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A new class of bivariate distributions with exponential and gamma conditionals

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Abstract.A new class of bivariate distributions is derived by specifying its conditionals as the exponential and gamma distributions. Some properties and relations with other distributions of the new class are studied. In particular, the estimation of parameters is considered by the methods of maximum likelihood and pseudolikelihood of a special case of the new class. An application using a real bivariate data is given for illustrating the flexibility of the new class in this context, and, also, for comparing the estimation results obtained by the maximum likelihood and pseudolikelihood methods.

Key Words:*Bivariate distribution, conditional distribution,exponential distribution, functional equation, gamma distribution, likelihood function, Pseudolikelihood function*

1. INTRODUCTION

In the last thirty years of the twenty$^{\text{th}}$ century a big attention has been made to the problem of constructing new classes of bivariate distributions with given marginals. Four meetings on this topic are held by Pall'Aglio et al. (1991), Fisher and Sen (1994), Benes and Stepan (1996), and Cuadras et al. (2002). Arnold et al (1999) (p. 1) contend that it is often easier to visualize conditional densities or features of conditional densities than marginal or joint densities. Conditionally specified distributions turn out to provide convenient conjugate prior families in some multiparameter Bayesian setting. This perhaps un expected home for such distributions is explained by the fact that conditionally specified densities are "tailor-made" for Gibbs sampling simulation algorithms (Arnold et al. (2001)). The technique of constructing new classes of bivariate distributions by specifying its conditionals has thus an increasing interest from the researchers in this area

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(Arnold (1987), Arnold and Strauss (1988a), Arnold et al. (1993), Castillo and Galambos(1989), Johnson and Wichern (1999) and Ratnaparkhi (1981)). One of the earliest contributions in this context was the work of Patil (1965) on the power series distribution. A major breakthrough in which the important role of functional equations in proving the results in this connection, was brought into focus, was provided by Castillo and Galambos (1987a, 1987b). For a comprehensive survey of the subject see the monograph of Balakrishnan and Lai (2009).

In this paper a new class of bivariate distributions with exponential and gamma conditionals is derived. Such a class will be useful in engineering applications as a lifetime distribution from one side due to the fact that the analyst often has better insight in modelling conditional distributions, and from the other side due to the desirable characteristics of each of the exponential and gamma distributions as life time distributions in reliability (Lee and Wang (2003)).

An absolutely continuous random variable (RV) X is said to have a one parameter exponential distribution if its density function is:

$$f(x) = \lambda \exp(-\lambda x), x > 0, \lambda > 0. \tag{1}$$

This will be denoted by $X \sim \mathrm{Exp}(\lambda)$. Also, by $Y \sim \mathrm{Ga}(\alpha, \beta)$ we mean that a RV Y has a two parameters gamma distribution with density function:

$$g(y) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} y^{\alpha-1} \exp(-\beta y), y > 0, \alpha > 0, \beta > 0. \tag{2}$$

2. THE BIVARIATE EXPONENTIAL AND GAMMA CONDITIONALS

Suppose that (X, Y) is a pair of RV's whose joint density $f_{X,Y}(x, y)$ exists and is positive over the first quadrant of R^2 and 0 elsewhere. To characterize the wider class of bivariate exponential and gamma distribution suppose that $Y \mid X = x$ having an exponential distribution for all $x > 0$, and $X \mid Y = y$ having a gamma distribution for all $y > 0$ with respective densities:

$$f_{Y|X}(y \mid x) = \lambda(x) \exp(-\lambda(x) y), y > 0, \tag{3}$$

and

$$f_{X|Y}(x \mid y) = \frac{\beta(y)^{\alpha(y)}}{\Gamma(\alpha(y))} x^{\alpha(y)-1} \exp(-\beta(y) x), x > 0, \tag{4}$$

where$\lambda(x)$, $\alpha(y)$ and $\beta(y)$ are some positive functions.

Denote the corresponding marginal densities of X and Y by $f_X(x)$ and $f_Y(y)$, respectively, where $f_X(x) > 0$ for all $x > 0$ and $f_Y(y) > 0$ for all $y > 0$. The general class of bivariate distributions with conditionals given by (3) and (4) is described by the following theorem.

