

# A Note on the Characteristic Function of Multivariate $t$ Distribution

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## Abstract

This study derives the characteristic functions of (multivariate/generalized)  $t$  distributions without contour integration. We extended Hursts method (1995) to (multivariate/generalized)  $t$  distributions based on the principle of randomization and mixtures. The derivation methods are relatively straightforward and are appropriate for graduate level statistics theory courses.

**Keywords:** Randomization and mixtures, modified Bessel function of the third kind, contour integration, Laplace-Stieltjes transform.

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## 1. Introduction

If  $X$  is a random variable, the characteristic function (cf) of  $X$  is defined by

$$\phi_X(t) = E[e^{itX}] = \int_{-\infty}^{\infty} e^{itX} dF_X(x), \quad t \in R,$$

where  $i = \sqrt{-1}$ ,  $F_X(x)$  is the distribution function of  $X$  and  $e^{itX} = \cos(tX) + i \sin(tX)$ . The focus of this note is on deriving the cf of the  $t$  distribution. Having heavier tails than the normal distribution,  $t$  distribution is widely applied in many areas of statistics such as tests for two sample means, linear regression analysis, and robust parametric modeling. Significant data is drawn from  $t$  distribution and studying  $t$  distribution has become increasingly important in financial analysis.

It is natural to focus on the cf of  $t$  distribution because they uniquely determine distribution functions. The cfs allow us to easily manipulate convolved probability density functions (pdfs) when they represent linear combinations of independent random variables. With regard to asymptotics, cfs play an important role in the Levy continuity theorem. Most importantly, the derivation of the central limit theorem rests on an asymptotic expansion of cfs. These topics are covered in all standard graduate level textbooks on probability (Durrett, 1996). Although cfs have many useful properties, most statistics theory courses avoid rigorous derivations of these functions because statistics students are not generally exposed to contour integration. A direct application of contour integration to develop the cf of (multivariate/generalized)  $t$  distribution is burdensome since derivation methods require more steps and more contour integration knowledge (Sutradhar, 1986) than the ones used in our study.

Students are typically advised to compute the moment generating function (when it is finite) of a random variable and then substitute  $it$  for  $t$  in the moment generating function to obtain the cf, where

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$i = \sqrt{-1}$ . This approach works for many distributions; however, this method does not work when the moment generating function is not finite (such as in  $t$  distribution with finite degrees of freedom). To resolve this problem, Datta and Ghosh (2007) suggested the easy and creative derivations of cfs for some basic probability density functions such as normal, double exponential, and Cauchy distributions. Their study did not include  $t$  distribution; however, the Cauchy distribution is a special case of the  $t$  distribution.

Owing to the property of odd function, the cf of the univariate  $t$  distribution (with degrees of freedom  $\nu$ ) is given by,

$$\phi_X(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right) \sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\cos(tx \sqrt{\nu})}{(1+x^2)^{\frac{\nu+1}{2}}} dx.$$

In 1956, Fisher and Healy tried to obtain a simple form of the cf of  $t$  distribution. However, they gave it only for odd degrees of freedom, as follows:

$$\phi_X(t) = e^{-|t \sqrt{\nu}|} S_{\nu}(|t \sqrt{\nu}|),$$

where

$$S_{\nu}(t) = \sum_{r=0}^{\frac{\nu-1}{2}} \frac{(\nu-r-1)! \left(\frac{\nu-1}{2}\right)! 2^r}{(\nu-1)! \left(\frac{\nu-1}{2}-r\right)! r!} t^r$$

with odd  $\nu$  ( $\nu = 1, 3, 5, \dots$ ). That is,

$$\begin{aligned} S_1 &= 1 \\ S_3 &= 1 + t \\ S_5 &= 1 + t + \frac{1}{3}t^2 \\ S_7 &= 1 + t + \frac{2}{5}t^2 + \frac{1}{15}t^3 \\ &\vdots \end{aligned}$$

Since their work, many statisticians have derived the cf of  $t$  distribution using various methods of their own. Ifram (1970) gave an expression of the cf for all degrees of freedom. Ifram derived the cf of  $F$  distribution and found the cf of  $t$  distribution to be a special case of  $F$  distribution; however, Pestana (1977) showed that the expression was incorrect. He indicated that Ifram (1970) made a mistake in the contour integration when he used various quadratic substitutions. Hurst (1995) also derived the cf of the univariate  $t$  distribution based on three facts: i) randomization and mixtures; ii) the cf of symmetric generalized hyperbolic distribution in terms of Bessel function; and iii) some well known approximations. A more recent derivation of the univariate case was presented by Dreier and Kotz (2002), based on the theory of positive definite densities.

