{\theta^3}{6} - \frac{\theta^5}{72} & 1 - \frac{\theta^2}{2} + \frac{5\theta^4}{72} - \frac{7\theta^6}{1728} \end{bmatrix}. \quad (27)$$

$$\text{tr } M_*(\theta) = 2 - \theta^2 + \frac{\theta^4}{12} - \frac{7\theta^6}{1728}. \quad (28)$$

The stability interval is  $[0, \theta_0^R]$ ,  $\theta_0^R \approx 2.50748$ , where  $\theta_0^R$  denotes a root of  $\text{tr } M_*(\theta) = -2$ .

To try to improve Ruth's method with respect to stability, we consider (5) for  $s = 4$  with  $\hat{b}_4 = 0$ , which is reduced to

$$\begin{cases} Q_1 = q_n + \Delta t b_1 f(p_n) \\ P_2 = p_n + \Delta t \hat{b}_1 g(t_n + c_1 \Delta t, Q_1) \\ Q_2 = Q_1 + \Delta t b_2 f(P_2) \\ P_3 = P_2 + \Delta t \hat{b}_2 g(t_n + c_2 \Delta t, Q_2) \\ Q_3 = Q_2 + \Delta t b_3 f(P_3) \\ p_{n+1} = P_3 + \Delta t \hat{b}_3 g(t + c_3 \Delta t, Q_3) \\ q_{n+1} = Q_3 + \Delta t b_4 f(p_{n+1}) \end{cases}. \quad (29)$$

At first glance, it appears that (29) needs more evaluation of  $f$  than (5) with  $s = 3$ , but  $f(p_{n+1})$  for the computation of  $q_{n+1}$  is again used for the computation of  $Q_1$  at the next step  $t = t_{n+1}$ . Hence, from the perspective of function evaluation, the work needed for (29) is the same as that for (5) with  $s = 3$ , e.g., Ruth's method. This idea is called first same as last and is often utilized in the numerical analysis of differential equations [2].

Method (29) is of the order 3 if the parameters satisfy

$$\begin{cases} b_1 + b_2 + b_3 + b_4 = 1, \hat{b}_1 + \hat{b}_2 + \hat{b}_3 = 1 \\ b_2 \hat{b}_1 + b_3(\hat{b}_1 + \hat{b}_2) + b_4(\hat{b}_1 + \hat{b}_2 + \hat{b}_3) = \frac{1}{2} \\ b_2 \hat{b}_1^2 + b_3(\hat{b}_1 + \hat{b}_2)^2 + b_4(\hat{b}_1 + \hat{b}_2 + \hat{b}_3)^2 = \frac{1}{3} \\ \hat{b}_1 b_1^2 + \hat{b}_2(b_1 + b_2)^2 + \hat{b}_3(b_1 + b_2 + b_3)^2 = \frac{1}{3} \end{cases}. \quad (30)$$

These are too complicated to treat. We thus introduce the simplifying condition

$$b_1 + b_4 = 0. \quad (31)$$

By virtue of this condition, the coefficient of  $\theta^6$  in  $\text{tr } M_*(\theta)$  becomes 0, and the trace is reduced to

$$\text{tr } M_*(\theta) = 2 - \theta^2 + \frac{\theta^4}{12}. \quad (32)$$

The stability interval becomes  $[0, 2\sqrt{3}]$ ,  $2\sqrt{3} \cong 3.46410$ , which is larger than that of Ruth's method.

Eqs. (30) and (31) form a system of 6 equations with 7 unknown variables, which has solutions with a free parameter, e.g.,  $b_1$ . Letting  $b_1 = 1/3$ , we obtain the following:

$$\begin{aligned} b &: 1/3, (\sqrt{13} + 3)/6, (3 - \sqrt{13})/6, -1/3 \\ \hat{b} &: (13 - \sqrt{13})/12, -1/2, (\sqrt{13} + 5)/12, 0 \end{aligned} \quad (33)$$

We refer to the corresponding method as the stabilized 3rd-order method. In Fig. 2, the functions  $\phi$  for Ruth's method and the stabilized 3rd-order method are presented. For  $\theta \leq 2.37$ ,  $\phi$  for Ruth's method is smaller than  $\phi$  for the stabilized 3rd-order method, but the latter has finite values up to  $\theta = 2\sqrt{3}$ .

Several symplectic methods of the order 4 are known. Among them, a method of the form (29) corresponding to the parameter values (see, e.g., [4], p. 109)

$$\begin{aligned} b &: \beta/2, (1 - \beta)/2, (1 - \beta)/2, \beta/2 \\ \hat{b} &: \beta, 1 - 2\beta, \beta, 0 \end{aligned}, \quad (34)$$

needs the least function evaluation, where

$\beta = (2 + 2^{1/3} + 2^{-1/3})/3$ , a root of  $\beta^3 - 2\beta^2 + \beta - 1/6 = 0$ .