THEOREM 1. The general class of bivariate distributions with conditional distributions as (3) and (4) is given by:

$$f_{X,Y}(x, y) = (x)^{-1} \exp\left(a_{11} + a_{12} y - (a_{31} + a_{32} y) x + (a_{21} + a_{22} y) \log x\right),$$
$$x > 0, y > 0,\ a_{12} \le 0, a_{31},\ a_{21} > 0,\ a_{32} \ge 0, a_{22} \ge 0, \tag{5}$$

where $\{a_{ij}\}$, $i = 1, 2, 3$, $j = 1, 2$ are the parameters, with a_{11} as the normalizing constant and is such that:

$$a_{11} = -\log\left\{\int_0^\infty \exp(a_{12}y)\,(a_{31}+a_{32}y)^{-a_{21}-a_{22}y}\Gamma(a_{21}+a_{22}y)dy\right\}.$$

The parameters $a_{12} \le 0, a_{31} > 0, a_{32} \ge 0, a_{21} > 0$ and $a_{22} \ge 0$ must be selected to satisfy

$$\int_0^\infty f_X(x)dx = 1, \int_0^\infty f_Y(y)dy = 1.$$

Proof. According to (3) and (4) we can write the joint density $f_{X,Y}(x,y)$ as a product of a marginal and a conditional density in both ways to get

$$f_Y(y)\frac{(\beta(y))^{\alpha(y)}}{\Gamma(\alpha(y))}x^{\alpha(y)}\exp(-\beta(y)x) = x\, f_X(x)\lambda(x)\exp(-\lambda(x)y), \tag{6}$$

Denoting

$$n(y) = \log\left(f_Y(y)\frac{(\beta(y))^{\alpha(y)}}{\Gamma(\alpha(y))}\right), \tag{7}$$

and

$$m(x) = \log(x\, f_X(x)\, \lambda(x)). \tag{8}$$

Eq. (6) readily reduces to

$$n(y) + \alpha(y)\log(x) - \beta(y)x = m(x) - \lambda(x)y. \tag{9}$$

The general solution of (9) is such that (see the Appendix for the details)

$$\begin{aligned}
n(y) &= a_{11} + a_{12}y,\\
\alpha(y) &= a_{21} + a_{22}y,\\
\beta(y) &= a_{31} + a_{32}y,\\
m(x) &= a_{11} + a_{21}\log(x) - a_{31}x,\\
\lambda(x) &= -a_{12} - a_{22}\log(x) + a_{32}x.
\end{aligned}$$

Substituting these expressions in (7) and (8), we can get the marginal densities as:

$$f_X(x) = -\frac{x^{a_{21}-1}\exp(a_{11}-a_{31}x)}{(a_{12}+a_{22}\log(x)-a_{32}x)},$$

$$x > 0, a_{12} \le 0, a_{31} > 0, a_{32} \ge 0, a_{21} > 0, a_{22} \ge 0, \tag{10}$$

$$f_Y(y) = \frac{\Gamma(a_{21}+a_{22}y)}{(a_{31}+a_{32}y)^{a_{21}+a_{22}y}}\exp(a_{11}+a_{12}y),$$

$$y > 0, a_{12} \le 0, a_{31} > 0, a_{32} \ge 0,\ a_{21} > 0, a_{22} \ge 0. \tag{11}$$

Finally in accordance with (10) and (11) the class of bivariate distributions with exponential and gamma conditionals is that given by (5).

Remark.

For $a_{22} = a_{32} = 0$ the marginal distribution of X and Y will, respectively, be $Ga(a_{21}, a_{31})$ and $Exp(a_{12})$. i.e. in this special case both the marginal and conditional distributions for X and Y are of the same type.

