

A Simultaneous Test Procedure

Seungman Hong^a, Joong-Jae Cho^b, Hyo-II Park^{1,c}

^aDepartment of Informational Statistics, Korea University, Korea

^bDepartment of Information Statistics, Chungbuk National University, Korea

^cDepartment of Statistics, Chonju University, Korea

Abstract

In this study, we propose a simultaneous test procedure based on the individual - values for each sub-null hypothesis with several well-known combining functions. We then compare the efficiency of our procedure with existing tests by obtaining empirical powers through a simulation study. Finally, we discuss some interesting features related to simultaneous test and point out a misconduct for the simulation study published in the previous work.

Keywords: Combining function, location-scale model, permutation principle, sub-null hypothesis, two-sample problem.

1. Introduction

The expected values of the two populations are required to be different for the Wilcoxon test to be powerful versus the Kolmogorov-Smirnov test that is well suited if the variances are different (Baumgartner *et al.*, 1998). Then which test should be chosen when one does not know in advance whether the mean or variance or both might be different? Baumgartner *et al.* (1998) proposed a test by extending the idea of Anderson and Darling's (1954) for the one-sample case to the two-sample problem. However, one can furnish another methodology by introducing the concept of the simultaneous test for location and scale parameters without further assumptions for the underlying distribution except the continuity. This topic has been initiated by Lepage (1971) with individual nonparametric tests for location and scale parameters and the quadratic form to combine two individual tests. Subsequently, many statisticians, such as Murakami (2007), Neuhäuser *et al.* (2011) and Marozzi (2012), have modified and proposed new procedures for this topic with various nonparametric tests for location and scale parameters. All the reviewed results are based on the quadratic form to combine individual nonparametric test statistics. Park (2011, 2013) considered some nonparametric procedures for this theme but used different combining functions with the standardized form of statistics.

The null distribution for any given nonparametric statistic can be achieved by applying the permutation principle for a small sample case. For a large sample case, it is customary to consider deriving the limiting null distribution with the large sample approximation theorem; however, high performance computers and efficient software have now made it possible to apply the permutation principle with the Monte-Carlo method.

In this study, we propose a nonparametric simultaneous test procedure and compare its efficiency with those of the several existing tests. This paper will be organized in the following order. In Section 2, we propose nonparametric simultaneous tests based on the *p*-values of the individual tests using the

¹ Corresponding author: Professor, Department of Statistics, Chonju University, Chungbuk 360-764, Korea.
E-mail: hipark2013@gmail.com

various combining functions. Then we compare their efficiencies with other existing tests by obtaining empirical powers through a simulation study in Section 3. We apply the permutation principle to obtain the null distribution in the simulation study. In Section 4, we discuss some interesting features related to the simultaneous tests and comment on the errors of the previous work.

2. A New Nonparametric Simultaneous Test Procedure

Let X_1, \dots, X_m and Y_1, \dots, Y_n be two independent random samples from populations with distribution functions F and G , respectively. In this study, we assume a following location-scale model such as for all $x \in (-\infty, \infty)$

$$G(x) = F\left(\frac{x}{\eta} - \delta\right)$$

for some $\delta \in (-\infty, \infty)$ and $\eta \in (0, \infty)$. We note that δ is the location translation parameter and η , the scale parameter. Then it is of our interest to propose a nonparametric test procedure for testing

$$H_0 : \{\delta = 0\} \cap \{\eta = 1\} \quad \text{against} \quad H_1 : \{\delta \neq 0\} \cup \{\eta \neq 1\}. \quad (2.1)$$

According to the approach of Lepage (1971), one may choose the Wilcoxon rank sum test for the sub-null hypothesis $H_0^1 : \delta = 0$, which is for the location translation parameter only and the Ansari-Bradley test (1960) for $H_0^2 : \eta = 1$, for the scale parameter only. In addition, one may use the Mood test (1954). We note that the Mood and Ansari-Bradley statistics are even translation invariant and the Wilcoxon score statistic is the odd translation invariant (*cf.* Duran *et al.*, 1976). Furthermore it is known that the even and odd translation invariant statistics are uncorrelated (Randall and Hogg, 1971). Therefore, all the proposed statistics for testing (2.1) have the form of a sum of squares of standardized statistics, which is a quadratic form. However, in this study we will use the p -values of two individual tests to construct the test statistics for testing (2.1) instead of statistics themselves in the spirit of Pesarin (2001). For this, let λ_1 and λ_2 be the p -values of any chosen two nonparametric individual tests for each sub-null hypothesis for $H_0^1 : \delta = 0$ and $H_0^2 : \eta = 1$, respectively. Then we consider to apply the nonparametric combination (NPC) test approach (Pesarin, 2001). For this, we first review several combination functions briefly.

- (1) The Fisher omnibus combining function (Fisher, 1932) is based on the statistic

$$F = -2 \sum_{i=1}^2 \log(\lambda_i),$$

where \log means the natural logarithm. Consequently, the null distribution of F follows a chi-square distribution with 4 degrees of freedom if the individual test statistics are independent.

