

Predictions for Progressively Type-II Censored Failure Times from the Half Triangle Distribution

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Abstract

This paper deals with the problem of predicting censored data in a half triangle distribution with an unknown parameter based on progressively Type-II censored samples. We derive maximum likelihood predictors and some approximate maximum likelihood predictors of censored failure times in a progressively Type-II censoring scheme. In addition, we construct the shortest-length predictive intervals for censored failure times. Finally, Monte Carlo simulations are used to assess the validity of the proposed methods.

Keywords: Approximate maximum likelihood predictor, Approximate predictive maximum likelihood estimator, Half triangle distribution, Prediction interval, Progressively Type-II censored sample.

1. Introduction

A triangle distribution was applied to kernel function in a non-parametric density estimation. Johnson (1997) studied triangular distribution as a proxy for beta distribution. Balakrishnan and Nevzorov (2003) developed some properties for triangular distribution. Kang (2007) derived explicit estimators for the scale parameter in a half triangle distribution (HTD) based on multiply Type-II censored samples. Han and Kang (2008) proposed some approximate maximum likelihood estimators (AMLEs) for the scale parameter in HTD based on progressively Type-II censored samples. Kang *et al.* (2009) presented methods to derive explicit estimators for the scale parameter in HTD by the approximation of the likelihood equation based on Type-I hybrid censored samples.

The cumulative distribution function (cdf) and probability density function (pdf) for random variable X with HTD can be given by

$$F(x) = 1 - \left(1 - \frac{x}{\sigma}\right)^2 \quad (1.1)$$

and

$$f(x) = \frac{2}{\sigma} \left(1 - \frac{x}{\sigma}\right), \quad 0 < x < \sigma. \quad (1.2)$$

The prediction problems for unobserved or censored failure times have attracted the attention of many fields such as medical and engineering sciences. Basak and Balakrishnan (2009) gave a detailed account of the usefulness to predict censored data from a progressively Type-II censoring scheme.

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Asgharzadeh and Valiollahi (2010) discussed methods to obtain predictive intervals for censored failure times for a progressively Type-II censored sample from proportional hazard rate models. Raqab *et al.* (2010) derived some predictors for censored failure times in a progressively Type-II censored sample from Pareto distribution and provided a method to construct the highest conditional density predictive intervals for censored failure times.

We are interested in predicting censored failure times at each stage of a progressively Type-II censoring scheme when a progressively Type-II censored sample comes from HTD. Section 2 derives maximum likelihood predictors (MLPs) and approximate MLPs (AMLPs) for censored failure times, and discusses some methods to obtain predictive intervals (PIs) for censored failure times. Section 3 assesses the validity of the proposed methods, and Section 4 provides the conclusions.

2. Prediction for Censored Failure Times

2.1. Progressively Type-II censoring scheme

Let us consider the following progressively Type-II censoring scheme. Suppose n randomly selected units were placed on a life test, only m units are completely observed until the experiment is terminated. At the time of the first failure, R_1 units among $n - 1$ surviving units are randomly withdrawn (or censored) from the life testing experiment. At each time of the next failure, R_2 units among $n - 2 - R_1$ surviving units are randomly withdrawn. Finally, at the time of m^{th} failure, all the remaining $R_m = n - m - R_1 - \dots - R_{m-1}$ units are censored. Here, the R_1, \dots, R_m are fixed. Note that the case $m = n$, in which case $R_1 = \dots = R_m = 0$, corresponds to the complete sample situation, while the case $R_1 = \dots = R_{m-1} = 0$, $R_m = n - m$ corresponds to the conventional Type-II censored scheme. Inferences for a continuous probability distribution based on progressively Type-II censored samples were discussed by several authors. Balakrishnan *et al.* (2003) suggested point and interval estimation methods for Gaussian distribution based on progressively Type-II censored samples. Kang *et al.* (2008) obtained AMLEs of the scale parameter in a half logistic distribution based on progressively Type-II censored samples. Kang and Seo (2011) derived the AMLEs of the scale parameter and the reliability function in a exponentiated half logistic distribution based on progressively Type-II censored samples.

Let

$$X_{1:m:n} \leq X_{2:m:n} \leq \dots \leq X_{m:m:n} \quad (2.1)$$

denote a progressively Type-II censored sample from the HTD with censoring scheme (R_1, \dots, R_m) . In addition, let $Y_{j;R_k}$ denotes the j^{th} order statistic out of R_k removed units at stage k , for $j = 1, 2, \dots, R_k$ and $k = 1, 2, \dots, m$. Note that $X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}$ are only observed and $Y_{j;R_k}$ is unobserved (censored). We develop prediction methods for the censored failure time $Y_{j;R_k}$ in the following subsections.