3. PROPERTIES OF THE NEW CLASS

In this section, the basic properties of the new bivariate class (5) will be discussed.We first express that the class (5) has the five parameters $a_{12}, a_{21}, a_{22}, a_{31}$ and a_{32}. The specific forms of the conditional distributions for the new class are:

$$f_{Y|X}(y \mid x) = \left(a_{32}x - a_{12} - a_{22}\log(x)\right)\exp\left[\left(a_{12} - a_{32}x + a_{22}\log(x)\right)y\right],$$
$$x > 0, y > 0, a_{12} \le 0, a_{32} \ge 0, a_{22} \ge 0, \quad (12)$$

$$f_{X|Y}(x \mid y) = \frac{(a_{31} + a_{32}y)^{a_{21}+a_{22}y}}{\Gamma(a_{21} + a_{22}y)} x^{a_{21}+a_{22}y-1} \exp\left(-(a_{31} + a_{32}y)x\right),$$
$$x > 0, y > 0, a_{31} > 0, a_{32} \ge 0, a_{21} > 0, a_{22} \ge 0. \quad (13)$$

i.e.

$$Y \mid X = x \sim Exp(a_{32}x - a_{12} - a_{22}\log(x)),$$
$$X \mid Y = y \sim Ga(a_{21} + a_{22}y, a_{31} + a_{32}y).$$

The conditional distributions given by (12) and (13) satisfy the compatibility theorem (Arnold et al. (1999), sec. 1.6) for the existence of the new class of bivariate distributions (5).

The regression functions corresponding to (12) and (13) are:

$$E(X \mid Y = y) = \frac{(a_{21} + a_{22}y)}{(a_{31} + a_{32}y)}, y > 0, a_{31} > 0, a_{32} \ge 0, a_{21} > 0, a_{22} \ge 0,$$

$$E(Y \mid X = x) = \frac{1}{a_{32}x - a_{12} - a_{22}\log(x)}, x > 0, a_{12} \le 0, a_{32} \ge 0, a_{22} \ge 0.$$

We note that these regression functions are nonlinear. Further $E(Y \mid X = x)$ is increasing if $x < \frac{a_{22}}{a_{32}}$ and decreasing if $x > \frac{a_{22}}{a_{32}}$ (fig. 1), while $E(X \mid Y = y)$ is decreasing for all y (fig. 2).

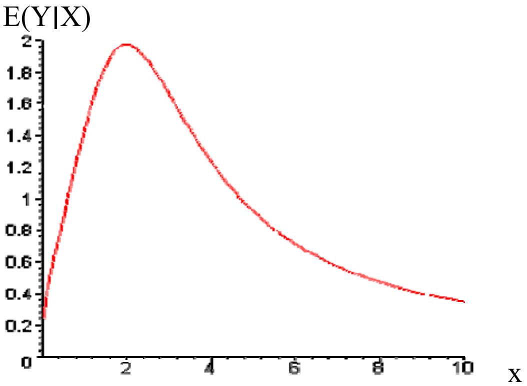

Figure. 1 The regression curve of y on x of class (5).

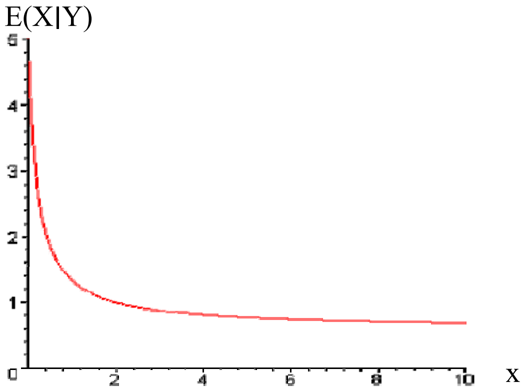

Figure. 2 The regression curve of x on y of class (5).

The following theorem expresses the independence between the two RV's X and Y.

THEOREM 2. For the class (5), the two RV's X and Y are independent iff $a_{22} = 0, a_{32} = 0$.

PROOF. Necessity:Putting $a_{22} = 0, a_{32} = 0$ in (5) we get, according to (10) and (11), that$f_{X,Y}(x, y) = f_X(x)f_Y(y)$.Hence X and Y are independent.
Sufficiency:Assume that X and Y are independent then, we must have

$$f_{X,Y}(x, y) = f_X(x)f_Y(y).$$

Now, the validity of this equality implies that $a_{22} = 0, a_{32} = 0$.

