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Necessity of step-stress accelerated life testing experiment at higher steps

N. Chandra* **and Mashroor Ahmad Khan**
Department of Statistics, Pondicherry University,
Ramanujan School of Mathematical Sciences, Pondicherry-605 014, India

M. Pandey
Department of Zoology and DST Center of Interdisciplinary Mathematical Sciences,
Banaras Hindu University, Varanasi 221 005, India

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Abstract. Accelerated life testing (ALT) is a well famous technique in life testing and reliability studies, this is particularly used to induce so high stress leading to failure of the highly reliable units quickly under stipulated duration of time. The step-stress ALT is one of the systematic experimental strategy of ALT applied to fail the units in steps. In this article we focus on two important issues (i) necessity of life tests at higher steps with relevant causes (ii) to develop a new optimum test plan for 3-step SSALT under the modified cumulative exposure model proposed by Khamis and Higgins (1998). It is assumed that the lifetime of test units follows Rayleigh distribution and its scale parameter at constant stress level is assumed to be a log-linear function of the stress. The maximum likelihood estimates of the parameters involved in the step-stress ALT model are obtained. A simulation study is performed for numerical investigation of the proposed new optimum plan 3-step, step-stress ALT. The necessity of the life test units at 3-step step-stress is also numerically examined in comparison to simple step-stress setup.

Key Words: *Accelerated life testing, compound linear plan, cumulative exposure model, fisher information matrix, maximum likelihood estimate, Rayleigh distribution*

1. INTRODUCTION

In present era of global markets have immense pressure on manufacturers to develop new, highly sophisticated, higher technology products in record time, while improving productivity, product field reliability, and overall quality. However, in order to guarantee

*Corresponding Author.
E-mail address: *nc.stat@gmail.com*

such service life and performance of the products, life-testing under normal operating conditions is obviously most desirable but, in using standard life-testing, one may often experience difficulty in obtaining the sufficient information about the failure time of the products. To overcome such problems, ALT is preferable. It is popular to obtain failure time data more quickly of highly reliable product at a higher than usual level of stress (e.g., temperature, vibration, voltage, pressure, humidity, cycling rate etc.). The failure time data obtained at accelerated conditions are analyzed by selecting a physical model relating to the lifetime and stress; and extrapolated to the design stress to estimate the lifetime distribution. The most commonly used model to analyze failure time data is cumulative exposure model introduced by Sedyakin (1966) and discussed further by Bagdonavicius (1978) and Nelson (1980, 1990). The cumulative exposure model given by Nelson (1990) for *m*-stress level is defined as follows:

$$G(t) = \begin{cases} G(t;\theta_1), & 0 \le t < \tau_1, \\ G(t-\tau_1+s_1;\theta_2), & \tau_1 \le t < \tau_2, \\ \quad \cdots & \\ G(t-\tau_{m-1}+s_{m-1};\theta_m) & \tau_{m-1} \le t < \infty, \end{cases} \tag{1}$$

where s_i is the solution of

$$G(s_i;\theta_{i+1}) = G(\tau_i - \tau_{i-1} + s_{i-1};\theta_i), \quad i=1, 2, ..., m-1 \tag{2}$$

The step-stress model is a special class of ALT; which allows the experimenters to apply the stresses in such a way that the stress will be changed at pre-specified time. Generally, a test unit starts at a specified low stress. If the unit does not fail at a specified time, stress on it is raised and held for a specified time. Further, stress is repeatedly increased until the entire test units fail(s) or the censoring time is reached. Miller and Nelson (1983), Bai et al. (1989), Khamis and Higgins (1996, 1998), Kateri and Balakrishnan (2008), Xiong and Milliken (1999), Balakrishnan et al. (2007, 2009b) are the key references in area of step-stress ALT.