For this method, we have the following:

$$M_*(\theta) = \begin{bmatrix} 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} + p_6\theta^6 & \theta - \frac{\theta^3}{6} + q_5\theta^5 + q_7\theta^7 \\ -\theta + \frac{\theta^3}{6} + r_5\theta^5 & 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} + p_6\theta^6 \end{bmatrix}. \quad (35)$$

$$\begin{aligned} \text{tr } M_*(\theta) &= 2 - \theta^2 + \frac{\theta^4}{12} + 2p_6\theta^6, \\ p_6 &= \frac{9\beta^2 - 6\beta + 1}{144}, \quad q_5 = -\frac{\beta^2 - \beta}{24}, \\ q_7 &= \frac{24\beta^2 - 16\beta + 3}{576}, \quad r_5 = \frac{2\beta^2 - \beta}{24}. \end{aligned} \quad (36)$$

The stability interval is  $[0, \theta_0^F]$ ,  $\theta_0^F \approx 1.57340$ , where  $\theta_0^F$  is a root of  $\text{tr } M_*(\theta) = 2$ . The stability interval is smaller than that of the symplectic Euler method (Fig. 2).

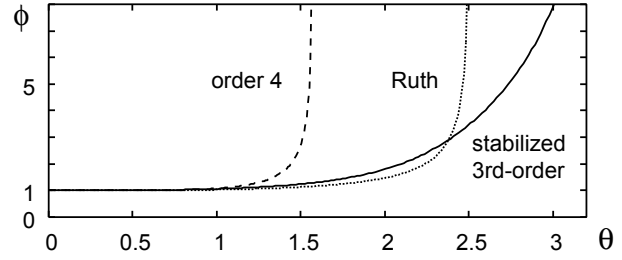


Fig. 2. Functions  $\phi$  for the three symplectic methods.

## V. NUMERICAL ILLUSTRATION

To test our numerical method, we consider the sine-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin u = 0. \quad (37)$$

This equation has the solitary wave solution (see, e.g., [10], chapter 17).

$$u(t, x) = \arctan\left(\exp \frac{x - \gamma t}{\sqrt{1 - \gamma^2}}\right), \quad (38)$$

where  $\gamma$  denotes a real number with  $|\gamma| < 1$ .

By introducing a new variable  $v = \partial u / \partial t$  and restricting the space variable  $x$  to  $-5 \leq x \leq 5$ , we get the problem

$$\begin{cases} \frac{\partial u}{\partial t} = v, & \frac{\partial v}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \sin u \quad (-5 \leq x \leq 5) \\ u(t, -5) = \varphi_0(t), & u(t, 5) = \varphi_1(t), \end{cases} \quad (39)$$

where  $\varphi_0(t)$  and  $\varphi_1(t)$  are given so that (38) satisfies (39). Moreover, we apply the method of lines approximation to problem (39) by using a mesh of the form  $x_j = -5 + j\Delta x$ ,  $j = 0, 1, \dots, M$ ,  $\Delta x = 10/M$ , where  $M$  denotes a positive integer. As usual, we denote approximate functions to  $u(t, x_j)$  and  $v(t, x_j)$  by  $u_j(t)$  and  $v_j(t)$ , respectively. By approximating  $\partial^2 u / \partial x^2$  with the standard central difference scheme, we get a Hamiltonian equation

$$\frac{dq}{dt} = p, \quad \frac{dp}{dt} = L_{\Delta x} q + g_0(t, q), \quad (40)$$

where  $q(t) = [u_1(t), u_2(t), \dots, u_{M-1}(t)]^T$ ,  $p(t) = [v_1(t), v_2(t), \dots, v_{M-1}(t)]^T$ ,

$$L_{\Delta x} = \frac{1}{\Delta x^2} \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -2 \end{bmatrix}. \quad (41)$$

$$g_0(t, q) = \frac{1}{\Delta x^2} [\varphi_0(t), \dots, \varphi_1(t)]^T - [\sin u_1, \sin u_2, \dots, \sin u_{M-1}]^T. \quad (42)$$

The matrix  $L_{\Delta x}$  has eigenvalues represented as

$$-\omega_j^2, \quad \omega_j = \frac{2}{\Delta x} \sin\left(\frac{j\pi}{2M}\right), \quad 1 \leq j \leq M-1. \quad (43)$$

By using a linear transform, we change the linear part of (40) into equations of the form

$$\frac{d\tilde{q}_j}{dt} = \omega_j \tilde{p}_j, \quad \frac{d\tilde{p}_j}{dt} = -\omega_j \tilde{q}_j, \quad 1 \leq j \leq M-1. \quad (44)$$

Since  $\omega_{M-1}$  is the largest among  $\omega_j$ 's, a symplectic method is stable for the linear part of (40) if  $\omega_{M-1}\Delta t$  is included in the interior of the stability interval. Denoting the stability interval by  $[0, \theta_0]$ , we express this condition as

$$\Delta t < \frac{\theta_0 \Delta x}{2 \sin\left[\frac{(M-1)\pi}{2M}\right]}, \quad (45)$$

which gives, as  $M \rightarrow \infty$ , a CFL condition

$$\Delta t \leq \frac{\theta_0 \Delta x}{2}. \quad (46)$$

We now consider time step sizes of the form  $\Delta t = 10/N$ , where  $N$  is a positive integer, and assume  $3N = 2M$  for  $M$  and  $N$ . Then, since  $\Delta t/\Delta x = 3/2$ , among the specific symplectic methods in Sections 2 and 3, only the stabilized 3rd-order method satisfies the CFL condition (46).