2.2. Point prediction

This subsection derives the predictors of censored failure times based on an observed progressively Type-II censored sample from HTD with censoring scheme (R_1, \dots, R_m) . In addition, we propose some approximate methods to obtain explicit estimators of the scale parameter because the derived predictors depend on the scale parameter.

By the Markov property of progressively Type-II censored order statistics, the conditional density

function of $Y = Y_{j:R_k}$, given $X_{1:m:n} = x_{1:m:n}, X_{2:m:n} = x_{2:m:n}, \dots, X_{m:m:n} = x_{m:m:n}$ is given by

$$\begin{aligned} f_{Y|X_{k:m:n}}(y|x_{k:m:n}) &= \frac{R_k!}{(j-1)!(R_k-j)!} f(y)[F(y) - F(x_{k:m:n})]^{j-1} \\ &\quad \times [1 - F(y)]^{R_k-j} [1 - F(x_{k:m:n})]^{-R_k}, \quad y \geq x_{k:m:n}, \end{aligned} \quad (2.2)$$

which is the conditional density function of Y , given $X_{k:m:n} = x_{k:m:n}$. Then, the predictive likelihood function of Y is given by

$$L = C \left[\prod_{i=1}^m f(x_{i:m:n}) [1 - F(x_{i:m:n})]^{R_i} \right] f_{Y|X_{k:m:n}}(y|x_{k:m:n}), \quad (2.3)$$

where $C = n(n-1-R_1)(n-2-R_1-R_2)\cdots(n-m+1-R_1-\cdots-R_{m-1})$.

From (1.1), (1.2), and (2.3), the log-predictive likelihood function of Y is written as

$$\begin{aligned} \log L &\propto -(m+1) \log \sigma + [1 + 2(R_k - j)] \log \left(1 - \frac{y}{\sigma}\right) + 2(j-1-R_k) \log \left(1 - \frac{x_{k:m:n}}{\sigma}\right) \\ &\quad + (j-1) \log \left[1 - \left(\frac{1-y/\sigma}{1-x_{k:m:n}/\sigma}\right)^2\right] + \sum_{i=1}^m (2R_i + 1) \log \left(1 - \frac{x_{i:m:n}}{\sigma}\right). \end{aligned} \quad (2.4)$$

From (2.4), the predictive likelihood equation for Y is given by

$$\begin{aligned} \frac{\partial}{\partial y} \log L &= -\frac{1}{\sigma} \left[\frac{1 + 2(R_k - j)}{1 - y/\sigma} - 2(j-1) \frac{\frac{1-y/\sigma}{(1-x_{k:m:n}/\sigma)^2}}{1 - \left(\frac{1-y/\sigma}{1-x_{k:m:n}/\sigma}\right)^2} \right] \\ &= 0. \end{aligned} \quad (2.5)$$

By solving the equation (2.5), the MLP of Y is obtained as

$$\hat{Y} = \sigma \left[1 - \left(1 - \frac{X_{k:m:n}}{\sigma}\right) \sqrt{\frac{2(R_k - j) + 1}{2R_k - 1}} \right], \quad (2.6)$$

which depends on the scale parameter σ as well as the observed sample $X_{k:m:n}$. Now we estimate the parameter σ and then plug it in.

From (2.4), the predictive equation for σ is given by

$$\begin{aligned} \frac{\partial}{\partial \sigma} \log L &= -\frac{1}{\sigma} \left[m + 1 - [1 + 2(R_k - j)] \frac{y/\sigma}{1 - y/\sigma} - 2(j-1-R_k) \frac{x_{k:m:n}/\sigma}{1 - x_{k:m:n}/\sigma} \right. \\ &\quad \left. - 2(j-1) \frac{\left(\frac{1-y/\sigma}{1-x_{k:m:n}/\sigma}\right)^2}{1 - \left(\frac{1-y/\sigma}{1-x_{k:m:n}/\sigma}\right)^2} \left(\frac{x_{k:m:n}/\sigma}{1 - x_{k:m:n}/\sigma} - \frac{y/\sigma}{1 - y/\sigma}\right) - \sum_{i=1}^m (2R_i + 1) \frac{x_{i:m:n}/\sigma}{1 - x_{i:m:n}/\sigma} \right] \\ &= 0. \end{aligned} \quad (2.7)$$