Most of the available literature on a step-stress ALT deal with two types of the problems: Data analysis and Test design. The data analysis problems give more attention on modeling data obtained from SSALT, and deriving life performance under typical conditions. A plenty of problems on data analysis studies by several authors, viz., Xiong (1998) and Xiong and Milliken (1999) discussed inference for the exponential step-stress model under type-II censored data by assuming that the mean lifetime of the experimental units is a log-linear function of the stress level. Balakrishnan et al. (2009b) and Balakrishnan and Xie (2007a, 2007b) have all developed exact inferential procedures through the use of conditional moment generating functions for the simple step-stress exponential model under different forms of the censoring. Gouno et al. (2004), Han et al. (2006), Balakrishnan and Han (2009a), Xie et al. (2008), and Fan et al. (2008) have all discussed inferential methods for step-stress ALT under the exponential distribution for progressively censored data.

The test design problem is developed for the determination of the optimal duration of stress changing to achieve certain objectives, such as obtaining the estimate of the parameters of interest and reliability. The development of test design under step-stress ALT model has been attempted by many researchers and they commonly used to design an optimal plan by minimizing the asymptotic variance of the maximum likelihood estimator of the log mean life or some percentile of life at a specified stress level. Miller nd Nelson (1983) obtained the optimum simple step-stress accelerated life test plans when the test units have exponentially distributed life times. Bai et al. (1989) extended the Miller and Nelson (1983) work for censored data and they used the nomographs techniques for obtaining the optimum times for time-step stress test. Khamis and Higgins (1996) derived the optimum 3-step SSALT plan by considering the existence of quadratic stress–life relation. Khamis (1997) generalized the optimum plans for m-step step-stress ALT design with k stress variables by assuming complete knowledge of the stress–life relationship with multiple stress variables. These studies were based on the assumption that the failure time follows exponential distribution because of its simplicity. Khamis and Higgins (1998) proposed a new model known as KH model for step-stress ALT as an alternative to the Weibull cumulative exposure model. Alhadeed and Yang (2002) obtained the optimal design for the simple step-stress ALT using the Khamis–Higgins (K–H) model. They assumed constant shape parameter and a log-linear life-stress relationship between the scale parameter and the stress. Hunt and Xu (2012) derived the optimum stress-changing time for generalized K-H model by assuming that the lifetime of a test unit follows a Weibull distribution, and both its shape and scale parameters are functions of the stress level, for type-I censored data. Chandra and Khan (2013) derived an optimum simple step-stress ALT plan for Lomax distribution when the available data are Type-I censored. Some more references are (Al-Haj Ebrahem and Al-Masri (2007), Srivastava and Shukla (2008), Fard and Li (2009), Hassan and Al-Ghamdi (2009), Chandra and Khan (2012)). Recently, Chandra, Khan and Pandey (2014) studied a problem for 3-step, SSALT by assuming the existence of linear as well as quadratic relationship between the log mean failure time and stress, and also they developed an optimum plan.

In view of the proposed aim of this article, some researchers (see, Lin et al. (2013), Shen et al. (2011), Guan and Tang (2012)) have developed the optimum test plans for m-step step-stress ALT but they attempted numerical investigations for validation of derived plan for $m \geq 3$ with the assumption that changing stress level time from one to another steps are equal i.e., it is reduces to exactly optimum simple step stress ALT problem (Lin et al. (2013)), even, few authors viz., Gouno et al. (2004), Balakrishnan and Han (2009a) and Wu et al. (2006, 2008) derived the expressions for optimum test plan with equal case. It means that the proposed test plans is dependent on only two extreme stresses (X_1, X_m) that is infeasible because there is loss of information. Therefore, Khamis and Higgins (1996) introduced compound linear plan to come out from this problem.

The main purpose of this article is to rectifying the possible causes which insist to perform the life tests experiment of highly reliable equipments / units at three or more steps. It is assumed that sufficient amount of failure time data of such units can be obtained at three or more steps used. The life testing experiment is required up to three or more steps, step-stress ALT taking in to considerations the following situations.

(i) The units are highly reliable in nature and;
(ii) Optimum simple step stress ALT plan having some practical limitations; highly depends on linear relationship between stress and time-to-failure; and use only two extreme stresses that may cause the irrelevant failure of units (see Khamis and Higgins (1996)).

(iii) The amount of stresses used in the two steps may not be sufficient to make the units fail.