- (2) The Liptak combining function (Liptak, 1958) is based on the statistic

$$L = \sum_{i=1}^2 \Phi^{-1}(1 - \lambda_i),$$

where Φ is the standard normal distribution function and Φ^{-1} , its inverse.

(3) The Tippett combining function (Tippett, 1931) is given by

$$T = \max\{1 - \lambda_1, 1 - \lambda_2\}.$$

As an alternative form for the Tippett combining function, one may consider the following form:

$$T' = \min\{\lambda_1, \lambda_2\}.$$

Then in order to complete the test with the NPC test approach, we should have the null distribution for any chosen combining function. For this we first note that the individual two nonparametric statistics are uncorrelated if they are the even and odd statistics for the location and scale, respectively. This consequently implies that the two individual nonparametric tests are asymptotically independent. Then we can summarize the asymptotic distributions of the combining functions in the following lemma.

Lemma 1. *If the two chosen nonparametric statistics are odd and even translation invariant for testing $H_0^1 : \delta = 0$ and $H_0^2 : \eta = 1$, respectively, under H_0 , the limiting null distribution for each combining function as follows:*

- (1) *The distribution of F is a chi-square with 4 degrees of freedom.*
- (2) *The distribution of L is a normal with mean 0 and variance 2.*
- (3) *The distribution of T is the product of two independent uniform on (0, 1).*

The proof of Lemma can be achieved easily by applying Slutsky's Theorem.

Then by using the limiting null distribution for each combining function, one may complete the test (2.1) by obtaining an overall p -value. In addition, there is another way to obtain an overall p -value for each combining function by applying the permutation principle. It is well-known that the permutation principle yields an exact but conditional test (Good, 2000). In the next section, we will carry out a simulation study to compare the efficiency of the proposed test procedure with other tests based on the permutation principle for the reasons discussed in Section 4.

3. Simulation Results

In this section, we consider a simulation study to compare the performance among several test procedures with our proposed one. For this subject, we consider the normal, Cauchy, uniform, exponential and double exponential distributions as the underlying one. For the comparison study with our procedure, we consider the Kolmogorov-Smirnov, Baumgartner, Wilcoxon and Mood tests. For each distribution, we consider three cases separately as follows: first both location and scale parameters change (Tables 1, 4, 7, 10 and 13); second only the location parameter changes (Tables 2, 5, 8, 11 and 14) and finally only the scale parameter changes (Tables 3, 6, 9, 12 and 15). The values of (δ, η) vary from $(0.0, 1.0)$ up to $(1.0, 2.0)$ with the increment 0.2 for both parameters. However, the other value is fixed at the sub-null hypothesis when only one parameter value varies. In our simultaneous procedure, we choose the Wilcoxon test for the location parameter and the Mood test for the scale parameter. To obtain the null distribution of each procedure, we apply the permutation principle as follows: for each simulation, we repeat 5000 simulations with a Monte-Carlo approach. In addition, we repeat 5000 simulations again with the Monte-Carlo approach. We use the SAS/IML with PC version and the nominal significance level is 0.05.

Table 1: Empirical powers for the normal distribution(both location and scale)

Test	(m, n)	(δ, η)					
		0.0, 1.0	0.2, 1.2	0.4, 1.4	0.6, 1.6	0.8, 1.8	1.0, 2.0
Kolmogorov-Smirnov	10, 10	0.0364	0.0562	0.0796	0.1272	0.1818	0.2624
	15, 10	0.0386	0.0674	0.1178	0.1808	0.2540	0.3356
	10, 15	0.0366	0.0598	0.1070	0.1726	0.2502	0.3402
Baumgartner	10, 10	0.0418	0.0604	0.1054	0.1728	0.2488	0.3412
	15, 10	0.0422	0.0750	0.1362	0.2190	0.3096	0.3998
	10, 15	0.0446	0.0852	0.1326	0.2020	0.3064	0.3976
Wilcoxon	10, 10	0.0500	0.1078	0.1804	0.2542	0.3314	0.3984
	15, 10	0.0612	0.1188	0.1990	0.2926	0.3632	0.4356
	10, 15	0.0564	0.1154	0.1792	0.2698	0.3528	0.4312
Mood	10, 10	0.0494	0.1076	0.1726	0.2434	0.3126	0.3626
	15, 10	0.0416	0.1202	0.2244	0.3312	0.4334	0.5228
	10, 15	0.0512	0.1274	0.2012	0.2690	0.3308	0.3880
Tippett	10, 10	0.0470	0.1158	0.1998	0.3142	0.4282	0.5410
	15, 10	0.0512	0.1402	0.2704	0.4006	0.5386	0.6386
	10, 15	0.0574	0.1374	0.2476	0.3502	0.4734	0.5664
Liptak	10, 10	0.0538	0.1384	0.2578	0.4334	0.5656	0.6800
	15, 10	0.0566	0.1646	0.3456	0.5456	0.6834	0.7846
	10, 15	0.0568	0.1492	0.2836	0.4726	0.6476	0.7502
Fisher	10, 10	0.0510	0.1348	0.2454	0.3938	0.5412	0.6676
	15, 10	0.0524	0.1624	0.3258	0.5196	0.6602	0.7628
	10, 15	0.0546	0.1636	0.2834	0.4564	0.5976	0.7354