Here, by replacing y with the MLP \hat{Y} in (2.7), the predictive likelihood equation (2.7) is written as

$$\begin{aligned} \frac{\partial}{\partial \sigma} \log L &= -\frac{1}{\sigma} \left[m + 1 + \frac{x_{k:m:n}/\sigma}{1 - x_{k:m:n}/\sigma} - \sum_{i=1}^m (2R_i + 1) \frac{x_{i:m:n}/\sigma}{1 - x_{i:m:n}/\sigma} \right] \\ &= 0. \end{aligned} \quad (2.8)$$

From (2.8), once the predictive maximum likelihood estimator (PMLE) $\hat{\sigma}$ is obtained, the MLP of Y is given by

$$\hat{Y} = \hat{\sigma} \left[1 - \left(1 - \frac{X_{k:m:n}}{\hat{\sigma}} \right) \sqrt{\frac{2(R_k - j) + 1}{2R_k - 1}} \right]. \quad (2.9)$$

However, the equation (2.8) does not admit an explicit solution for σ . Hence, we propose approximate PMLEs (APMLEs) of σ as

$$\tilde{\sigma}_1 = \frac{A}{B} \quad (2.10)$$

and

$$\tilde{\sigma}_2 = \frac{-C + \sqrt{C^2 - 4(m+1)D}}{2(m+1)}, \quad (2.11)$$

where

$$\begin{aligned} A &= -\frac{x_{k:m:n}}{q_{k:m:n}} + \sum_{i=1}^m (2R_i + 1) \frac{x_{i:m:n}}{q_{i:m:n}}, \\ B &= m + 1 - \frac{(1 - \sqrt{q_{k:m:n}})^2}{q_{k:m:n}} + \sum_{i=1}^m (2R_i + 1) \frac{(1 - \sqrt{q_{i:m:n}})^2}{q_{i:m:n}}, \\ C &= \frac{1 - 2(1 - \sqrt{q_{k:m:n}})}{q_{k:m:n}} x_{k:m:n} - \sum_{i=1}^m (2R_i + 1) \frac{1 - 2(1 - \sqrt{q_{i:m:n}})}{q_{i:m:n}} x_{i:m:n}, \\ D &= \frac{x_{k:m:n}^2}{q_{k:m:n}} - \sum_{i=1}^m (2R_i + 1) \frac{x_{i:m:n}^2}{q_{i:m:n}}. \end{aligned}$$

Note that the APMLEs $\tilde{\sigma}_i$ ($i = 1, 2$) are explicit estimators, and they always positive since $A > 0$, $B > 0$, and $D < 0$. Refer to Appendix for detailed proofs of the APMLEs.

By replacing σ with $\tilde{\sigma}_i$ ($i = 1, 2$) in (2.6), AMLPs of Y are given by

$$\tilde{Y}_1 = \tilde{\sigma}_1 \left[1 - \left(1 - \frac{X_{k:m:n}}{\tilde{\sigma}_1} \right) \sqrt{\frac{2(R_k - j) + 1}{2R_k - 1}} \right] \quad (2.12)$$

and

$$\tilde{Y}_2 = \tilde{\sigma}_2 \left[1 - \left(1 - \frac{X_{k:m:n}}{\tilde{\sigma}_2} \right) \sqrt{\frac{2(R_k - j) + 1}{2R_k - 1}} \right]. \quad (2.13)$$

However, we do not consider this case since the predictors \hat{Y} and \tilde{Y}_i ($i = 1, 2$) are the same as $X_{k:m:n}$ for $j = 1$.

2.3. Predictive interval

In this subsection, we construct two types of PIs for censored failure times based on an observed progressively Type-II censored sample from a HTD with censoring scheme (R_1, \dots, R_m) : one is the equal-tails PI (ETPI) the other is the shortest-length PI (SLPI).