Further, we also developed an optimum plan for unequal 3-step step-stress ALT under modified cumulative exposure model proposed by Khamis and Higgins (1998) as a case study of necessity of life tests experiments at higher steps.
In the subsequent sections, the proposed model and assumptions are discussed in section 2. The MLE and the Fisher information matrix are derived for the parameter estimation in section 3. The compound linear plan is given in section 4. Confidence interval for the model parameter and testing of hypothesis for the existence of linearity is described in section 5. The simulation study is performed with example in section 6 and the conclusion of the proposed study is summarized in section 7.

Notations

X_0, X_1, X_2, X_3	test stresses (design, low, medium and high)
ξ	extrapolation amount where $\xi = (X_1 - X_0)/(X_2 - X_1)$
n	number of test units
n_i	number of failed units at stress X_i, i=1,2,3
t_{ij}	failure time of test unit j at stress X_i, i=1,2,3; j=1,2,……n_i
θ_i	mean life at X_i, i=1,2,3
τ	stress changing point
τ^*	optimum time of changing stress
$F_i(t)$	c.d.f. of Rayleigh distribution with mean θ_i, i=1,2,3
F(t)	c.d.f. of a test unit under step-stress
β_0, β_1	parameters of log-linear relationship between stress X_i and mean life θ_i

2. PROPOSED MODEL AND ASSUMPTIONS

The Rayleigh distribution was originally derived in connection with a problem in acoustics, and has been used in modeling certain features of electronic waves and as the distance distribution between individuals in a spatial Poisson process. Most frequently however it appears as a suitable model in life testing and reliability theory. For more details on the Rayleigh distribution the reader is referred to Johnson et al. (1994).
The probability density function and distribution function of the Rayleigh distribution are given, respectively, by:

$$f(t;\theta) = \frac{t}{\theta^2}\exp\left(-\frac{t^2}{2\theta^2}\right), \quad t > 0, \theta > 0 \tag{3}$$

and

$$F(t;\theta) = 1 - \exp\left(-\frac{t^2}{2\theta^2}\right), \quad t > 0, \theta > 0 \tag{4}$$

Initially n units are tested at a lower stress level X_1. The test is run until time τ_1, also known as hold time, when the stress level is increased to X_2. The test is continued until all units fail or until a predetermined censoring time reached, whichever occurs first.

Assumptions

(i) Testing is done at stresses X_1, X_2 and X_3 where $X_1<X_2< X_3$.

(ii) Under any constant stress, the time to failure of a test unit follows a Rayleigh distribution with distribution function is given in (4).

(iii) The scale parameter θ_i at stress level i, i= 0, 1, 2, 3 is a log-linear function of stress, i.e.,

$$\log(\theta_i) = \beta_0 + \beta_1 X_i \tag{5}$$

where, β_0 and β_1 are the unknown parameters depending on the nature of the product and method of the test.

(iv) The lifetimes of test units are independent and identically distributed.

From (2) and (4), it is easy to obtain the value of s_i with τ_0=0 and s_0=0, given as

$$s_i = \frac{\theta_{i+1}}{\theta_i}(\tau_i - \tau_{i-1} + s_{i-1}), \qquad i = 1, 2, \dots, m-1$$

Hence, the Rayleigh cumulative distribution function for a 3-step, step-stress ALT using K-H model is given by

$$G(t) = \begin{cases} 1-\exp\left(-\dfrac{t_1^2}{2\theta_1^2}\right), & 0 \le t < \tau_1 \\ 1-\exp\left(-\dfrac{t_2^2-\tau_1^2}{2\theta_2^2} - \dfrac{\tau_1^2}{2\theta_1^2}\right), & \tau_1 \le t < \tau_2 \\ 1-\exp\left(-\dfrac{t_3^2-\tau_2^2}{2\theta_3^2} - \dfrac{\tau_2^2-\tau_1^2}{2\theta_2^2} - \dfrac{\tau_1^2}{2\theta_1^2}\right), & \tau_2 \le t < \infty \end{cases} \tag{6}$$