Table 2: Empirical powers for the normal distribution(location only)

Test	(m, n)	δ				
		0.2	0.4	0.6	0.8	1.0
Kolmogorov-Smirnov	10, 10	0.0502	0.0818	0.1522	0.2754	0.4226
	15, 10	0.0676	0.1240	0.2134	0.3396	0.4748
	10, 15	0.0568	0.1156	0.2168	0.3412	0.5238
Baumgartner	10, 10	0.0688	0.1134	0.2188	0.3686	0.5272
	15, 10	0.0706	0.1254	0.2524	0.4030	0.5790
	10, 15	0.0890	0.1612	0.2876	0.4456	0.6376
Wilcoxon	10, 10	0.1204	0.2166	0.3706	0.5328	0.6876
	15, 10	0.1156	0.2394	0.4020	0.5926	0.7454
	10, 15	0.1286	0.2468	0.3998	0.5892	0.7646
Tippett	10, 10	0.0922	0.1560	0.2604	0.4150	0.5792
	15, 10	0.1028	0.1684	0.3174	0.4796	0.6484
	10, 15	0.1090	0.1806	0.3086	0.4650	0.6544
Liptak	10, 10	0.0854	0.1412	0.2156	0.3376	0.4696
	15, 10	0.1010	0.1778	0.2878	0.4506	0.5968
	10, 15	0.0932	0.1524	0.2326	0.3186	0.4142
Fisher	10, 10	0.0946	0.1516	0.2598	0.3966	0.5388
	15, 10	0.1078	0.1856	0.3160	0.4812	0.6512
	10, 15	0.1074	0.1840	0.2922	0.4234	0.5870

The proposed tests generally show relatively high performance for the normal, Cauchy and double exponential distributions when both values vary. The Wilcoxon test achieves high efficiency in terms of empirical power for the uniform and exponential distributions. The Kolmogorov-Smirnov and Baumgartner tests achieve mediocre performance. We note that even when only the scale changes, the performance of Mood test shows the inferiority for the uniform and exponential distributions. In addition, the Mood test indicates some serious biasedness for uniform and exponential distributions. For the location only change case, the Wilcoxon test performs the best efficiency for all distributions.

Table 3: Empirical powers for the normal distribution(scale only)

Test	(m, n)	η				
		1.2	1.4	1.6	1.8	2.0
Kolmogorov -Smirnov	10, 10	0.0422	0.0444	0.0514	0.0586	0.0644
	15, 10	0.0470	0.0654	0.0812	0.0970	0.1138
	10, 15	0.0394	0.0436	0.0508	0.0606	0.0776
Baumgartner	10, 10	0.0482	0.0542	0.0634	0.0722	0.0846
	15, 10	0.0536	0.0698	0.0926	0.1104	0.1378
	10, 15	0.0428	0.0470	0.0570	0.0702	0.0850
Mood	10, 10	0.1086	0.1976	0.2864	0.3826	0.4602
	15, 10	0.1126	0.2212	0.3426	0.4758	0.5562
	10, 15	0.1232	0.2206	0.3348	0.4274	0.5300
Tippett	10, 10	0.0822	0.1434	0.2186	0.2968	0.3756
	15, 10	0.1018	0.1756	0.2762	0.3806	0.4814
	10, 15	0.0920	0.1658	0.2534	0.3350	0.4296
Liptak	10, 10	0.0902	0.1458	0.2024	0.2572	0.3178
	15, 10	0.1026	0.1640	0.2372	0.3342	0.4018
	10, 15	0.1034	0.1592	0.2146	0.2816	0.3596
Fisher	10, 10	0.0884	0.1474	0.2198	0.3044	0.3782
	15, 10	0.1012	0.1752	0.2634	0.3766	0.4734
	10, 15	0.0978	0.1676	0.2476	0.3398	0.4160

Table 4: Empirical powers for the Cauchy distribution(both location and scale)