Local measures of dependence

To identify the dependence type of the two RV's X and Y, having joint density (5) we will use the local measure of dependence defined by (Holland and Wang (1987)).

$$\gamma_f(x, y) = \frac{\partial^2}{\partial x\, \partial y} \ln f_{x|y}\,(x \mid y), \tag{14}$$

$f_{X,Y}(x, y)$ is said to be totally positive of order two (TP2) (totally negative of order two (TN2)) iff$\gamma_f(x, y) > 0$ (< 0) (Holland and Wang (1987)).

THEOREM 3. The joint density function (5) is TP2 (TN2) iff$x < \frac{a_{22}}{a_{32}}\left(x > \frac{a_{22}}{a_{32}}\right)$.

Proof. From (5) and (14) we have

$$\gamma_f(x, y) = \left(\frac{a_{22} - a_{32}x}{x}\right).$$

Now $\gamma_f(x, y) > 0$, if $x < \frac{a_{22}}{a_{32}}$, and hence (5) is TP2.

Also $\gamma_f(x, y) > 0$, if $x > \frac{a_{22}}{a_{32}}$, and hence (5) is TN2.

Remarks.

(i) It follows from theorem 3 that if the density (5) is TP2 (TN2) then the two RV's X and Y are positively (negatively) correlated.

(ii) It is natural to find that the conditional densities (12) and (13) have the same local dependence functions, as (5). i. e.,

$$\gamma_{f_{X|Y}}(x, y) = \gamma_{f_{Y|X}}(x, y) = \gamma_f(x, y).$$

(iii) $\gamma_f(x, y) > 0$,iff$a_{22} = a_{32} = 0$ then X and Y are independent as given by Theorem 2.

Relations to other distributions

Rewrite the joint density function (5) as:

$$f_{X,Y}(x, y) = (x)^{a_{21}-1}\exp(a_{11} + a_{12}y - (a_{31} + a_{32}y)x + (a_{22}y)\log(x).$$

1)Putting $a_{21} = 1$and $a_{22} = 0$, then $f_{X,Y}(x, y)$ reduces to:

$$f_{X,Y}(x, y) = \exp(a_{11} + a_{12}y - a_{31}x - a_{32}xy), x > 0, y > 0,$$

which is the bivariate distribution with exponential conditionals obtained by Arnold and Strauss (1988a).
2)Now consider (5) putting $a_{21} = 1, a_{32} = a_{22} = 0$ and renaming $a_{12} = -\lambda_1, a_{31} = \lambda_2, a_{11} = \log(\lambda_1\lambda_2)$ we get the bivariate exponential distribution with independent components as follows:

$$f_{X,Y}(x,y) = \lambda_1\lambda_2\exp(-(\lambda_1 y + \lambda_2 x).$$

3) The bivariate gamma exponential distribution obtained by Nadarajah (2007) can be obtained from (5) by choosing $a_{21} = 1, a_{12} = 0, a_{22} = 0$.

Two subclasses of the new class (5)
The new class (5) can be classified by suitable selections for the parameters $a_{12}, a_{21}, a_{22}, a_{31}$and a_{32} into the following two subclasses:

Subclass I: (The class has four parameters only).
By choosing $a_{22} = 0$, (5) reduces to

$$f_{X,Y}(x,y) = (x)^{a_{21}-1}\exp(a_{11} + a_{12}y - (a_{31} + a_{32}y)x),$$
$$x > 0, y > 0, a_{12} \le 0, a_{31} > 0, a_{32} \ge 0, a_{21} > 0. \quad (15)$$

The conditional densities of (15) are given by:

$$Y \mid X = x \sim \mathrm{Exp}(a_{32}x - a_{12}),$$

and

$$X \mid Y = y \sim \mathrm{Ga}(a_{21}, a_{31} + a_{32}y).$$

The regression functions for these conditional distributions are respectively:

$$E(X \mid Y = y) = \frac{a_{21}}{a_{31} + a_{32}y}, y > 0, a_{31} > 0, a_{32} \ge 0, a_{21} > 0,$$
$$E(Y \mid X = x) = \frac{1}{a_{32}x - a_{12}}, x > 0, a_{12} \le 0, a_{32} \ge 0.$$

These two regression functions are nonlinear and decreasing for all $y > 0$ and $x > 0$ with curves both similar that given by fig. 2.