and hence the corresponding probability density function (p.d.f.) is given by

$$g(t) = \begin{cases} \dfrac{t_1}{\theta_1^2}\exp\left(-\dfrac{t_1^2}{2\theta_1^2}\right), & 0 \le t < \tau_1 \\ \dfrac{t_2}{\theta_2^2}\exp\left(-\dfrac{t_2^2-\tau_1^2}{2\theta_2^2} - \dfrac{\tau_1^2}{2\theta_1^2}\right), & \tau_1 \le t < \tau_2 \\ \dfrac{t_3}{\theta_3^2}\exp\left(-\dfrac{t_3^2-\tau_2^2}{2\theta_3^2} - \dfrac{\tau_2^2-\tau_1^2}{2\theta_2^2} - \dfrac{\tau_1^2}{2\theta_1^2}\right), & \tau_2 \le t < \infty \end{cases} \tag{7}$$

3. MAXIMUM LIKELIHOOD ESTIMATION AND FISHER INFORMATION MATRIX

The MLE method is used for parameter estimation and analysis of failure time's t_{ij} from the step-stress ALT. The likelihood function from observations t_{ij}, i=1,2,3; j=1,2,…..n_i and the pdf of T given in (7) is derived as follows:

$$L(\mathbf{t}; \theta_1, \theta_2) = \prod_{j=1}^{n_1}\frac{t_{1j}}{\theta_1^2}\exp\left(-\frac{t_{1j}^2}{2\theta_1^2}\right) \times \prod_{j=1}^{n_2}\frac{t_{2j}}{\theta_2^2}\exp\left(-\frac{t_{2j}^2-\tau_1^2}{2\theta_2^2} - \frac{\tau_1^2}{2\theta_1^2}\right) \times$$

$$\prod_{j=1}^{n_3} \frac{t_{3j}}{\theta_3^2} \exp\left(-\frac{t_{3j}^2-\tau_2^2}{2\theta_3^2}-\frac{\tau_2^2-\tau_1^2}{2\theta_2^2}-\frac{\tau_1^2}{2\theta_1^2}\right) \tag{8}$$

After taking log of (8), the likelihood function becomes

$$\log L(\mathbf{t};\theta_1,\theta_2) = -2n_1\log\theta_1 - 2n_2\log\theta_2 - 2n_3\log\theta_3 + \sum_{l=1}^{3}\sum_{j=1}^{n_l} t_{lj} - \sum_{j=1}^{n_1}\left(\frac{t_{1j}^2}{2\theta_1^2}\right)$$
$$-\sum_{j=2}^{n_2}\left(\frac{t_{2j}^2-\tau_1^2}{2\theta_2^2}+\frac{\tau_1^2}{2\theta_1^2}\right)-\sum_{j=1}^{n_3}\left(\frac{t_{3j}^2-\tau_2^2}{2\theta_3^2}+\frac{\tau_2^2-\tau_1^2}{2\theta_2^2}+\frac{\tau_1^2}{2\theta_1^2}\right) \tag{9}$$

From assumption (iii),

$$\log L(\mathbf{t};\beta_0,\beta_1) = -2n\beta_0 - 2\beta_1(n_1X_1+n_2X_2+n_3X_3) + \sum_{l=1}^{3}\sum_{j=1}^{n_l} t_{lj} - \frac{1}{2}\sum_{j=1}^{n_1}\left(\frac{t_{1j}^2}{e^{2(\beta_0+\beta_1X_1)}}\right)$$
$$-\frac{1}{2}\sum_{j=2}^{n_2}\left(\frac{t_{2j}^2-\tau_1^2}{e^{2(\beta_0+\beta_1X_2)}}+\frac{\tau_1^2}{e^{2(\beta_0+\beta_1X_1)}}\right)$$
$$-\frac{1}{2}\sum_{j=1}^{n_3}\left(\frac{t_{3j}^2-\tau_2^2}{e^{2(\beta_0+\beta_1X_3)}}+\frac{\tau_2^2-\tau_1^2}{e^{2(\beta_0+\beta_1X_2)}}+\frac{\tau_1^2}{e^{2(\beta_0+\beta_1X_1)}}\right) \tag{10}$$