Test	(m, n)	(δ, η)					
		0.0, 1.0	0.2, 1.2	0.4, 1.4	0.6, 1.6	0.8, 1.8	1.0, 2.0
Kolmogorov -Smirnov	10, 10	0.0406	0.0462	0.0676	0.0930	0.1230	0.1576
	15, 10	0.0388	0.0542	0.0688	0.1092	0.1472	0.1852
	10, 15	0.0534	0.0648	0.0858	0.1126	0.1572	0.2042
Baumgartner	10, 10	0.0490	0.0610	0.0850	0.1090	0.1398	0.1638
	15, 10	0.0540	0.0576	0.0736	0.1078	0.1444	0.1842
	10, 15	0.0594	0.0708	0.0982	0.1266	0.1646	0.2130
Wilcoxon	10, 10	0.0682	0.0884	0.1254	0.1624	0.1956	0.2180
	15, 10	0.0466	0.0742	0.1272	0.1794	0.2178	0.2504
	10, 15	0.0708	0.1008	0.1298	0.1712	0.2022	0.2366
Mood	10, 10	0.0564	0.0836	0.1204	0.1632	0.1984	0.2152
	15, 10	0.0486	0.0874	0.1292	0.1826	0.2256	0.2744
	10, 15	0.0624	0.0894	0.1306	0.1788	0.2066	0.2464
Tippett	10, 10	0.0620	0.1048	0.1460	0.1886	0.2412	0.2898
	15, 10	0.0482	0.0926	0.1446	0.2076	0.2714	0.3346
	10, 15	0.0652	0.1090	0.1598	0.2174	0.2778	0.3212
Liptak	10, 10	0.0614	0.1084	0.1694	0.2240	0.2990	0.3506
	15, 10	0.0486	0.1050	0.1852	0.2654	0.3464	0.4112
	10, 15	0.0476	0.1034	0.1834	0.2566	0.3174	0.3678
Fisher	10, 10	0.0596	0.1142	0.1786	0.2358	0.2920	0.3436
	15, 10	0.0448	0.0936	0.1788	0.2462	0.3322	0.4062
	10, 15	0.0594	0.1134	0.1710	0.2532	0.3178	0.3714

In addition, the Baumgartner test considerably shows its power for the exponential distribution. Finally, for the scale change only, Mood test only achieves best performance for the normal, Cauchy and double exponential distributions. In general, it is recommended to use our proposed procedure by choosing a suitable combining function when one cannot be sure when the difference between two populations comes from the location or scale or both.

Table 5: Empirical powers for the Cauchy distribution(location only)

Test	(m, n)	δ				
		0.2	0.4	0.6	0.8	1.0
Kolmogorov -Smirnov	10, 10	0.0436	0.0688	0.1036	0.1686	0.2234
	15, 10	0.0464	0.0732	0.1244	0.1888	0.2678
	10, 15	0.0678	0.0912	0.1424	0.2164	0.3138
Baumgartner	10, 10	0.0600	0.0896	0.1218	0.1722	0.2264
	15, 10	0.0602	0.0764	0.1180	0.1880	0.2556
	10, 15	0.0726	0.1012	0.1530	0.2156	0.2942
Wilcoxon	10, 10	0.0968	0.1414	0.1974	0.2546	0.3100
	15, 10	0.0844	0.1434	0.2182	0.2898	0.3754
	10, 15	0.1064	0.1548	0.2078	0.2924	0.3648
Tippett	10, 10	0.0826	0.1160	0.1552	0.2056	0.2534
	15, 10	0.0752	0.0982	0.1604	0.2334	0.2976
	10, 15	0.0920	0.1310	0.1756	0.2342	0.3018
Liptak	10, 10	0.0848	0.1146	0.1498	0.1880	0.2310
	15, 10	0.0736	0.1150	0.1726	0.2308	0.2786
	10, 15	0.0792	0.1164	0.1590	0.2066	0.2442
Fisher	10, 10	0.0848	0.1242	0.1694	0.2042	0.2436
	15, 10	0.0698	0.1098	0.1722	0.2352	0.3034
	10, 15	0.0854	0.1216	0.1838	0.2346	0.2898

Table 6: Empirical powers for the Cauchy distribution(scale only)

Test	(m, n)	η				
		1.2	1.4	1.6	1.8	2.0
Kolmogorov -Smirnov	10, 10	0.0442	0.0434	0.0458	0.0520	0.0626
	15, 10	0.0426	0.0496	0.0576	0.0664	0.0768
	10, 15	0.0486	0.0520	0.0540	0.0646	0.0722
Baumgartner	10, 10	0.0528	0.0542	0.0604	0.0670	0.0744
	15, 10	0.0510	0.0608	0.0652	0.0728	0.0796
	10, 15	0.0576	0.0598	0.0582	0.0658	0.0716
Mood	10, 10	0.0938	0.1244	0.1676	0.2012	0.2480
	15, 10	0.0806	0.1276	0.1638	0.2146	0.2518
	10, 15	0.0882	0.1280	0.1724	0.2164	0.2624
Tippett	10, 10	0.0784	0.1112	0.1312	0.1522	0.1780
	15, 10	0.0742	0.0934	0.1380	0.1688	0.2006
	10, 15	0.0860	0.1062	0.1346	0.1662	0.1976
Liptak	10, 10	0.0856	0.1076	0.1360	0.1696	0.1908
	15, 10	0.0698	0.0988	0.1254	0.1634	0.1934
	10, 15	0.0704	0.1026	0.1332	0.1644	0.1912
Fisher	10, 10	0.0842	0.1098	0.1388	0.1656	0.1956
	15, 10	0.0706	0.0960	0.1246	0.1698	0.2008
	10, 15	0.0796	0.1082	0.1342	0.1690	0.2112