Subclass II: (The class has three parameters only).
By choosing $a_{32} = a_{22} = 0$, (5) reduces to

$$f_{\{X,Y\}(x,y)} = (x)^{a_{21}-1}\exp(a_{11} + a_{12}y - a_{31}x),$$
$$x, y > 0,\ a_{12} \le 0, a_{31} > 0, a_{21} > 0. \quad (16)$$

This class is, also, a subclass of subclass I (the case $a_{32} = 0$).
The marginal distributions are:

$$f_X(x) = -\frac{x^{a_{21}-1}\exp(a_{11} - a_{31}x)}{a_{12}}, x > 0, a_{12} \le 0, a_{31} > 0, a_{21} > 0,$$
$$f_Y(y) = \frac{\Gamma(a_{21})}{(a_{31})^{a_{21}}}\exp(a_{11} + a_{12}y), y > 0, a_{12} \le 0, a_{31} > 0,\ a_{21} > 0.$$

In the following section some additional detailed analysis of subclass I is considered.

4. ANALYSIS OF SUBCLASS I OF EXPONENTIAL AND GAMMA CONDITIONALS

Renaming the parameters of the joint density (15) of subclass I can be written as:

$$f_{X,Y}(x,y) = \{N(\theta)\}^{-1}(x)^{\theta_4-1}\exp(\theta_1 y - (\theta_2+\theta_3 y)x),$$
$$x,y>0,\theta_1\le 0,\theta_2>0,\theta_3\ge 0,\theta_4>0, \quad (17)$$

with $\theta_1,\theta_2,\theta_4$ are scale parameters, and θ_3 is the dependence parameter, ($\theta_3=0$ corresponds to the independence case), and

$$N(\theta) = \Gamma(\theta_4)\int_0^\infty \exp(\theta_1 y)(\theta_2+\theta_3 y)^{-\theta_4}dy = \frac{\Gamma(\theta_4)}{\theta_3\theta_2^{\theta_4-1}}\exp\left(-\frac{\theta_1\theta_2}{\theta_3}\right)E\left[\theta_4,-\left(\frac{\theta_1\theta_2}{\theta_3}\right)\right], \quad (18)$$

is the normalizing constant, where $E(n,z)$ is the exponential integral function, $E(n,z)=\int_1^\infty \frac{\exp(-zt)}{t^n}dt$.

The conditional densities are:

$$X \mid Y = y \sim Ga(\theta_4,\theta_2+\theta_3 y), \quad (19)$$
$$Y \mid X = x \sim Exp(\theta_3 x - \theta_1). \quad (20)$$

The product moments for the subclass I can be obtained, using (4.11), as:

$$E(X^mY^n) = \frac{\{N(\theta)\}^{-1}}{\theta_3}(-\theta_1)^{-1-n}\theta_2^{-\theta_4-m}\pi\csc[(\theta_4+m-n)\pi]\left\{\theta_3\left(-\frac{\theta_1\theta_2}{\theta_3}\right)^{\theta_4+m}\Gamma(\theta_4+m)U_{mn}+\theta_1\theta_2\left(-\frac{\theta_1\theta_2}{\theta_3}\right)^n\Gamma(1+n)V_{mn}\right\},\quad m,n=0,1,2,\dots,$$

where

$$U_{mn} = \frac{{}_1F_1\left(\theta_4+m,\theta_4+m-n,-\frac{\theta_1\theta_2}{\theta_3}\right)}{\Gamma(\theta_4+m-n)},$$
$$V_{mn} = \frac{{}_1F_1\left(1+n,2-\theta_4-m+n,-\frac{\theta_1\theta_2}{\theta_3}\right)}{\Gamma(2-\theta_4-m+n)},$$

and ${}_1F_1$ is the confluent hypergeometric function and $\csc(z)$ is the cosecant of z.