Sutradhar (1986, 1988) derived the first version of the cf of multivariate  $t$  distribution based on a series representation composed of three types. The first is the case when the degree of freedom,  $\nu$ , is odd. The second case is when the degree of freedom,  $\nu$ , is even. The third case deals with a fractional value of  $\nu$ . However, these representations are too complex. See Kotz and Nadarajah (2004) for a general overview of the cfs of multivariate  $t$  distribution.

The rest of this paper is organized as follows. Section 2 presents some basic materials essential to deriving the cf of  $t$  distribution. In Section 3, we suggest a simple method to derive the cf of the  $t$  distribution for both the univariate and multivariate cases. The method of Hurst (1995) primarily used the idea of Laplace-Stieltjes transform based on mixtures. Here, we extend his method to (multivariate/generalized)  $t$  distribution. Finally, Section 4 concludes the paper.

## 2. Preliminaries

In this section, we present some basic materials essential to deriving the cf of  $t$  distribution. Let  $F$  be a distribution function depending on a parameter  $\theta$ , and  $u$  a probability density. Then

$$W(x) = \int_{-\infty}^{\infty} F(x, \theta) u(\theta) d\theta$$

is a monotone function of  $x$  increasing from 0 to 1 and hence a distribution function, so we will denote  $W(x)$  as  $F_X(x)$  to emphasize the fact that it is a distribution function of  $X$ . If  $F$  has a continuous density  $f$ , then  $W$  has a density  $w$  given by

$$w(x) = \int_{-\infty}^{\infty} f(x, \theta) u(\theta) d\theta,$$

so we will denote  $w(x)$  as  $f_X(x)$  to emphasize the fact that it is a density function of  $X$ . This process may be described probabilistically as randomization and mixtures or simply mixtures (Feller, 1966). One example of the mixtures is called the variance-mean mixture (Barndorff-Nielsen *et al.*, 1982). Let  $X = \mu + \beta U + \sigma \sqrt{U}Z$ , where  $\sigma > 0$ ,  $Z \sim N(0, 1)$ , and  $U$  follows a distribution function  $F$  on  $[0, \infty)$ . Further, assume that  $Z$  and  $U$  are independent. Then,  $X|U \sim N(\mu + \beta U, \sigma^2 U)$  and we say that the distribution of  $X$  is a normal variance-mean mixture with position  $\mu$ , drift  $\beta$ , structure  $\sigma$ , and mixing distribution  $F(u)$ . Note that if  $\beta = 0$ , it is also called the scale mixtures of normal distributions. Hence, the pdf of  $X$  is given by

$$\exp\left\{\frac{(x-\mu)\beta}{\sigma^2}\right\} \int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 u}} \exp\left\{-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2 u} - \frac{1}{2} \frac{u\beta^2}{\sigma^2}\right\} dF_U(u). \quad (2.1)$$

It is well-known that the (multivariate/generalized)  $t$  distributions can be expressed as a normal variance-mean mixture and as the scale mixtures of normal distributions. Let the random variable

$$X = \sqrt{U}Z, \quad (2.2)$$

where  $U \sim IG(v/2, \delta/2)$ ,  $Z \sim N(0, 1)$ , and  $U$  is independent of  $Z$ . The pdf of  $U \sim IG(v/2, \delta/2)$  is given by

$$f_U(u) = \frac{\left(\frac{\delta}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} u^{-\frac{v}{2}-1} \exp\left\{-\frac{\delta}{2u}\right\}, \quad u > 0.$$

Then, the random variable  $X$  has a univariate generalized  $t$  distribution or Pearson type VII distribution (see Fang *et al.*, 1990, Section 3.3). We denote this as  $X \sim gt(v, \delta)$  with the following pdf

$$f_X(x) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right) \sqrt{\delta\pi}} \left(1 + \frac{x^2}{\delta}\right)^{-\frac{v+1}{2}}, \quad x \in \mathbb{R}, \quad v, \delta > 0.$$

If  $\nu = \delta$ , then  $X$  follows the univariate  $t$  distribution; that is,  $X \sim t(\nu)$ .