From (1.1) and (1.2), the conditional density of $Y|X_k = x_k$ given in (2.2) is written as

$$f_{Y|x_{k:m:n}}(y|x_{k:m:n}) = \frac{R_k!}{(j-1)!(R_k-j)!} \left[1 - \left(\frac{1-y/\sigma}{1-x_{k:m:n}/\sigma} \right)^2 \right]^{j-1} \left(\frac{1-y/\sigma}{1-x_{k:m:n}/\sigma} \right)^{2(R_k-j)} \times \frac{2(1-y/\sigma)}{\sigma(1-x_{k:m:n}/\sigma)^2}. \quad (2.14)$$

Let

$$U = \left(\frac{1-Y/\sigma}{1-X_{k:m:n}/\sigma} \right)^2. \quad (2.15)$$

Then, the density function of U is

$$g(u) = \frac{R_k!}{(j-1)!(R_k-j)!} u^{R_k-j} (1-u)^{j-1}, \quad 0 < u < 1, \quad (2.16)$$

which is the pdf of the Beta distribution with parameters (R_k-j+1, j) , and denote by $\text{Beta}(R_k-j+1, j)$. For $0 < \alpha < 1$, assuming that the scale parameter σ is known, a $100(1-\alpha)\%$ PI for Y based on U can be constructed as

$$\begin{aligned} 1 - \alpha &= P[\beta_1(\alpha) < U < \beta_2(\alpha)] \\ &= P\left[\sigma\left(1 - \left(1 - \frac{x_{k:m:n}}{\sigma}\right) \sqrt{\beta_2(\alpha)}\right) < Y < \sigma\left(1 - \left(1 - \frac{x_{k:m:n}}{\sigma}\right) \sqrt{\beta_1(\alpha)}\right)\right]. \end{aligned} \quad (2.17)$$

We can choose $\beta_1(\alpha)$ and $\beta_2(\alpha)$ to obtain a $100(1-\alpha)\%$ ETPI for Y based on U . But it is not the shortest possible PI. Hence, we propose the following $100(1-\alpha)\%$ PI for Y based on U :

$$\left(\sigma\left(1 - \left(1 - \frac{x_{k:m:n}}{\sigma}\right) \sqrt{b_2}\right), \sigma\left(1 - \left(1 - \frac{x_{k:m:n}}{\sigma}\right) \sqrt{b_1}\right) \right). \quad (2.18)$$

Here b_1 and b_2 satisfy the equations

$$\left(\frac{b_1}{b_2} \right)^{R_k-j+\frac{1}{2}} = \left(\frac{1-b_2}{1-b_1} \right)^{j-1} \quad (2.19)$$

and

$$\begin{aligned} 1 - \alpha &= \int_{b_1}^{b_2} g(u) du \\ &= G(b_2) - G(b_1), \end{aligned} \quad (2.20)$$

where $G(\cdot)$ is the cdf of the Beta (R_k-j+1, j) and it has the form of the incomplete beta function. Since the Beta (R_k-j+1, j) is unimodal for $R_k > j$ and $j > 1$, we consider the cases of $j = 2, \dots, R_k - 1$.

Note that the proposed interval is a $100(1-\alpha)\%$ SLPI for Y based on U , which is derived using the ideas of Juola (1993). Refer to Appendix for detailed proofs of the SLPI for Y based on U .

If σ is unknown, the PMLE $\hat{\sigma}$ and APMLEs $\tilde{\sigma}_i$ ($i = 1, 2$) can be used as an alternative.

Table 1: Means and variances of maximum likelihood predictors (MLPs) and approximate MLPs (AMLPs)