Now, the marginal densities corresponding to (17) are:

$$f_X(x) = \frac{\{N(\theta)\}^{-1}x^{\theta_4-1}}{\theta_3 x-\theta_1}\exp(-\theta_2 x),\ x>0,\theta_3\le 0,\theta_2>0,\theta_3\ge 0,\theta_4>0,$$
$$f_Y(y) = \frac{\{N(\theta)\}^{-1}\Gamma(\theta_4)}{(\theta_2+\theta_3 y)^{\theta_4}}\exp(\theta_1 y),\ y>0,\theta_1\le 0,\theta_2>0,\theta_3\ge 0,\theta_4>0.$$

Hence, using these marginals, we have:

$$E(X^m) = \frac{\{N(\theta)\}^{-1}}{\theta_3}\left(-\frac{\theta_2^{-\theta_4-m}}{\theta_1}\right)^{-1}\pi\csc[(\theta_4+m)\pi]\left\{\theta_3\left(-\frac{\theta_1\theta_2}{\theta_3}\right)^{\theta_4+m}\Gamma(\theta_4+m)U_{m0}+\theta_1\theta_2V_{m0}\right\},\ m=1,2,\dots,$$

and

$$E(Y^n) = \frac{\{N(\theta)\}^{-1}}{\theta_3}(-\theta_1)^{-1-n}\theta_2^{-\theta_4}\pi\csc[(\theta_4-n)\pi]$$

$$\left\{\theta_3\left(-\frac{\theta_1\theta_2}{\theta_3}\right)^{\theta_4}\Gamma(\theta_4)U_{0n}+\theta_1\theta_2\left(-\frac{\theta_1\theta_2}{\theta_3}\right)^{n}\Gamma(1+n)V_{0n}\right\}, n=1,2,\ldots.$$

Also the conditional moments can be easily obtained using (19) and (20) as:

$$E(X^k \mid Y=y)=\frac{\Gamma(k+\theta_4)}{\Gamma(\theta_4)}(\theta_2+\theta_3 y)^{-k}, k=1,2,\ldots,$$

and

$$E\left(Y^k \mid X=x\right)=\Gamma(1+k)(\theta_3 x-\theta_1)^{-k}, k=1,2,\ldots.$$

We notice that the conditional moments are rational functions of the conditioned variable.

4.1. The maximum likelihood estimation (MLE) for theparameters of subclass I

Suppose that we have n observations $(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$ which represent a random sample of size n from the bivariate class (17).
Hence, the log-likelihood function $l(\theta)$ is given by:

$$l(\theta)=-n\log\big(N(\theta)\big)-\sum_{i=1}^{n}\log(x_i)^{(\theta_4-1)}+\theta_1\sum_{i=1}^{n}y_i-\sum_{i=1}^{n}(\theta_2+\theta_3 y_i)\,x_i.$$

Differentiating with respect to the parameters $\theta_1,\theta_2,\theta_3$ and θ_4 and then setting the partial derivatives equal to zero, we obtain:

$$\frac{\frac{\partial N(\theta)}{\partial\theta_1}}{N(\theta)}=\frac{1}{n}\sum_{i=1}^{n}y_i, \tag{21}$$

$$\frac{\frac{\partial N(\theta)}{\partial\theta_2}}{N(\theta)}=-\frac{1}{n}\sum_{i=1}^{n}x_i, \tag{22}$$

$$\frac{\frac{\partial N(\theta)}{\partial\theta_3}}{N(\theta)}=-\frac{1}{n}\sum_{i=1}^{n}x_i\,y_i, \tag{23}$$

$$\frac{\frac{\partial N(\theta)}{\partial\theta_4}}{N(\theta)}=\frac{1}{n}\sum_{i=1}^{n}\log(x_i), \tag{24}$$

where $N(\theta)$ is given by (18). The implicit nature of the system (21)-(24) suggests the numerical derivation of the MLE of the parameters $\theta_1,\theta_2,\theta_3$and θ_4. Also the evaluation of the integral $E(n,z)$ imposes the complicated nature of the normalizing constant $N(\theta)$.
A possible approach for solving the system (21)-(24) using numerical solution involves selecting, for example, initial values for $\theta_1,\theta_2,\theta_3$ then searching for a value for θ_4 to make (24) holds. Then this value of θ_4 with the previous values of θ_2,θ_3 are used to search a value for θ_1 that make (21) holds, and similarly for θ_2 and θ_3.