where, $n = n_1 + n_2 + n_3$

The MLEs $\hat{\beta}_0$ and $\hat{\beta}_1$ of the parameters β_0 and β_1 are the values which maximize the log-likelihood function. The first derivative of the log-likelihood functions given in (10) with respect to β_0 and β_1 are obtained as follows:

$$\frac{\partial \log L}{\partial \beta_0} = -2n + \sum_{j=1}^{n_1}\left(\frac{t_{1j}^2}{e^{2(\beta_0+\beta_1X_1)}}\right)+\sum_{j=1}^{n_2}\left(\frac{t_{2j}^2-\tau_1^2}{e^{2(\beta_0+\beta_1X_2)}}+\frac{\tau_1^2}{e^{2(\beta_0+\beta_1X_1)}}\right)$$
$$+\sum_{j=1}^{n_3}\left(\frac{t_{3j}^2-\tau_2^2}{e^{2(\beta_0+\beta_1X_3)}}+\frac{\tau_2^2-\tau_1^2}{e^{2(\beta_0+\beta_1X_2)}}+\frac{\tau_1^2}{e^{2(\beta_0+\beta_1X_1)}}\right) = 0 \tag{11}$$

$$\frac{\partial \log L}{\partial \beta_1} = -2(n_1X_1+n_2X_2+n_3X_3) + \sum_{j=1}^{n_1}\left(\frac{X_1t_{1j}^2}{e^{2(\beta_0+\beta_1X_1)}}\right)$$
$$+\sum_{j=1}^{n_2}\left(\frac{X_2\left(t_{2j}^2-\tau_1^2\right)}{e^{2(\beta_0+\beta_1X_2)}}+\frac{X_1\tau_1^2}{e^{2(\beta_0+\beta_1X_1)}}\right)$$
$$+\sum_{j=1}^{n_3}\left(\frac{X_3\left(t_{3j}^2-\tau_2^2\right)}{e^{2(\beta_0+\beta_1X_3)}}+\frac{X_2(\tau_2^2-\tau_1^2)}{e^{2(\beta_0+\beta_1X_2)}}+\frac{X_1\tau_1^2}{e^{2(\beta_0+\beta_1X_1)}}\right) = 0 \tag{12}$$

Obviously, it is not easy to obtain the closed form solution for these two non-linear equations (11) and (12). So, to solve these equations numerically, Newton-Raphson iterative method is used to obtain the MLE of β_0 and β_1.

The Fisher information matrix for n samples is obtain by taking the expectation of second and mixed partial derivatives of (10) with respect to β_0 and β_1 is given as follows

$$F = n\begin{bmatrix} E\left(-\frac{\partial^2 \log L}{\partial \beta_o^2}\right) & E\left(-\frac{\partial^2 \log L}{\partial \beta_0\, \partial \beta_1}\right) \\ E\left(-\frac{\partial^2 \log L}{\partial \beta_1\, \partial \beta_0}\right) & E\left(-\frac{\partial^2 \log L}{\partial \beta_1^2}\right) \end{bmatrix} \tag{13}$$

The elements of the Fisher information matrix are

$$E\left(-\frac{\partial^2 \log L}{\partial \beta_o^2}\right) = 4$$

$$E\left(-\frac{\partial^2 \log L}{\partial \beta_0\, \partial \beta_1}\right) = 4\left[X_1 + (X_2 - X_1)A_1 + (X_3 - X_2)A_2\right]$$

$$E\left(-\frac{\partial^2 \log L}{\partial \beta_1^2}\right) = 4\left[X_1^2 + \left(X_2^2 - X_1^2\right)A_1 + \left(X_3^2 - X_2^2\right)A_2\right]$$

where, $A_1 = \exp\left(-\frac{\tau_1^2}{2\theta_1^2}\right)$ and $A_2 = \exp\left(-\frac{\tau_2^2 - \tau_1^2}{2\theta_2^2} - \frac{\tau_1^2}{2\theta_1^2}\right)$.