n	m	Scheme	k	j	\bar{Y}		\bar{Y}_1		\bar{Y}_2				
					Mean	Variance	Mean	Variance	Mean	Variance			
15	10	(4*0, 3, 3*0, 2, 0)	5	2	0.3287	0.0082	0.3199	0.0076	0.3256	0.0075			
				3	0.5544	0.0191	0.5330	0.0158	0.5469	0.0159			
				9	0.5957	0.0224	0.5767	0.0187	0.5897	0.0189			
	10	(0, 3, 6*0, 2, 0)	2	2	0.2466	0.0043	0.2380	0.0038	0.2433	0.0038			
				3	0.5109	0.0151	0.4897	0.0121	0.5029	0.0122			
				9	0.6086	0.0222	0.5888	0.0183	0.6020	0.0185			
20	15	(3*0, 2, 4*0, 3, 6*0)	4	2	0.4373	0.0067	0.4197	0.0054	0.4291	0.0054			
				9	0.4092	0.0076	0.3996	0.0070	0.4048	0.0069			
				3	0.6116	0.0129	0.5879	0.0101	0.6009	0.0101			
	10	(5, 2*0, 5, 6*0)	1	2	0.1227	0.0012	0.1161	0.0010	0.1200	0.0010			
				3	0.2367	0.0032	0.2227	0.0024	0.2309	0.0024			
				4	0.3771	0.0077	0.3537	0.0053	0.3674	0.0054			
				5	0.5809	0.0181	0.5441	0.0120	0.5657	0.0123			
				4	0.2136	0.0044	0.2069	0.0042	0.2109	0.0041			
	10	(2*0, 3, 0, 2, 0, 2, 2*0, 3)	3	2	0.2602	0.0048	0.2559	0.0048	0.2586	0.0048			
				3	0.5304	0.0172	0.5198	0.0167	0.5264	0.0170			
				5	0.4588	0.0135	0.4504	0.0132	0.4556	0.0134			
				7	0.5041	0.0162	0.4952	0.0158	0.5007	0.0161			
				10	0.5196	0.0172	0.5135	0.0167	0.5172	0.0170			
30	20	(3*0, 5, 3*0, 5, 12*0)	4	2	0.6636	0.0269	0.6487	0.0256	0.6577	0.0264			
				3	0.2840	0.0022	0.2752	0.0019	0.2794	0.0019			
				4	0.4270	0.0042	0.4124	0.0033	0.4194	0.0032			
				5	0.6348	0.0086	0.6118	0.0064	0.6228	0.0064			
				8	0.2470	0.0029	0.2429	0.0028	0.2449	0.0028			
	20	(2*0, 10, 17*0)	3	2	0.3507	0.0038	0.3418	0.0034	0.3461	0.0034			
				4	0.4782	0.0056	0.4634	0.0046	0.4705	0.0045			
				5	0.6633	0.0097	0.6401	0.0073	0.6513	0.0072			
				7	0.3920	0.0033	0.3786	0.0025	0.3848	0.0025			
				8	0.4738	0.0046	0.4573	0.0034	0.4650	0.0034			
15	15	(5, 6*0, 10, 7*0)	1	2	0.5745	0.0066	0.5541	0.0048	0.5636	0.0048			
				3	0.7208	0.0103	0.6947	0.0073	0.7068	0.0073			
				4	0.1202	0.0007	0.1153	0.0006	0.1180	0.0006			
				5	0.2404	0.0022	0.2299	0.0017	0.2356	0.0017			
				8	0.3882	0.0054	0.3708	0.0040	0.3803	0.0040			
				3	0.6029	0.0127	0.5755	0.0095	0.5905	0.0094			
				4	0.2063	0.0032	0.2040	0.0032	0.2053	0.0032			
				5	0.2478	0.0037	0.2431	0.0036	0.2457	0.0035			
				6	0.2921	0.0043	0.2849	0.0041	0.2889	0.0040			
				7	0.3400	0.0052	0.3300	0.0048	0.3356	0.0047			
10				8	0.3925	0.0064	0.3794	0.0056	0.3867	0.0055			
				9	0.4512	0.0079	0.4348	0.0067	0.4440	0.0066			
				10	0.5191	0.0101	0.4987	0.0082	0.5101	0.0081			
				1	0.6027	0.0132	0.5775	0.0103	0.5916	0.0102			
				2	0.7241	0.0187	0.6918	0.0140	0.7098	0.0140			

Continue

Continue

n	m	Scheme	k	j	\hat{Y}		\tilde{Y}_1		\tilde{Y}_2	
					Mean	Variance	Mean	Variance	Mean	Variance
30	15	(10, 6*0, 5, 7*0)	1	2	0.0640	0.0004	0.0617	0.0003	0.0629	0.0003
				3	0.1146	0.0006	0.1100	0.0005	0.1124	0.0005
				4	0.1687	0.0011	0.1615	0.0009	0.1653	0.0009
				5	0.2271	0.0018	0.2172	0.0014	0.2225	0.0014
				6	0.2912	0.0029	0.2783	0.0022	0.2852	0.0022
				7	0.3629	0.0044	0.3466	0.0032	0.3553	0.0032
				8	0.4458	0.0065	0.4256	0.0048	0.4363	0.0048
				9	0.5477	0.0097	0.5228	0.0071	0.5361	0.0071
				10	0.6959	0.0157	0.6639	0.0113	0.6809	0.0114
			8	2	0.2903	0.0050	0.2852	0.0049	0.2880	0.0048
				3	0.3832	0.0064	0.3723	0.0059	0.3782	0.0058
				4	0.4975	0.0091	0.4794	0.0076	0.4893	0.0075
				5	0.6634	0.0148	0.6350	0.0112	0.6505	0.0111