The multivariate extension is similar to the univariate case. Let the random vector

$$\mathbf{X} = \sqrt{U}\mathbf{Z}, \quad (2.3)$$

where  $U \sim IG(\nu/2, \delta/2)$  independently of  $\mathbf{Z} \sim N_p(0, \mathbf{I})$ . Then, the random vector  $\mathbf{X}$  follows the multivariate generalized  $t$  distribution. We denote this as  $\mathbf{X} \sim gt_p(\nu, \delta)$  with a pdf given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\Gamma\left(\frac{\nu+p}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\delta\pi)^{\frac{p}{2}}} \left(1 + \frac{\mathbf{x}^T \mathbf{x}}{\delta}\right)^{-\frac{\nu+p}{2}}, \quad \mathbf{x} \in \mathbb{R}^p, \nu, \delta > 0.$$

If  $\nu = \delta$ , then  $\mathbf{X}$  follows the multivariate  $t$  distribution; that is,  $\mathbf{X} \sim t_p(\nu)$ . Deriving the univariate  $t$  densities is straightforward using gamma integration, with  $\mu = 0, \beta = 0$ , and  $\sigma = 1$  in Equation (2.1). The multivariate case is similar, but uses a density of  $N_p(0, \mathbf{I})$  and gamma integration.

Now, we define the integral representation of the modified Bessel function of the third kind (Watson, 1966, p.182) and examine some of its properties (Abramowitz and Stegun, 1972).

**Definition 1.** *The integral representation of the modified Bessel function of the third kind is defined by*

$$K_{\lambda}(w) = \frac{1}{2} \int_0^{\infty} x^{\lambda-1} \exp\left\{-\frac{1}{2}w\left(x + \frac{1}{x}\right)\right\} dx, \quad w > 0.$$

*Some properties of the modified Bessel function of the third kind are*

- I.  $K_{\lambda}(w) = K_{-\lambda}(w), w > 0, \lambda \in \mathbb{R}$  and
- II.  $K_{\lambda}(w) \cong \Gamma(\lambda)2^{\lambda-1}w^{-\lambda}$  as  $w \rightarrow 0^+, \lambda > 0$ .

**Proof:** Because Abramowitz and Stegun (1972) skipped the proofs of these properties, we include them here. We can prove property I using a transformation. Let  $y = 1/x$ , then  $K_{\lambda}(w)$  can be expressed in terms of  $y$ , as follows:

$$\begin{aligned} K_{\lambda}(w) &= \frac{1}{2} \int_0^{\infty} y^{-\lambda-1} \exp\left\{-\frac{1}{2}w\left(\frac{1}{y} + y\right)\right\} dy \\ &= K_{-\lambda}(w). \end{aligned}$$

To prove property II, we show that

$$\lim_{w \rightarrow 0^+} \frac{K_{\lambda}(w)}{\Gamma(\lambda)2^{\lambda-1}w^{-\lambda}} \cong 1.$$

To do so, we first show

$$\lim_{w \rightarrow 0^+} \int_0^{\infty} f_w(x) dx = \int_0^{\infty} \lim_{w \rightarrow 0^+} f_w(x) dx,$$

where

$$f_w(x) = \left(\frac{wx}{2}\right)^{\lambda-1} \exp\left\{-\frac{wx}{2}\right\} \left(\exp\left\{-\frac{w}{2x}\right\} - 1\right) \frac{w}{2}.$$

Now, let  $k = 1/w$ , then

$$f_w(x) = \left(\frac{x}{2k}\right)^{\lambda-1} \exp\left\{-\frac{x}{2k}\right\} \left(\exp\left\{-\frac{1}{2kx}\right\} - 1\right) \frac{1}{2k}.$$

Note that  $\lim_{w \rightarrow 0^+} f_w(x) = \lim_{k \rightarrow \infty} f_w(x) = 0$  and  $|f_w(x)| \leq g(x)$ , where  $g(x) = (wx/2)^{\lambda-1} \exp\{-wx/2\} w/2$ , since  $|e^{-w/(2x)} - 1| \leq 1$ . The application of the dominated convergence theorem allows us to interchange the order of the limit and the integration since it is integrable. That is,  $\int_0^\infty g(x)dx = \Gamma(\lambda)$ . Thus,  $\lim_{w \rightarrow 0^+} \int_0^\infty f_w(x)dx = \int_0^\infty \lim_{w \rightarrow 0^+} f_w(x)dx$ . From these results, we have the following:

$$\begin{aligned} \lim_{w \rightarrow 0^+} \frac{K_\lambda(w)}{\Gamma(\lambda)2^{\lambda-1}w^{-\lambda}} &= \lim_{w \rightarrow 0^+} \frac{\int_0^\infty \frac{1}{2} \left(\frac{x}{w}\right)^\lambda \left(\frac{wx}{2}\right)^{\lambda-1} \exp\left\{-\frac{wx}{2}\right\} \exp\left\{-\frac{w}{2x}\right\} \frac{w}{2} dx}{\Gamma(\lambda)2^{\lambda-1}w^{-\lambda}} \\ &= \lim_{w \rightarrow 0^+} \frac{1}{\Gamma(\lambda)} \int_0^\infty \left(\frac{wx}{2}\right)^{\lambda-1} \exp\left\{-\frac{wx}{2}\right\} \left(1 + \exp\left\{-\frac{w}{2x}\right\} - 1\right) \frac{w}{2} dx \\ &= \lim_{w \rightarrow 0^+} \frac{\left[\Gamma(\lambda) + \int_0^\infty \left(\frac{wx}{2}\right)^{\lambda-1} \exp\left\{-\frac{wx}{2}\right\} \left(\exp\left\{-\frac{w}{2x}\right\} - 1\right) \frac{w}{2} dx\right]}{\Gamma(\lambda)} \\ &= \lim_{w \rightarrow 0^+} \left[1 + \frac{\int_0^\infty f_w(x)dx}{\Gamma(\lambda)}\right] = 1 + \frac{\int_0^\infty \lim_{w \rightarrow 0^+} f_w(x)dx}{\Gamma(\lambda)} = 1. \end{aligned}$$

Hence, we have proved property II. □

### 3. Extension of Hurst's Method

Using Feller's randomization and mixture, Hurst (1995) verified the relationship between the cf of  $X|U = u$  and the Laplace-Stieltjes transform of  $U$ . The relationship can be described briefly as follows. Let  $X = \sqrt{U}Z$ , where  $Z \sim N(0, 1)$  and the non-negative random variable  $U$  are independent. Then  $X|U \sim N(0, U)$ . Hence, the cf of  $X$  is given by

$$\begin{aligned} \phi_X(t) &= E[e^{itX}] = \int_{-\infty}^\infty e^{itx} f(x)dx = \int_{-\infty}^\infty e^{itx} \int_0^\infty f(x|u)f(u)dudx \\ &= \int_0^\infty \int_{-\infty}^\infty e^{itx} f(x|u)dx f(u)du = \int_0^\infty \phi_{X|u}(t)f(u)du \\ &= \int_0^\infty e^{-\frac{u}{2}t^2} f(u)du = L_u\left(\frac{t^2}{2}\right), \quad t \in \mathbb{R}. \end{aligned} \tag{3.1}$$

Note that the Laplace-Stieltjes transform  $L_u(t)$  of  $u$  is defined as  $E(e^{-tu}) = \int_0^\infty e^{-tu} dF_U(u)$  when  $U$  is a non-negative random variable (Feller, 1966). Hurst (1995) derived the cf of the symmetric generalized hyperbolic distribution. He then found the cf of the univariate  $t$  distribution using this relationship and the fact that  $t$  distribution is a special case of the generalized hyperbolic distribution.

We employ a similar idea, but use a simple transformation to derive the Laplace-Stieltjes transform. We then extend this to the (multivariate/generalized) versions. The pdf of the generalized inverse Gaussian distribution with parameter  $(\lambda, \nu, \xi)$ ,  $X \sim GIG(\lambda, \nu, \xi)$ , is given in Barndorff-Nielsen

(1978) as

$$f_X(x) = \frac{\left(\frac{\xi}{\nu}\right)^{\frac{\lambda}{2}}}{2K_\lambda(\sqrt{\nu\xi})} x^{\lambda-1} \exp\left\{-\frac{1}{2}\left(\xi x + \frac{\nu}{x}\right)\right\}, \quad x \in \mathbb{R}^+,$$

where  $\lambda \in \mathbb{R}$ , and  $\xi, \nu \geq 0$ .

Next, we express  $L_u(t)$  in terms of the modified Bessel function of the third kind. When  $U$  follows  $GIG(\lambda, \nu, \xi)$ , the Laplace-Stieltjes transform is expressed as follows:

$$\begin{aligned} L_u(t) &= \int_0^\infty e^{-tu} f(u) du \\ &= \int_0^\infty e^{-tu} \frac{\left(\frac{\xi}{\nu}\right)^{\frac{\lambda}{2}} u^{\lambda-1}}{2K_\lambda(\sqrt{\nu\xi})} \exp\left\{-\frac{1}{2}\left(\xi u + \frac{\nu}{u}\right)\right\} du \\ &= \frac{\left(\frac{\xi}{\nu}\right)^{\frac{\lambda}{2}}}{2K_\lambda(\sqrt{\nu\xi})} \int_0^\infty u^{\lambda-1} \exp\left\{-\frac{1}{2}\left((2t + \xi)u + \frac{\nu}{u}\right)\right\} du. \end{aligned}$$