4.2. Pseudolikelihood estimation for the parameters of subclass I

The complicated nature of the normalizing constant $N(\theta)$ appears had in (18) motivated us to seek alternatives to the maximum likelihood estimation technique such as the maximum pseudolikelihood estimation (MPLE), introduced by Besag [(1975), (1977)] and Arnold & Strauss (1988b).

The (MPLE) scheme for bivariate distributions is based on the maximization of the product of the corresponding conditional densities and hence the problem is tackled without call to the normalizing constant as it is clear from (19) and (20). Therefore the pseudolikelihood function PL(θ) is given after taking natural logs by:

$$\log PL(\theta) = \sum_{i=1}^{n} \log(\theta_3 x_i - \theta_1) - \sum_{i=1}^{n} (\theta_3 x_i - \theta_1) y_i - n \log\left(\Gamma(\theta^4)\right)$$
$$+(\theta_4 - 1)\sum_{i=1}^{n} \log(x_i) + \theta_4 \sum_{i=1}^{n} \log(\theta_2 + \theta_3 y_i) - \sum_{i=1}^{n} (\theta_2 + \theta_3 y_i)\, x_i.$$

Differentiating logPL(θ) with respect to $\theta_1, \theta_2, \theta_3$, and θ_4 and setting the partial derivatives equal to zero, we get the pseudolikelihood equations as:

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \frac{1}{(\theta_3 x_i - \theta_1)}, \tag{25}$$

$$\sum_{i=1}^{n} x_i = \theta_4 \sum_{i=1}^{n} \frac{1}{(\theta_2 + \theta_3 y_i)}, \tag{26}$$

$$2\sum_{i=1}^{n} x_i\, y_i = \sum_{i=1}^{n} \frac{x_i}{(\theta_3 x_i - \theta_1)} + \theta_4 \sum_{i=1}^{n} \frac{y_i}{(\theta_2 + \theta_3 y_i)}, \tag{27}$$

$$\sum_{i=1}^{n} \log(x_i) = n\frac{\partial}{\partial \theta_4} \log\left(\Gamma(\theta_4)\right) - \sum_{i=1}^{n} \log(\theta_2 + \theta_3 y_i). \tag{28}$$

The system (25)-(28) can be solved by iteration. In the following section we use a set of real data to compare the maximum likelihood and the maximum pseudolikelihood schemes for estimating the parameters of the bivariate class (17).

5. APPLICATION

The following data has been obtained from Johnson and Wichern (1999) (p. 374). It represents the bone mineral density (BMD) measured in gm/cm² for 24 children after one year of birth. This bivariate data represent the BMD for Dominant Ulna (X) and Ulna bones (Y) and they are as follows:

(0.869,0.964), (0.602,0.689), (0.765,0.738), (0.761,0.698), (0.551,0.619), (0.753,0.515), (0.708,0.787), (0.687,0.715), (0.844,0.656), (0.869,0.789), (0.654,0.726), (0.692,0.526), (0.670,0.580), (0.823,0.773), (0.746,0.729), (0.656,0.506), (0.693,0.740), (0.883,0.785), (0.577,0.627),(0.802,0.769), (0.540,0.498), (0.804,0.779),(0.570,0,634), (0.585,0.640).

The complete sufficient statistics for the given data are:

$$\frac{1}{24}\sum_{i=1}^{n} y_i = 0.68675, \quad \frac{1}{24}\sum_{i=1}^{n} x_i = 0.712667,$$

$$\frac{1}{24}\sum_{i=1}^{n} x_i\, y_i = 0.496653, \qquad \frac{1}{24}\sum_{i=1}^{n} \log\,(x_i) = -0.34985.$$

These numerical results are used to obtain each of MLE using Eqs. (21)-(24), and the MPLE using Eqs. (25)-(28).
Table 1 contains the true (initial) values of $\theta_i (i = 1,2,3,4)$, their estimators and their mean squared errors.