3. Simulation Study

To compare the performance of the proposed estimators, we simulate their mean squared errors (MSEs) and biases through Monte Carlo simulations. By using the algorithm presented in Balakrishnan and Sandhu (1995), we generate progressively Type-II censored samples from a standard HTD for various censoring schemes used by Han and Kang (2008). For each scheme, we compute the PMLE $\hat{\sigma}$ and APMLEs $\tilde{\sigma}_i$ ($i = 1, 2$) at each stage. Here, the PMLE $\hat{\sigma}$ is obtained by using the Newton-Raphson method whose convergence depends on the choice of the initial value, which can be solved using the APMLEs $\tilde{\sigma}_i$ ($i = 1, 2$) as starting values for the iterations. Based on the estimators, we obtain means of the MLP \hat{Y} and AMLPs \tilde{Y}_i ($i = 1, 2$) and their variances over 10,000 simulations. In addition, we compute average lengths of the ETPI and SLPI for Y based on U over 10,000 simulations. Table 1 and Table 2 provide the results. In this study, we focus on the problem of predicting censored data in the situation where the sample is progressively Type-II censored; therefore, we do not report the results for the PMLE $\hat{\sigma}$ and APMLEs $\tilde{\sigma}_i$ ($i = 1, 2$).

Table 1 shows that the variances of the AMLPs \tilde{Y}_i ($i = 1, 2$) are smaller than the MLP \hat{Y} . In addition, variances of all predictors increase as j increases in the each scheme. From the Table 2, we see that average lengths of PIs based on the APMLEs $\tilde{\sigma}_i$ ($i = 1, 2$) are shorter than the PIs based on the PMLE $\hat{\sigma}$, especially in the APMLE $\tilde{\sigma}_1$. In addition, the average length of PIs increase as j increases in each scheme. As expected, the SLPI has a shorter average length than the ETPI for $R_k > j$ and $j > 1$.

4. Concluding Remarks

This paper discusses predictors and PIs for progressively Type-II censored failure times from a HTD. The predictors and PIs depend on a scale parameter whose PMLE cannot be solved explicitly. We propose two types of APMLEs as an alternative to PMLE. In addition, we derive the ETPIs and SLPIs for the censored failure times and compare them using Monte Carlo simulations. Based on the simulation results, we suggest predictive methods based on the APMLEs when the scale parameter is unknown and recommend the use of SLPI when $R_k > j$ and $j > 1$.

Table 2: Average length of 95% predictive intervals (PIs) for censored failure times based on U