If we now let  $y = \sqrt{(2t + \xi)/\nu} u$ , then  $L_u(t)$  becomes

$$\frac{\left(\frac{\xi}{\nu}\right)^{\frac{\lambda}{2}}}{2K_\lambda(\sqrt{\nu\xi})} \frac{2K_\lambda(\sqrt{\nu(2t + \xi)})}{\left(\frac{2t + \xi}{\nu}\right)^{\frac{\lambda}{2}}} = \frac{K_\lambda(\sqrt{\nu(2t + \xi)})}{K_\lambda(\sqrt{\nu\xi})} \left(\frac{\xi}{2t + \xi}\right)^{\frac{\lambda}{2}}, \quad t \in \mathbb{R}^+. \quad (3.2)$$

Hence, when  $\lambda \in \mathbb{R}$  and  $\nu = \delta^2, \xi = \alpha^2 \geq 0$ , the cf of the symmetric generalized hyperbolic distribution can be obtained using (3.1) and (3.2), as follows:

$$\phi_X(t) = L_u\left(\frac{t^2}{2}\right) = \frac{K_\lambda(\delta \sqrt{t^2 + \alpha^2})}{K_\lambda(\delta \alpha)} \left(\frac{\alpha^2}{t^2 + \alpha^2}\right)^{\frac{\lambda}{2}}, \quad t \in \mathbb{R}. \quad (3.3)$$

Note that this cf was derived in an alternative way in Barndorff-Nielsen and Blaesild (1981) using an appropriate Fourier cosine transform. Hurst (1995) derived the cf using the Laplace-Stieltjes transform of  $u$ .

Now we show that a generalized inverse Gaussian distribution with parameters,  $(-\nu/2, \nu, \xi \rightarrow 0^+)$ , is an inverse gamma distribution with parameters  $(\nu/2, \nu/2)$ . Let  $\lambda = -\nu/2$ , then the pdf of the generalized inverse Gaussian distribution becomes

$$f_X(x) = \frac{\left(\frac{\xi}{\nu}\right)^{-\frac{\nu}{4}}}{2K_{-\frac{\nu}{2}}(\sqrt{\nu\xi})} x^{-\frac{\nu}{2}-1} \exp\left\{-\frac{1}{2}\left(\xi x + \frac{\nu}{x}\right)\right\}.$$

For  $\nu > 0$ , using properties I and II of the modified Bessel function of the third kind, we have

$$\frac{\left(\frac{\xi}{\nu}\right)^{-\frac{\nu}{4}}}{2K_{-\frac{\nu}{2}}(\sqrt{\nu\xi})} = \frac{\left(\frac{\xi}{\nu}\right)^{-\frac{\nu}{4}}}{2K_{\frac{\nu}{2}}(\sqrt{\nu\xi})} \cong \frac{\left(\frac{\nu}{\xi}\right)^{\frac{\nu}{4}}}{2\Gamma\left(\frac{\nu}{2}\right) 2^{\frac{\nu}{2}-1} (\nu\xi)^{-\frac{\nu}{4}}} = \frac{\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}$$

as  $\xi \rightarrow 0^+$ . The remaining part is

$$x^{-\frac{\nu}{2}-1} \exp \left\{ -\frac{1}{2} \left( \xi x + \frac{\nu}{x} \right) \right\} \cong x^{-\frac{\nu}{2}-1} e^{-\frac{\nu/2}{x}}$$

as  $\xi \rightarrow 0^+$ . Hence,  $f_X(x)$  with  $\lambda = -\nu/2$  goes to

$$\frac{\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} x^{-\frac{\nu}{2}-1} e^{-\frac{\nu/2}{x}}, \quad x > 0 \quad (3.4)$$

as  $\xi \rightarrow 0^+$ , which is the density of  $IG(\nu/2, \nu/2)$ .

Therefore, we can now derive the cf of the  $t$  distribution, with degree of freedom  $\nu$ . This is explained in the next Section.

### 3.1. Univariate results

We first calculate the cf of the  $t$  distribution and then derive the cf of the generalized  $t$  distribution.