Thus, the Fisher information matrix can be written as:

$$F = 4n\begin{bmatrix} 1 & X_1 + (X_2 - X_1)A_1 + (X_3 - X_2)A_2 \\ X_1 + (X_2 - X_1)A_1 + (X_3 - X_2)A_2 & X_1^2 + \left(X_2^2 - X_1^2\right)A_1 + \left(X_3^2 - X_2^2\right)A_2 \end{bmatrix} \tag{14}$$

The asymptotic variance of log of mean life at normal use stress X_0 is given by

$$\begin{aligned} n\mathrm{AVar}\log\left(\hat{\theta}_0\right) &= \mathrm{AVar}\log\left(\hat{\beta}_0 + \hat{\beta}_1 X_0\right) \\ &= \frac{1}{4}\left[\frac{\xi^2 + \eta^2 A_1 + 2\eta\xi A_1 + \left(1 - \eta^2 + 2\xi - 2\eta\xi\right)A_2}{\eta^2(A_1 - A_2) + A_2 - \{\eta(A_1 - A_2) + A_2\}^2}\right] \end{aligned} \tag{15}$$

where, $\xi = \frac{X_1 - X_0}{X_3 - X_1}$ and $\eta = \frac{X_2 - X_1}{X_3 - X_1}$.

The optimum linear plan can be obtained by minimizing the asymptotic variance of the log mean life estimate in (15) at normal use stress when $\tau_1 = \tau_2$, so that only two extreme stresses X_1 and X_3 are used in testing. Hence, the optimum stress changing time is obtained as

$$\tau^* = \theta_1\sqrt{2\log\left(\frac{1 + 2\xi}{\xi}\right)} \tag{16}$$

The above optimum linear plan uses only two stresses, which is infeasible. Since three stresses X_1, X_2 and X_3 are required for optimum plan under a 3-step, step-stress ALT experiment, therefore, we prefer the compound linear plan to obtain the stress changing times given in section 4.

4. COMPOUND LINEAR PLAN

The compound linear plan is an alternative of simple linear optimum plan and was initially introduced by Khamis and Higgins (1996). The compound linear plan for the Rayleigh

distribution under a 3-step, step-stress accelerated life test using K-H model is derived as follows.
We fix stress levels and choose τ_1 and τ_2 at which to change stresses. Our compromise plan uses the optimum simple step stress plan twice.

(a) Choose τ_1 to be the optimum value for changing stress in the step-stress model with stresses X_0, X_1, X_2 is given by

$$\tau_1^* = \theta_1 \sqrt{2\log\left(\frac{1+2\xi_1}{\xi_1}\right)}$$

where, $\xi_1 = \frac{X_1-X_0}{X_2-X_1}$

(b) Let $\tau_1^{'}$ be the optimum time for changing stress in the simple step-stress model with stresses X_0, X_2, X_3 is given below

$$\tau_1^{'} = \theta_2 \sqrt{2\log\left(\frac{1+2\xi_2}{\xi_2}\right)}$$

where, $\xi_2 = \frac{X_2-X_0}{X_3-X_2}$

(c) Let $\tau_2^* = \tau_1^* + \tau_1^{'}$, be the optimum stress changing time with stresses X_0, X_1, X_2, X_3.

The numerical value of the optimum stress changing times (τ_1^* and τ_2^*) are calculated by an example is given in section 6.

5. CONFIDENCE INTERVAL AND HYPOTHESIS TESTING

The confidence intervals of population parameters β_0 and β_1 are given as:

$$\left.\begin{aligned} L_{\beta_0} &= \hat{\beta}_0 - z\sqrt{\mathrm{Var}(\hat{\beta}_0)}, \quad U_{\beta_0} = \hat{\beta}_0 + z\sqrt{\mathrm{Var}(\hat{\beta}_0)} \\ L_{\beta_1} &= \hat{\beta}_1 - z\sqrt{\mathrm{Var}(\hat{\beta}_1)}, \quad U_{\beta_1} = \hat{\beta}_1 + z\sqrt{\mathrm{Var}(\hat{\beta}_1)} \end{aligned}\right\} \tag{17}$$