n	m	Scheme	k	j	By using $\hat{\sigma}$		By using $\tilde{\sigma}_1$		By using $\tilde{\sigma}_2$	
					ETPI	SLPI	ETPI	SLPI	ETPI	SLPI
15	10	(4*0, 3, 3*0, 2, 0)	5	2	0.4445	0.3315	0.4194	0.3128	0.4357	0.3249
				3	0.5168	-	0.4877	-	0.5066	-
				9	0.3277	-	0.2916	-	0.3163	-
	10	(0, 3, 6*0, 2, 0)	2	2	0.5204	0.3881	0.4957	0.3697	0.5110	0.3811
				3	0.6050	-	0.5764	-	0.5941	-
				9	0.3085	-	0.2708	-	0.2959	-
20	15	(3*0, 2, 4*0, 3, 6*0)	4	2	0.6405	-	0.6070	-	0.6248	-
				9	0.3985	0.2972	0.3709	0.2766	0.3860	0.2879
				3	0.4634	-	0.4313	-	0.4488	-
	10	(5, 2*0, 5, 6*0)	1	2	0.3682	0.2239	0.3438	0.2090	0.3581	0.2177
				3	0.4518	0.3752	0.4219	0.3503	0.4394	0.3649
				4	0.5152	0.4924	0.4811	0.4598	0.5011	0.4789
				5	0.5441	-	0.5081	-	0.5292	-
				4	0.3196	0.1943	0.2949	0.1793	0.3096	0.1882
			4	3	0.3922	0.3257	0.3619	0.3005	0.3799	0.3155
				3	0.4472	0.4274	0.4126	0.3944	0.4333	0.4141
				5	0.4723	-	0.4358	-	0.4575	-
				3	0.5320	0.3968	0.5196	0.3875	0.5273	0.3933
	10	(2*0, 3, 0, 2, 0, 2, 2*0, 3)	3	2	0.6186	-	0.6042	-	0.6132	-
				3	0.6080	-	0.5920	-	0.6020	-
				5	0.5345	-	0.5175	-	0.5280	-
			7	2	0.2836	0.2115	0.2662	0.1985	0.2768	0.2064
				3	0.3298	-	0.3095	-	0.3218	-
30	20	(3*0, 5, 3*0, 5, 12*0)	4	2	0.3753	0.2282	0.3601	0.2189	0.3674	0.2234
				3	0.4606	0.3824	0.4419	0.3669	0.4508	0.3743
				4	0.5252	0.5019	0.5039	0.4816	0.5141	0.4913
				5	0.5547	-	0.5322	-	0.5429	-
			8	2	0.3344	0.2033	0.3191	0.1940	0.3265	0.1985
				3	0.4104	0.3408	0.3915	0.3251	0.4006	0.3326
				4	0.4680	0.4472	0.4465	0.4267	0.4568	0.4366
				5	0.4942	-	0.4715	-	0.4824	-
				3	0.2111	0.1101	0.2028	0.1058	0.2066	0.1078
	20	(2*0, 10, 17*0)	3	2	0.2611	0.1857	0.2509	0.1785	0.2556	0.1818
				3	0.3028	0.2447	0.2910	0.2351	0.2965	0.2396
				4	0.3391	0.2949	0.3259	0.2834	0.3320	0.2887
				5	0.3713	0.3397	0.3568	0.3265	0.3635	0.3326
				6	0.3998	0.3808	0.3843	0.3659	0.3915	0.3728
			8	7	0.4245	0.4188	0.4079	0.4025	0.4156	0.4100
				8	0.4427	0.4329	0.4255	0.4252	0.4335	0.4310
				9	0.4399	-	0.4228	-	0.4307	-
				10	0.3879	0.2358	0.3697	0.2248	0.3797	0.2308
				3	0.4760	0.3952	0.4537	0.3767	0.4659	0.3868
15	(5, 6*0, 10, 7*0)	1	2	4	0.5427	0.5187	0.5174	0.4944	0.5313	0.5077
				5	0.5732	-	0.5464	-	0.5610	-
			8	2	0.1751	0.0913	0.1650	0.0860	0.1706	0.0890
				3	0.2166	0.1541	0.2041	0.1452	0.2111	0.1502
				4	0.2513	0.2030	0.2367	0.1912	0.2448	0.1978
				5	0.2814	0.2447	0.2651	0.2305	0.2742	0.2384
				6	0.3081	0.2819	0.2902	0.2655	0.3002	0.2747
			10	7	0.3318	0.3159	0.3125	0.2976	0.3233	0.3079
				8	0.3522	0.3475	0.3318	0.3273	0.3432	0.3386
				9	0.3673	0.3658	0.3460	0.3440	0.3579	0.3462
				10	0.3650	-	0.3439	-	0.3557	-

Continue

Continue

n	m	Scheme	k	j	By using $\hat{\sigma}$		By using $\tilde{\sigma}_1$		By using $\tilde{\sigma}_2$	
					ETPI	SLPI	ETPI	SLPI	ETPI	SLPI
30	15	(10, 6*0, 5, 7*0)	1	2	0.2137	0.1115	0.2037	0.1062	0.2090	0.1090
				3	0.2644	0.1881	0.2520	0.1792	0.2586	0.1839
				4	0.3067	0.2478	0.2923	0.2361	0.2999	0.2423
				5	0.3434	0.2986	0.3273	0.2846	0.3359	0.2921
				6	0.3760	0.3440	0.3583	0.3278	0.3677	0.3365
				7	0.4049	0.3856	0.3859	0.3675	0.3960	0.3771
				8	0.4298	0.4241	0.4096	0.4042	0.4204	0.4148
				9	0.4483	0.4386	0.4272	0.4271	0.4384	0.4285
				10	0.4455	-	0.4246	-	0.4357	-
			8	2	0.2998	0.1823	0.2810	0.1709	0.2913	0.1771
				3	0.3679	0.3055	0.3448	0.2863	0.3574	0.2968
				4	0.4195	0.4009	0.3932	0.3758	0.4076	0.3895
				5	0.4430	-	0.4153	-	0.4304	-

Appendix A: The APMLEs of σ

Let $Z = X/\sigma$. Then the equation (2.8) is written as

$$\begin{aligned} \frac{\partial}{\partial \sigma} \log L &= -\frac{1}{\sigma} \left[m + 1 + \frac{Z_{k:m:n}}{1 - Z_{k:m:n}} - \sum_{i=1}^m (2R_i + 1) \frac{Z_{i:m:n}}{1 - Z_{i:m:n}} \right] \\ &= 0. \end{aligned} \quad (\text{A.1})$$