**Result 1:** Let  $X \sim t(\nu)$ , then the cf of the  $t$  distribution is

$$\phi_X(t) = \frac{K_{\frac{\nu}{2}}(\sqrt{\nu}|t|)(\sqrt{\nu}|t|)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right) 2^{\frac{\nu}{2}-1}}, \quad t \in \mathbb{R} \text{ and } \nu > 0.$$

**Proof:** Since the  $t$  distribution can be expressed as a normal variance-mean mixture using equation (2.2), with  $\nu = \delta$  and because  $GIG(-\nu/2, \nu, \xi \rightarrow 0^+)$  becomes  $IG(\nu/2, \nu/2)$ , we have

$$L_u\left(\frac{t^2}{2}\right) = \frac{K_{\lambda}(\sqrt{(t^2 + \xi)\nu})}{K_{\lambda}(\sqrt{\nu\xi})} \left(\frac{\xi}{t^2 + \xi}\right)^{\frac{\lambda}{2}} \cong \phi_X(t)$$

as  $\xi \rightarrow 0^+$  when  $\lambda = -\nu/2$ . By properties I and II of the modified Bessel function of the third kind, we obtain the cf of the  $t$  distribution as

$$\begin{aligned} L_u\left(\frac{t^2}{2}\right) &\cong \frac{K_{\frac{\nu}{2}}(\sqrt{(t^2 + \xi)\nu})}{\Gamma\left(\frac{\nu}{2}\right) 2^{\frac{\nu}{2}-1} (\nu\xi)^{-\frac{\nu}{4}}} \left(\frac{\xi}{[t^2 + \xi]}\right)^{-\frac{\nu}{4}} \\ &= \frac{K_{\frac{\nu}{2}}(\sqrt{(t^2 + \xi)\nu})}{\Gamma\left(\frac{\nu}{2}\right) 2^{\frac{\nu}{2}-1}} \left(\nu[t^2 + \xi]\right)^{\frac{\nu}{4}} \\ &= \frac{K_{\frac{\nu}{2}}(\sqrt{\nu}|t|)(\sqrt{\nu}|t|)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right) 2^{\frac{\nu}{2}-1}} = \phi_X(t) \end{aligned}$$

by substituting in  $\xi = 0$ . □

Extending this result to the generalized  $t$  distribution is straightforward. The generalized  $t$  distribution (Arellano-Valle and Bolfarine, 1995) retains its conditional distribution, but not in the  $t$  distribution.

**Result 2:** Let  $X \sim gt(\nu, \delta)$ , then the cf is

$$\phi_X(t) = \frac{K_{\frac{\nu}{2}}(\sqrt{\delta}|t|)(\sqrt{\delta}|t|)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)2^{\frac{\nu}{2}-1}}, \quad t \in \mathbb{R} \text{ and } \nu, \delta > 0.$$

**Proof:** Using the stochastic representation in (2.2) and that  $GIG(-\nu/2, \delta, \xi \rightarrow 0^+)$  becomes  $IG(\nu/2, \delta/2)$  by a simple adjustment of previous result in (3.4), we have

$$\begin{aligned} L_u\left(\frac{t^2}{2}\right) &\cong \frac{K_{\frac{\nu}{2}}(\sqrt{(t^2 + \xi)\delta})}{\Gamma\left(\frac{\nu}{2}\right)2^{\frac{\nu}{2}-1}(\delta\xi)^{-\frac{\nu}{4}}}\left(\frac{\xi}{[t^2 + \xi]}\right)^{-\frac{\nu}{4}} \\ &= \frac{K_{\frac{\nu}{2}}(\sqrt{(t^2 + \xi)\delta})}{\Gamma\left(\frac{\nu}{2}\right)2^{\frac{\nu}{2}-1}}(\delta[t^2 + \xi])^{\frac{\nu}{4}} \\ &= \frac{K_{\frac{\nu}{2}}(\sqrt{\delta}|t|)(\sqrt{\delta}|t|)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)2^{\frac{\nu}{2}-1}} = \phi_X(t) \end{aligned}$$

by substituting in  $\xi = 0$ . □

### 3.2. Multivariate results

Let  $\mathbf{X} = \sqrt{U}\mathbf{Z}$ , where  $\mathbf{Z} \sim N_p(\mathbf{0}, \mathbf{I})$  and a non-negative random variable  $U$  are independent. Then  $\mathbf{X}|U \sim N_p(\mathbf{0}, U\mathbf{I})$ . Hence, the cf of  $\mathbf{X}$  is given by

$$\begin{aligned} \phi_{\mathbf{X}}(\mathbf{t}) &= E[e^{it^T \mathbf{X}}] = \int_{\mathbb{R}^p} e^{it^T \mathbf{x}} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^p} e^{it^T \mathbf{x}} \int_0^\infty f_{\mathbf{X}|U}(\mathbf{x}|u) f_U(u) du d\mathbf{x} \\ &= \int_0^\infty \int_{\mathbb{R}^p} e^{it^T \mathbf{x}} f_{\mathbf{X}|U}(\mathbf{x}|u) d\mathbf{x} f_U(u) du = \int_0^\infty \phi_{\mathbf{X}|U}(\mathbf{t}) f_U(u) du \\ &= \int_0^\infty \exp\left(-\frac{1}{2}u\mathbf{t}^T \mathbf{t}\right) f_U(u) du = L_u\left(\frac{\mathbf{t}^T \mathbf{t}}{2}F\right).cd \end{aligned}$$