Test of hypothesis for parameters can be performed either by using the likelihood ratio test or the approximate normality of the MLEs in large sample sizes. In the latter case, it is most convenient to use approximation

$$(\hat{\beta}_0, \hat{\beta}_1) \sim N\left((\hat{\beta}_0, \hat{\beta}_1), F^{-1}\right)$$

An important inference problem concerning the regression coefficient (β_0, β_1) is the test of hypothesis H_0: $\beta_1 = 0$ against H_1: $\beta_1 < 0$. To test H_0 against H_1 one can use the likelihood ratio statistic given as

$$\Lambda = -2\log\left[\frac{L(\tilde{\beta}_0, 0)}{L(\hat{\beta}_0, \hat{\beta}_1)}\right] \tag{18}$$

The test with approximate size γ is given by the following; reject H_0 if and only if, $\Lambda > \chi^2_{1-\gamma,1}$ where $\chi^2_{1-\gamma,1}$ is the $(1-\gamma)^{th}$ quantile or percentile of the Chi-square distribution with one degree of freedom. Similarly, we can test for H'_0: $\beta_0=0$ against H'_1: $\beta_0 \neq 0$.

6. SIMULATION STUDIES: AN EXAMPLE

We simulated n=50 observations from Rayleigh cumulative exposure model given in (6) defined in section 2. Assume the initial values that are used to simulate the data as given below: $\beta_0=2$, $\beta_1=1$, $X_1=0.7$, $X_2=0.8$, $X_3=1.0$ then both the changing stress time are obtained as: $\tau_1^* = 13.75$ and $\tau_2^* = 31.68$.

Table 1. Simulated Rayleigh failure life time data

Level of Stresses	Failure times							
$X_1=0.7$	2.116	3.396	3.781	4.159	5.507	7.046	7.690	8.935
	9.739	10.498	11.634	11.975	12.324	12.514	12.931	13.047
	13.234	13.401	13.540					
$X_2=0.8$	16.002	16.289	16.965	16.966	17.322	18.185	18.331	19.970
	21.780	22.030	22.480	23.829	24.870	24.893	24.897	25.686
	25.962	26.582	27.696	28.085	28.305	31.049	31.365	
$X_3=1.0$	32.687	33.754	34.862	36.107	36.244	36.967	47.902	49.478

From the simulated data, we fit the following model

$$\log(\theta_i) = \beta_0 + \beta_1 X_i, \quad i = 1,2,3$$

The MLEs of β_0 and β_1 are obtained using R software (codes are given in appendix). The MLEs are

$$\hat{\beta}_0 = 3.17845437, \text{ and } \hat{\beta}_1 = -0.04579409$$

The asymptotic variance-covariance matrix is obtained as

$$\hat{F}^{-1} = \begin{bmatrix} 0.3591328 & -0.4416003 \\ -0.4416003 & 0.5506703 \end{bmatrix}$$

Table 2. Hypothesis is tested for parameters in the model $\log(\theta_i) = \beta_0 + \beta_1 X_i, i = 1,2,3$

Model	LogL	Λ	d.f.	$\chi_1^2(0.05)$
Full model(β_0, β_1)	-122.0040	-	-	-
$\beta_0=0$	-150.7103	57.4126	1	3.84
$\beta_1=0$	-130.4296	16.8512	1	3.84

The observed value of F^{-1}, i.e. $\hat{F}^{-1}$, is obtained by substituting the estimated values of parameters $\hat{\beta}_0$ and $\hat{\beta}_1$, for the true parameters in the asymptotic Fisher information matrix. To find the standard error of $\hat{\beta}_0$ and $\hat{\beta}_1$, we take the square root of diagonal element of $\hat{F}^{-1}$. The 95% confidence limits for the estimate of the parameters are

$$2.003872 \le \beta_0 \le 4.353037 \text{ and } -1.500254 \le \beta_1 \le 1.408666.$$

Table 2 shows the likelihood ratio statistics Λ for the test of various sub-models vs. full model. The 4th column of the table indicates the degree of freedom of the Chi-square approximation. The test $H_1: \beta_1 = 0$ and $H_0': \beta_0 = 0$ are rejected.