Table 1. Estimation of parameters of the bivariate class (17).

true values	MLE	MSE	MPLE	MSE
θ_1=-0.5	-1.1856	0.4700	-1.1723	0.4519
θ_2=1	1.1661	0.0276	1.1298	0.0168
θ_3=0.4	0.2430	0.0246	0.3614	0.0015
θ_4=1	2.3397	1.7948	1.4315	0.1862

6. CONCLUSION

In this paper, a bivariate Exponential and Gamma distribution is introduced by specifying its conditionals as the exponential and gamma distributions. In addition, the parameters of a special case of the new class via the method of MLE and MPLE are shown. Based on the results, table 1 the MPLE is generally better than the results of the MLE because the MPLE technique uses the conditional distributions which in our case does not suffer from the problem caused by the normalizing constant of the class (17).

APPENDIX
(Solution of the Functional Equation)

In this appendix, we present the following theorem which provides solutions to the general functional equation (A.1). This theorem is due to Aczel(1996).

THEOREM. All solutions of the equation

$$\sum_{k=1}^{n} f_k(x)g_k(y) = 0, x \in S(X),\ \ y \in S(Y). \tag{A.1}$$

can be written in the form

$$\begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nr} \end{bmatrix} \begin{bmatrix} \Phi_1(x) \\ \Phi_2(x) \\ \vdots \\ \Phi_r(x) \end{bmatrix} \tag{A.2}$$

$$\begin{bmatrix} g_1(y) \\ g_2(y) \\ \vdots \\ g_n(y) \end{bmatrix} = \begin{bmatrix} b_{1r+1} & b_{1r+2} & \cdots & b_{1n} \\ b_{2r+1} & b_{2r+2} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{nr+1} & b_{nr+2} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} \Psi_{r+1}(y) \\ \Psi_{r+2}(y) \\ \vdots \\ \Psi_n(y) \end{bmatrix}$$

where$0 < r < n$ is an integer, and $\Phi_1(x), \Phi_2(x), \ldots, \Phi_r(x)$ on the one hand and $\Psi_{r+1}(y)$, $\Psi_{r+2}(y), \ldots, \Psi_n(y)$ on the other are arbitrary systems of mutually independent functions and the constants a_{ij} and b_{ij} satisfy

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1r} & a_{2r} & \cdots & a_{nr} \end{bmatrix} \begin{bmatrix} b_{1r+1} & b_{1r+2} & \cdots & b_{1n} \\ b_{2r+1} & b_{2r+2} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{nr+1} & b_{nr+2} & \cdots & b_{nn} \end{bmatrix} = 0. \qquad (A.3)$$

Applying this theorem to solve (2.7), we get (using (A.1) and (A.2)),

$$\begin{bmatrix} n(y) \\ \alpha(y) \\ \beta(y) \\ 1 \\ y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix}, \qquad (A.4)$$

and, $$\begin{bmatrix} 1 \\ \log(x) \\ -x \\ -m(x) \\ \lambda(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \begin{bmatrix} 1 \\ \log(x) \\ x \end{bmatrix}. \qquad (A.5)$$

Where

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} & 1 & 0 \\ a_{12} & a_{22} & a_{32} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = 0.$$

Yielding

$a_{11} + b_{11} = 0, \quad a_{21} + b_{12} = 0, \quad -a_{31} + b_{13} = 0,$

$a_{12} + b_{21} = 0, \quad a_{22} + b_{22} = 0, \quad -a_{32} + b_{23} = 0.$

These relations, along with the consideration of (A.4) and (A.5), lead to

$n(y) = a_{11} + a_{12}y, \quad \alpha(y) = a_{21} + a_{22}y, \quad \beta(y) = a_{31} + a_{32}y,$

$m(x) = a_{11} + a_{21}\log(x) - a_{31}x, \quad \lambda(x) = -a_{12} - a_{22}\log(x) + a_{32}x.$

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