If the non-negative random variable  $U$  has  $GIG(\lambda, \nu, \xi)$ , then the cf of  $\mathbf{X}$  is

$$\phi_{\mathbf{X}}(\mathbf{x}) = \frac{K_\lambda(\sqrt{\nu(\xi + \mathbf{t}^T \mathbf{t})})}{K_\lambda(\sqrt{\nu\xi})} \left(\frac{\xi}{[\xi + \mathbf{t}^T \mathbf{t}]}\right)^{\frac{\lambda}{2}}$$

by (3.2), and substituting  $\mathbf{t}^T \mathbf{t}/2$  for  $t$ . Using similar techniques for the univariate results, we can derive the cf of the multivariate  $t$  distribution.

**Result 3:** Let  $\mathbf{X} \sim t_p(\nu)$ . Then the cf of the multivariate  $t$  distribution is

$$\phi_{\mathbf{X}}(\mathbf{t}) = \frac{K_{\frac{\nu}{2}}(\|\sqrt{\nu}\mathbf{t}\|)(\|\sqrt{\nu}\mathbf{t}\|)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)2^{\frac{\nu}{2}-1}}, \quad \mathbf{t} \in \mathbb{R}^p \text{ and } \nu > 0,$$

where  $\|\mathbf{t}\| = \sqrt{\mathbf{t}^T \mathbf{t}}$ .

**Proof:** Since the multivariate  $t$  distribution can be expressed as a normal variance-mean mixture by (2.3), with  $\nu = \delta$ , and  $GIG(-\nu/2, \nu, \xi \rightarrow 0^+)$  becomes  $IG(\nu/2, \nu/2)$ ,  $L_u(\mathbf{t}^\top \mathbf{t}/2)$  converges to  $\phi_{\mathbf{X}}(\mathbf{t})$  as  $\xi \rightarrow 0^+$  when  $\lambda = -\nu/2$ . Using properties I and II of the modified Bessel function of the third kind, we obtain the cf of the multivariate  $t$  as follows:

$$\begin{aligned} L_u\left(\frac{\mathbf{t}^\top \mathbf{t}}{2}\right) &\cong \frac{K_{\frac{\nu}{2}} \sqrt{(\mathbf{t}^\top \mathbf{t} + \xi) \nu}}{\Gamma\left(\frac{\nu}{2}\right) 2^{\frac{\nu}{2}-1} (\nu \xi)^{-\frac{\nu}{4}}} \left(\frac{\xi}{[\mathbf{t}^\top \mathbf{t} + \xi]}\right)^{-\frac{\nu}{4}} \\ &= \frac{K_{\frac{\nu}{2}} \sqrt{(\mathbf{t}^\top \mathbf{t} + \xi) \nu}}{\Gamma\left(\frac{\nu}{2}\right) 2^{\frac{\nu}{2}-1}} \left(\nu [\mathbf{t}^\top \mathbf{t} + \xi]\right)^{\frac{\nu}{4}} \\ &= \frac{K_{\frac{\nu}{2}} (\|\sqrt{\nu} \mathbf{t}\|) (\|\sqrt{\nu} \mathbf{t}\|)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right) 2^{\frac{\nu}{2}-1}} = \phi_{\mathbf{X}}(\mathbf{t}) \end{aligned}$$

by substituting in  $\xi = 0$ . □

The cf of the multivariate generalized  $t$  distribution is obtained as follows.

**Result 4:** Let  $\mathbf{X} \sim gt_p(\nu, \delta)$ , then the cf of the multivariate generalized  $t$  distribution is

$$\phi_{\mathbf{X}}(\mathbf{t}) = \frac{K_{\frac{\nu}{2}} (\|\sqrt{\delta} \mathbf{t}\|) (\|\sqrt{\delta} \mathbf{t}\|)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right) 2^{\frac{\nu}{2}-1}}, \quad \mathbf{t} \in \mathbb{R}^p \text{ and } \nu, \delta > 0,$$

where  $\|\mathbf{t}\| = \sqrt{\mathbf{t}^\top \mathbf{t}}$ .