Table 3. Comparison of asymptotic Fisher information matrix of 2-step and 3-step stress ALT

Sample Size	2 step-stress ALT			3 step-stress ALT		
	Fisher Inf. Matrix	Trace	Determinant	Fisher Inf. Matrix	Trace	Determinant
30	2.0796 -2.1993 -2.1993 2.3639	4.4434	0.0788	0.4798 -0.3625 -0.3625 0.2944	0.7743	0.0098
50	1.6819 -1.7422 -1.7422 1.8265	3.5083	0.0366	0.4636 -0.3309 -0.3309 0.2468	0.7104	0.0049
100	1.1114 -1.1410 -1.1410 1.1820	2.2934	0.0118	0.1993 -0.1452 -0.1452 0.1114	0.3108	0.0011
150	0.5607 -0.5791 -0.5791 0.6052	1.1659	0.0040	0.1345 -0.0979 -0.0979 0.0750	0.2095	0.0005
200	0.4965 -0.5104 -0.5104 0.5299	1.0265	0.0027	0.1003 -0.0732 -0.0732 0.0563	0.1566	0.0003

7. CONCLUSION

Step-stress accelerated life test procedure is highly recommended for life testing of more reliable units or devices, when these units have a long life under the normal environmental situation. In this article, 3-step, step-stress accelerated life test plan is developed for the Rayleigh distribution. The Fisher information matrix for both 2-step and 3-step, step-stress ALT are obtained by calculating Trace and Determinant of the matrix; the confidence interval and hypothesis is tested for checking the linearity of the model (5) and their numerical values are also reported.

The purpose of life testing of highly reliable units at higher steps are given in section 1, and we justified numerically the necessity of life tests at 3-step (see table-3) and it is observed that life testing at 3-step have better Fisher's sample information in compare to that at two step. Therefore, we recommend the life testing at higher steps.

Some more interesting optimum plans for higher step-stress under quadratic relationship between lifetime and stress can be developed by using both classical and Bayesian techniques, for other life time distributions also.

APPENDIX

```
##### Simulation study for 3-step step-stress ALT for Rayleigh distribution
n=50;S1=0.7;S2=0.8;S3=1.0;tau1<-13.75859;tau2<- 31.6844
U<-runif(n)
Us<-sort(U)
F1tau<- prayleigh((tau1/theta1),1)
N1<-sum(Us<F1tau)
X1<- Us[1:N1]
t1<- theta1*qrayleigh(X1,1)
F2tau<-prayleigh(tau2/theta2,1)
w=N1+1
V<- Us[w:n]
N2<-sum(V<F2tau)
X2<-V[1:N2]
N2<- length(X2)
t2<- theta2*qrayleigh(X2,1)+tau1-(theta2/theta1)*tau1
w1<-N1+N2+1;V2<-Us[w1:n];N3<-length(V2)
V3<-theta3*qrayleigh(V2,1)
t3<-V3+tau2-(theta3/theta1)*tau1-(theta3/theta2)*(tau2-tau1)
c1<-matrix(t1,ncol=1);c2<-matrix(t2,ncol=1);c3<-matrix(t3,ncol=1)
cc<-data.frame(rbind(c1,c2,c3))[,1]
loglik<-function(param){
 a=param[1]; b=param[2]
ll<--2*n*a-2*b*(N1*S1+N2*S2+N3*S3)+sum(log(cc))-sum(t1^2)*exp(-2*(a+b*S1))-
sum((t2^2-tau1^2)*exp(-2*(a+b*S2))+tau1^2*exp(-2*(a+b*S1)))-sum((t3^2-
tau2^2)*exp(-2*(a+b*S3))+(tau2^2-tau1^2)*exp(-2*(a+b*S2))+tau1^2*exp(-2*(a+b*S1)))
ll
}
M<-maxNR(loglik,start=c(1,2))
summary(M)
```

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