4. Some Concluding Remarks

Several recent nonparametric test procedures have been proposed that apply the simultaneous use of multiple statistics to test whether the two samples come from the same population. One of them is the so-called versatile test to increase the power of test (Fleming and Harrington, 1991). In this approach, Park (2010) proposed a procedure based on a group of quantile statistics for the two sample problem. In addition, the multi-aspect test is another nonparametric procedure in this manner (Pesarin, 2001). This type of tests divides the null hypothesis into several aspects that intersects the divided aspects such as (2.1). By choosing a suitable nonparametric test for each aspect and combining them

Table 7: Empirical powers for the uniform distribution(both location and scale)

Test	(m, n)	(δ, η)					
		0.0, 1.0	0.2, 1.2	0.4, 1.4	0.6, 1.6	0.8, 1.8	1.0, 2.0
Kolmogorov -Smirnov	10, 10	0.0418	0.1086	0.2890	0.4966	0.6762	0.8174
	15, 10	0.0434	0.1268	0.3544	0.5784	0.7954	0.9026
	10, 15	0.0552	0.1424	0.3624	0.6588	0.8340	0.9384
Baumgartner	10, 10	0.0506	0.1654	0.4306	0.6598	0.8536	0.9292
	15, 10	0.0460	0.1820	0.4978	0.7584	0.9008	0.9614
	10, 15	0.0660	0.2002	0.5012	0.7856	0.9206	0.9848
Wilcoxon	10, 10	0.0622	0.2856	0.5414	0.7740	0.9002	0.9596
	15, 10	0.0574	0.3034	0.6152	0.8382	0.9382	0.9748
	10, 15	0.0608	0.3008	0.6080	0.8342	0.9414	0.9850
Mood	10, 10	0.0534	0.1024	0.1020	0.0724	0.0390	0.0172
	15, 10	0.0566	0.1350	0.1496	0.1230	0.0736	0.0386
	10, 15	0.0532	0.1046	0.0988	0.0616	0.0358	0.0188
Tippett	10, 10	0.0556	0.2398	0.4884	0.6754	0.8458	0.9308
	15, 10	0.0564	0.2664	0.5602	0.7868	0.9062	0.9612
	10, 15	0.0612	0.2646	0.5414	0.7734	0.9034	0.9690
Liptak	10, 10	0.0630	0.2888	0.5818	0.7696	0.8910	0.9586
	15, 10	0.0556	0.3664	0.6734	0.8534	0.9434	0.9776
	10, 15	0.0638	0.2776	0.5780	0.7778	0.9006	0.9568
Fisher	10, 10	0.0544	0.2822	0.5664	0.7606	0.8994	0.9598
	15, 10	0.0608	0.3414	0.6468	0.8404	0.9382	0.9762
	10, 15	0.0586	0.2856	0.5980	0.8322	0.9234	0.9820

Table 8: Empirical powers for the uniform distribution(location only)

Test	(m, n)	δ				
		0.2	0.4	0.6	0.8	1.0
Kolmogorov -Smirnov	10, 10	0.0490	0.0756	0.1212	0.2154	0.3178
	15, 10	0.0566	0.0888	0.1548	0.2598	0.4076
	10, 15	0.0742	0.1114	0.1786	0.2662	0.3900
Baumgartner	10, 10	0.0734	0.1236	0.2030	0.3450	0.4882
	15, 10	0.0668	0.1202	0.2214	0.3876	0.5564
	10, 15	0.0876	0.1628	0.2726	0.4178	0.5912
Wilcoxon	10, 10	0.1192	0.2190	0.3584	0.5040	0.6508
	15, 10	0.1192	0.2466	0.4006	0.5614	0.7324
	10, 15	0.1310	0.2444	0.3826	0.5482	0.7052
Tippett	10, 10	0.0916	0.1538	0.2534	0.4004	0.5294
	15, 10	0.1078	0.1618	0.2928	0.4396	0.6040
	10, 15	0.1084	0.1802	0.2966	0.4318	0.5958
Liptak	10, 10	0.0932	0.1542	0.2114	0.3076	0.4312
	15, 10	0.0986	0.1748	0.2944	0.4234	0.5484
	10, 15	0.0848	0.1280	0.1838	0.2604	0.3766
Fisher	10, 10	0.0970	0.1522	0.2376	0.3562	0.5106
	15, 10	0.1012	0.1816	0.3070	0.4428	0.5856
	10, 15	0.0996	0.1678	0.2492	0.3842	0.5348

together, one may complete the test by obtaining an overall-value. However, the simultaneous test is different from the previous two procedures in the sense that the previous two procedures are used for the enhancement of power, the simultaneous procedure should be used when one can not assure which parameter is different from the two distributions. The simulation results adequately show this consideration.