Han and Kang (2008) approximated the following terms by using Taylor series:

$$\frac{Z_{i:m:n}}{1 - Z_{i:m:n}} \approx -\frac{(1 - \sqrt{q_{i:m:n}})^2}{q_{i:m:n}} + \frac{1}{q_{i:m:n}} Z_{i:m:n} \quad (\text{A.2})$$

and

$$\frac{1}{1 - Z_{i:m:n}} \approx \frac{1 - 2(1 - \sqrt{q_{i:m:n}})}{q_{i:m:n}} + \frac{1}{q_{i:m:n}} Z_{i:m:n}, \quad (\text{A.3})$$

where

$$\begin{aligned} p_{i:m:n} &= 1 - \prod_{j=m-i+1}^m \frac{j + r_{m-i+1} + \dots + r_m}{j + 1 + r_{m-i+1} + \dots + r_m}, \\ q_{i:m:n} &= 1 - p_{i:m:n}, \quad i = 1, \dots, m. \end{aligned}$$

Then, by using (A.2) and (A.3), two approximate predictive likelihood equations for σ are obtained as

$$\begin{aligned} \frac{\partial}{\partial \sigma} \log L &\approx -\frac{1}{\sigma} \left[m + 1 - \frac{(1 - \sqrt{q_{k:m:n}})^2}{q_{k:m:n}} + \frac{1}{q_{k:m:n}} Z_{k:m:n} - \sum_{i=1}^m (2R_i + 1) \left(-\frac{(1 - \sqrt{q_{i:m:n}})^2}{q_{i:m:n}} + \frac{1}{q_{i:m:n}} Z_{i:m:n} \right) \right] \\ &= 0 \end{aligned} \quad (\text{A.4})$$

and

$$\begin{aligned} \frac{\partial}{\partial \sigma} \log L &\approx -\frac{1}{\sigma} \left[m + 1 + \frac{1 - 2(1 - \sqrt{q_{k:m:n}})}{q_{k:m:n}} Z_{k:m:n} + \frac{1}{q_{k:m:n}} Z_{k:m:n}^2 \right. \\ &\quad \left. - \sum_{i=1}^m (2R_i + 1) \left(\frac{1 - 2(1 - \sqrt{q_{i:m:n}})}{q_{i:m:n}} Z_{i:m:n} + \frac{1}{q_{i:m:n}} Z_{i:m:n}^2 \right) \right] \\ &= 0. \end{aligned} \quad (\text{A.5})$$

The proof is completed by solving the equations (A.4) and (A.5).

Appendix B: The SLPI for Y based on U

Let $\beta_1(\alpha) = b_1$ and $\beta_2(\alpha) = b_2$ in (2.17). Then, the length of the PI is

$$l = \sigma \left(1 - \frac{x_{k:m:n}}{\sigma} \right) \left(\sqrt{b_2} - \sqrt{b_1} \right). \quad (\text{B.1})$$

We minimize the length (B.1) subject to

$$\begin{aligned} 1 - \alpha &= \int_{b_1}^{b_2} g(u) du \\ &= G(b_2) - G(b_1), \end{aligned} \quad (\text{B.2})$$

where $G(\cdot)$ is the cdf of the Beta ($R_k - j + 1, j$) and it has the form of the incomplete beta function. Now

$$\frac{dl}{db_1} = \frac{\sigma}{2} \left(1 - \frac{x_{k:m:n}}{\sigma} \right) \left(\frac{1}{\sqrt{b_2}} \frac{db_2}{db_1} - \frac{1}{\sqrt{b_1}} \right) \quad (\text{B.3})$$

and

$$\frac{db_2}{db_1} = \frac{g(b_1)}{g(b_2)}, \quad (\text{B.4})$$

giving

$$\frac{dl}{db_1} = \frac{\sigma}{2} \left(1 - \frac{x_{k:m:n}}{\sigma} \right) \left(\frac{1}{\sqrt{b_2}} \frac{g(b_1)}{g(b_2)} - \frac{1}{\sqrt{b_1}} \right). \quad (\text{B.5})$$

Therefore, the $100(1 - \alpha)\%$ PI for Y based on U has the shortest length when

$$\left(\frac{b_1}{b_2} \right)^{R_k - j + \frac{1}{2}} = \left(\frac{1 - b_2}{1 - b_1} \right)^{j-1} \quad (\text{B.6})$$

and the proof is complete.

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