Table 2 shows the errors

$$\varepsilon_M = \max\{\varepsilon_{n,j} : 1 \leq n \leq N, 1 \leq j \leq M-1\}, \quad \varepsilon_{n,j} = \left| (u_j)_n - u(t_n, x_j) \right|, \quad (47)$$

for  $M = 150, 300, 600, 1200$ , in the case  $\gamma = 1/2$ . It is observed that the numerical solution converges to the exact solution (38) with  $\mathcal{O}(\Delta x^2)$ . For this selection of  $\Delta x$  and  $\Delta t$ , the other methods bring no significant numerical results because of overflow.

**Table 2.** Numerical results by the stabilized 3rd-order method

$M$	150	300	600	1200
$\varepsilon_M$	$2.4 \times 10^{-3}$	$6.0 \times 10^{-4}$	$1.5 \times 10^{-4}$	$3.7 \times 10^{-5}$

## ACKNOWLEDGMENTS

The authors would like to thank Ms. Wakana Tamaru, a student of Nanzan University for her help with checking the mathematical expressions and numerical results presented in the paper.

## REFERENCES

- [1] K. Feng, "On the difference schemes and symplectic geometry," in *Proceedings of the 1984 Beijing Symposium on Differential Geometry and Differential Equations*, Beijing, China, pp. 42-58, 1985.
- [2] E. Hairer, S. P. Norsett, and G. Wanner, *Solving Ordinary Differential Equations I: Nonstiff Problems*, 2nd ed. Heidelberg, Germany: Springer-Verlag, 1993.
- [3] B. Leimkuhler and S. Reich, *Simulating Hamiltonian Dynamics*. Cambridge, MA: Cambridge University Press, 2004.
- [4] J. M. Sanz-Serna and M. P. Calvo, *Numerical Hamiltonian Problems*. London: Chapman & Hall, 1994.
- [5] E. Hairer, "Backward analysis of numerical integrators and symplectic methods," *Annals of Numerical Mathematics*, vol. 1, pp. 107-132, 1994.
- [6] H. Yoshida, "Recent progress in the theory and application of symplectic integrators," *Celestial Mechanics and Dynamical Astronomy*, vol. 56, no. 1-2, pp. 27-4, 1993.
- [7] F. Y. Liu, X. Wu, and B. K. Lu, "On the numerical stability of some symplectic integrators," *Chinese Astronomy and Astrophysics*, vol. 31, no. 2, pp. 172-186, 2007.
- [8] M. A. Lopez-Marcos, J. M. Sanz-Serna, and R. D. Skeel, "An explicit symplectic integrator with maximal stability interval," in *Numerical Analysis: A. R. Mitchell 75th Birthday Volume*, Singapore: World Scientific, pp.163-175, 1996.
- [9] D. Murai and T. Koto, "Stability and convergence of staggered Runge-Kutta schemes for semilinear wave equations," *Journal of Computational and Applied Mathematics*, vol. 235, no. 14, pp. 4251-4264, 2011.
- [10] G. B. Whitham, *Linear and Nonlinear Wave*. New York, NY: John Wiley & Sons, 1974.



**Toshiyuki Koto**

received his B.S. degree in 1984 and M.S. degree in 1986 from Department of Mathematics, the University of Tokyo. In 1992, he received his Ph.D. in Engineering from Nagoya University. From April 1986 to March 1991, he was with Fujitsu Limited; from April 1991 to March 2004, with the University of Electro-Communications; and from April 2004 to March 2009, with Nagoya University. Since April 2009, he has been a professor at Nanzan University. His research interests include numerical analysis and applied mathematics. He is a member of the Mathematical Society of Japan, the Society of Information Processing, and the Japan Society of Industrial and Applied Mathematics.



**Eun-Jee Song**

received her B.S. degree from the Department of Mathematics, Sookmyung Women's University, in 1984. She earned her M.S. and Ph.D. degrees from the Department of Information Engineering, Nagoya University, Japan in 1988 and 1991, respectively. She was an exchange professor at the Department of Computer Science, the University of Auckland, New Zealand, in 2007. She is currently a full and tenured professor of the Department of Computer Science, Namseoul University, Cheonan, Korea.