The limiting distribution for each combining function in Lemma could be applied to the simulation

Table 9: Empirical powers for the uniform distribution(scale only)

Test	(m, n)	η				
		1.2	1.4	1.6	1.8	2.0
Kolmogorov-Smirnov	10, 10	0.0672	0.1516	0.2652	0.3688	0.4626
	15, 10	0.0778	0.1878	0.3148	0.4462	0.5582
	10, 15	0.0828	0.1934	0.3280	0.4894	0.6274
Baumgartner	10, 10	0.1074	0.2262	0.3592	0.4876	0.5910
	15, 10	0.1206	0.2570	0.4294	0.5576	0.6870
	10, 15	0.1170	0.2530	0.4294	0.5848	0.6992
Mood	10, 10	0.1418	0.2102	0.2408	0.2632	0.2678
	15, 10	0.1802	0.2748	0.3446	0.3720	0.4014
	10, 15	0.1464	0.2114	0.2392	0.2330	0.2086
Tippett	10, 10	0.1718	0.3378	0.4756	0.5892	0.6648
	15, 10	0.2236	0.3970	0.5568	0.6686	0.7562
	10, 15	0.1950	0.3546	0.5120	0.6238	0.7280
Liptak	10, 10	0.2274	0.4552	0.6334	0.7564	0.8452
	15, 10	0.2856	0.5374	0.7096	0.8192	0.8916
	10, 15	0.2248	0.4674	0.6726	0.7986	0.8708
Fisher	10, 10	0.2172	0.4352	0.6002	0.7280	0.8156
	15, 10	0.2684	0.5158	0.6718	0.8002	0.8724
	10, 15	0.2300	0.4460	0.6324	0.7714	0.8564

Table 10: Empirical powers for the exponential distribution(both location and scale)

Test	(m, n)	(δ, η)					
		0.0, 1.0	0.2, 1.2	0.4, 1.4	0.6, 1.6	0.8, 1.8	1.0, 2.0
Kolmogorov-Smirnov	10, 10	0.0404	0.1088	0.2740	0.4826	0.6794	0.8234
	15, 10	0.0556	0.1184	0.3602	0.6256	0.8396	0.9422
	10, 15	0.0392	0.1542	0.3722	0.6228	0.7858	0.9048
Baumgartner	10, 10	0.0490	0.1516	0.4016	0.6472	0.8088	0.9076
	15, 10	0.0630	0.1336	0.4318	0.7020	0.8824	0.9676
	10, 15	0.0496	0.2168	0.5126	0.7438	0.8992	0.9690
Wilcoxon	10, 10	0.0488	0.2524	0.5494	0.7604	0.8886	0.9522
	15, 10	0.0512	0.2804	0.5990	0.8356	0.9574	0.9868
	10, 15	0.0588	0.2942	0.6066	0.8108	0.9374	0.9792
Mood	10, 10	0.0536	0.0116	0.0034	0.0002	0.0000	0.0000
	15, 10	0.0562	0.0080	0.0022	0.0014	0.0000	0.0000
	10, 15	0.0530	0.0170	0.0042	0.0000	0.0000	0.0000
Tippett	10, 10	0.0484	0.1728	0.4446	0.6686	0.8112	0.9146
	15, 10	0.0516	0.1694	0.4766	0.7292	0.8962	0.9698
	10, 15	0.0618	0.2144	0.5030	0.7198	0.8894	0.9494
Liptak	10, 10	0.0492	0.0966	0.2172	0.3564	0.5008	0.6594
	15, 10	0.0574	0.1302	0.2734	0.4688	0.6556	0.8056
	10, 15	0.0682	0.0982	0.1948	0.3392	0.4982	0.6410
Fisher	10, 10	0.0452	0.1344	0.3246	0.5526	0.7204	0.8482
	15, 10	0.0550	0.1458	0.3892	0.6284	0.8278	0.9354
	10, 15	0.0614	0.1576	0.3664	0.6084	0.7852	0.8968

study in the previous section to obtain the empirical powers easily. However, we did not use this approach due to the permutation principle since the exact distributions of the Kolmogorov-Smirnov and Baumgartner statistics are very complicated and even the asymptotic approach is messy for the Kolmogorov-Smirnov statistic. Even the test based on the permutation principle is exact.

We should also note that we used p -values of the sub-null hypotheses instead of nonparametric statistics to construct the test statistics. By this approach, we can apply our procedure to any form

Table 11: Empirical powers for the exponential distribution(location only)

Test	(m, n)	δ			
		0.2	0.4	0.6	1.0
Kolmogorov-Smirnov	10, 10	0.0804	0.1756	0.3560	0.5242
	15, 10	0.0776	0.2428	0.4890	0.7214
	10, 15	0.1112	0.2602	0.4752	0.6646
Baumgartner	10, 10	0.0988	0.2498	0.4678	0.6412
	15, 10	0.0940	0.2536	0.4966	0.7098
	10, 15	0.1584	0.3542	0.6024	0.7608
Wilcoxon	10, 10	0.1656	0.3700	0.5684	0.7152
	15, 10	0.1712	0.3942	0.6266	0.7820
	10, 15	0.2012	0.4084	0.6122	0.7604
Tippett	10, 10	0.1098	0.2606	0.4508	0.6036
	15, 10	0.1084	0.2768	0.4946	0.6824
	10, 15	0.1506	0.3036	0.5024	0.6668
Liptak	10, 10	0.0490	0.0816	0.1602	0.2542
	15, 10	0.0704	0.1164	0.2034	0.3142
	10, 15	0.0676	0.1022	0.1740	0.2718
Fisher	10, 10	0.0734	0.1698	0.3066	0.4540
	15, 10	0.0808	0.1842	0.3558	0.5446
	10, 15	0.1042	0.1996	0.3588	0.5228