**Proof:** Using the stochastic representation (2.3) and that  $GIG(-\nu/2, \delta, \xi \rightarrow 0^+)$  becomes  $IG(\nu/2, \delta/2)$  by simple adjustment of previous result in (3.4), we have

$$\begin{aligned} L_u\left(\frac{\mathbf{t}^\top \mathbf{t}}{2}\right) &\cong \frac{K_{\frac{\nu}{2}} \left(\sqrt{(\mathbf{t}^\top \mathbf{t} + \xi) \delta}\right)}{\Gamma\left(\frac{\nu}{2}\right) 2^{\frac{\nu}{2}-1} (\delta \xi)^{-\frac{\nu}{4}}} \left(\frac{\xi}{[\mathbf{t}^\top \mathbf{t} + \xi]}\right)^{-\frac{\nu}{4}} \\ &= \frac{K_{\frac{\nu}{2}} \left(\sqrt{(\mathbf{t}^\top \mathbf{t} + \xi) \delta}\right)}{\Gamma\left(\frac{\nu}{2}\right) 2^{\frac{\nu}{2}-1}} \left(\delta [\mathbf{t}^\top \mathbf{t} + \xi]\right)^{\frac{\nu}{4}} \\ &= \frac{K_{\frac{\nu}{2}} (\|\sqrt{\delta} \mathbf{t}\|) (\|\sqrt{\delta} \mathbf{t}\|)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right) 2^{\frac{\nu}{2}-1}} = \phi_{\mathbf{X}}(\mathbf{t}) \end{aligned}$$

by substituting in  $\xi = 0$ . □

### 3.3. Introducing location-scale parameters

Once we have the Results 1–4, the location-scale extensions are straightforward using the properties of cf. Similar to normal distribution, for the univariate case, we have  $Y = \mu + \sigma X$ , where  $X \sim gt(\nu, \delta)$  and we have  $\mathbf{Y} = \boldsymbol{\mu} + \Sigma^{1/2} \mathbf{X}$ , where  $\mathbf{X} \sim gt_p(\nu, \delta)$  for the multivariate case. Then,  $Y \sim gt(\mu, \sigma^2, \nu, \delta)$  and  $\mathbf{Y} \sim gt_p(\boldsymbol{\mu}, \Sigma, \nu, \delta)$ , respectively. When  $\nu = \delta$ , we have the (multivariate)  $t$  distribution with location and scale parameters.

Since  $\phi_Y(t) = E[e^{itY}] = e^{it\mu} E[e^{it\sigma X}] = e^{it\mu} \phi_X(\sigma t)$ , we have the cf of  $Y$  as

$$\phi_Y(t) = \exp(it\mu) \frac{K_{\frac{\nu}{2}}(\sqrt{\delta}|\sigma t|) (\sqrt{\delta}|\sigma t|)^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2}) 2^{\frac{\nu}{2}-1}}, \quad t \in \mathbb{R} \text{ and } \nu, \delta, \sigma > 0.$$

Similarly  $\phi_Y(t) = e^{it^\top \mu} \phi_X(\Sigma^{1/2} \mathbf{t})$ . Hence, we have the cf of  $\mathbf{Y}$  as

$$\phi_Y(\mathbf{t}) = \exp(it^\top \mu) \frac{K_{\frac{\nu}{2}}(\|\sqrt{\delta} \Sigma^{\frac{1}{2}} \mathbf{t}\|) (\|\sqrt{\delta} \Sigma^{\frac{1}{2}} \mathbf{t}\|)^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2}) 2^{\frac{\nu}{2}-1}},$$

$\mathbf{t} \in \mathbb{R}^p$  and  $\nu, \delta > 0$ , where  $\|\mathbf{t}\| = \sqrt{\mathbf{t}^\top \mathbf{t}}$ . Note that  $(Y - \mu)/\sigma \sim gt(\nu, \delta)$  and  $\Sigma^{-1/2}(\mathbf{Y} - \mu) \sim gt_p(\nu, \delta)$ .

#### 4. Conclusion

The  $t$  distribution has properties that make it a useful alternative to normal distribution. The additional parameter of degrees of freedom governs the heaviness of the tails. This paper derived the cfs of (multivariate/generalized)  $t$  distributions based on the principle of randomization and mixtures as well as the cf of the symmetric generalized hyperbolic distribution. The expression of the cfs depend on the integral representation of the modified Bessel function of the third kind. The derivation methods are simple and independent of contour integration; consequently, the methods are suitable for graduate students.

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