Table 12: Empirical powers for the exponential distribution(scale only)

Test	(m, n)	η			
		1.2	1.4	1.6	1.8
Kolmogorov-Smirnov	10, 10	0.0556	0.0802	0.1064	0.1456
	15, 10	0.0514	0.0784	0.1198	0.1602
	10, 15	0.0760	0.1088	0.1466	0.1972
Baumgartner	10, 10	0.0612	0.0928	0.1306	0.1778
	15, 10	0.0718	0.0956	0.1388	0.1900
	10, 15	0.0888	0.1202	0.1750	0.2474
Mood	10, 10	0.0656	0.0764	0.0842	0.0936
	15, 10	0.0764	0.0916	0.1094	0.1312
	10, 15	0.0704	0.0780	0.0792	0.0908
Tippett	10, 10	0.0940	0.1356	0.1870	0.2406
	15, 10	0.1020	0.1494	0.2144	0.2944
	10, 15	0.1056	0.1558	0.2042	0.2806
Liptak	10, 10	0.0968	0.1492	0.2226	0.2938
	15, 10	0.1174	0.1746	0.2538	0.3392
	10, 15	0.1112	0.1558	0.2266	0.2822
Fisher	10, 10	0.0956	0.1524	0.2180	0.2834
	15, 10	0.1102	0.1726	0.2432	0.3378
	10, 15	0.1158	0.1612	0.2244	0.2890

of the alternatives which are one-sided or two-sided without any modification of the test statistics. This fact may be an advantage of our approach in this simultaneous test. However one should not misunderstand that the use of p -values for the construction of a test statistic has not been initiated by us. Pesarin (2001) has already used the p -values to construct the nonparametric test statistics for the multi-aspect test.

We would like to point out that Park (2013) provided the simulation results that the two tests based on T_n and S_n in Park(2013) showed some wide differences between the empirical powers under H_0 . However, during the preparation of this paper, we realized that the differences are because T_n is for the two-sided alternative while S_n , for the one-sided one. The third author apologizes for this error to

Table 13: Empirical powers for the double-exponential distribution(both location and scale)

Test	(m, n)	(δ, η)					
		0.0, 1.0	0.2, 1.2	0.4, 1.4	0.6, 1.6	0.8, 1.8	1.0, 2.0
Kolmogorov -Smirnov	10, 10	0.0418	0.0652	0.1554	0.2524	0.3482	0.4290
	15, 10	0.0554	0.0814	0.1684	0.2668	0.3716	0.4802
	10, 15	0.0430	0.0920	0.1742	0.2738	0.3714	0.4794
Baumgartner	10, 10	0.0508	0.0918	0.1620	0.2724	0.3934	0.4798
	15, 10	0.0664	0.0858	0.1648	0.2826	0.4168	0.5164
	10, 15	0.0462	0.0982	0.1846	0.2810	0.3990	0.5110
Wilcoxon	10, 10	0.0624	0.1356	0.2396	0.3504	0.4428	0.5178
	15, 10	0.0578	0.1464	0.2758	0.3912	0.4852	0.5674
	10, 15	0.0606	0.1428	0.2490	0.3584	0.4686	0.5476
Mood	10, 10	0.0536	0.1032	0.1632	0.2256	0.2734	0.3078
	15, 10	0.0568	0.1076	0.1786	0.2644	0.3338	0.4134
	10, 15	0.0534	0.1066	0.1474	0.1908	0.2276	0.2552
Tippett	10, 10	0.0552	0.1252	0.2488	0.3680	0.5154	0.5886
	15, 10	0.0564	0.1460	0.2826	0.4374	0.5758	0.6766
	10, 15	0.0612	0.1448	0.2570	0.3862	0.4908	0.6024
Liptak	10, 10	0.0638	0.1562	0.2982	0.4656	0.6112	0.7110
	15, 10	0.0550	0.1698	0.3536	0.5338	0.6984	0.7982
	10, 10	0.0636	0.1534	0.2864	0.4506	0.6106	0.7354
Fisher	10, 10	0.0542	0.1402	0.2928	0.4500	0.6108	0.7056
	15, 10	0.0606	0.1768	0.3524	0.5214	0.6792	0.7902
	10, 15	0.0584	0.1566	0.2902	0.4382	0.5920	0.7218

Table 14: Empirical powers for the double-exponential distribution(location only)

Test	(m, n)	δ				
		0.2	0.4	0.6	0.8	1.0
Kolmogorov -Smirnov	10, 10	0.0686	0.1676	0.3312	0.4828	0.6394
	15, 10	0.0808	0.1978	0.3648	0.5612	0.7144
	10, 15	0.1002	0.2104	0.3824	0.5568	0.7356
Baumgartner	10, 10	0.0892	0.1792	0.3410	0.5034	0.6738
	15, 10	0.0874	0.1926	0.3556	0.5612	0.7202
	10, 15	0.1050	0.2228	0.3986	0.5724	0.7450
Wilcoxon	10, 10	0.1420	0.2934	0.4758	0.6490	0.7970
	15, 10	0.1464	0.3256	0.5112	0.7012	0.8496
	10, 15	0.1598	0.3212	0.5008	0.6934	0.8338
Tippett	10, 10	0.1086	0.2042	0.3664	0.5416	0.6954
	15, 10	0.1298	0.2458	0.4164	0.6038	0.7582
	10, 15	0.1250	0.2370	0.4136	0.5770	0.7462
Liptak	10, 10	0.1188	0.2050	0.3268	0.4542	0.5618
	15, 10	0.1142	0.2334	0.3972	0.5428	0.6810
	10, 15	0.1104	0.1908	0.2880	0.4076	0.5156
Fisher	10, 10	0.1202	0.2204	0.3642	0.5196	0.6820
	15, 10	0.1208	0.2592	0.4176	0.5974	0.7482
	10, 15	0.1234	0.2134	0.3644	0.5254	0.6954

the readers.

Acknowledgement

The authors are very grateful to the two anonymous referees for pointing out errors and suggesting the constructive advices.

Table 15: Empirical powers for the double-exponential distribution(scale only)

Test	(m, n)	η				
		1.2	1.4	1.6	1.8	2.0
Kolmogorov-Smirnov	10, 10	0.0416	0.0448	0.0502	0.0510	0.0596
	15, 10	0.0476	0.0562	0.0614	0.0664	0.0778
	10, 15	0.0518	0.0582	0.0640	0.0732	0.0772
Baumgartner	10, 10	0.0594	0.0636	0.0688	0.0798	0.0894
	15, 10	0.0572	0.0680	0.0806	0.0928	0.1052
	10, 15	0.0590	0.0616	0.0644	0.0706	0.0800
Mood	10, 10	0.0970	0.1648	0.2256	0.2794	0.3544
	15, 10	0.1136	0.1608	0.2448	0.3298	0.4032
	10, 15	0.1082	0.1692	0.2482	0.3162	0.3786
Tippett	10, 10	0.0888	0.1282	0.1856	0.2290	0.2776
	15, 10	0.0924	0.1416	0.1960	0.2506	0.3258
	10, 15	0.0952	0.1414	0.1870	0.2438	0.3042
Liptak	10, 10	0.0934	0.1310	0.1716	0.2134	0.2556
	15, 10	0.0900	0.1314	0.1844	0.2376	0.2944
	10, 15	0.0884	0.1302	0.1732	0.2178	0.2628
Fisher	10, 10	0.0856	0.1298	0.1826	0.2330	0.2850
	15, 10	0.0968	0.1400	0.1966	0.2602	0.3284
	10, 15	0.0902	0.1386	0.1998	0.2524	0.3006

References

- Anderson, T. W. and Darling, D. A. (1954). A test of goodness of fit, *Journal of American Statistical Society*, **49**, 765–769.
- Ansari, A. R. and Bradley, R. A. (1960). Rank-sum tests for dispersions, *Annals of Mathematical Statistics*, **31**, 1174–1189.
- Baumgartner, W., Wei, P. and Schindler, H. (1998). A nonparametric test for the general two-sample problem, *Biometrics*, **54**, 1129–1135.
- Duran, B. S., Tsai, W. S. and Lewis, T. O. (1976). A class of location-scale nonparametric tests, *Biometrika*, **63**, 173–176.
- Fisher, R. A. (1932). *Statistical Methods for Research Workers*, 4th Ed., Oliver and Boyd, Edinburgh.
- Fleming, T. R. and Harrington, D. P. (1991). *Counting Processes and Survival Analysis*, Wiley, New York.
- Good, P. (2000). *Permutation Tests-A Practical Guide to Resampling Methods Testing Hypotheses*, Springer-Verlag, New York.
- Lepage, Y. (1971). A combination of Wilcoxon's and Ansari-Bradley's statistics, *Biometrika*, **58**, 213–217.
- Liptak, I. (1958). On the combination of independent tests, *Magyar Tudomanyos Akademia Matematikai Kutato Intezetenek Kozlomonyei*, **3**, 127–141.
- Marozzi, M. (2012). A modified Cucconi test for location and scale change alternatives, *Revista Colombiana de Estadistica*, **35**, 371–384.
- Mood, A. M. (1954). On the asymptotic efficiency of certain nonparametric two-sample tests, *Annals of Mathematical Statistics*, **25**, 514–522.
- Murakami, H. (2007). Lepage type statistic based on the modified Baumgartner statistic, *Computational Statistics and Data Analysis*, **51**, 5061–5067.
- Neuhäuser, M., Leuchs, A.-K. and Ball, D. (2011). A new location-scale test based on a combination of the ideas of Levene and Lepage, *Biometrical Journal*, **53**, 525–534.

- Park, H. I. (2010). A study on the nonparametric tests for the two-sample problem, *Journal of Korean Data Analysis Society*, **12**, 2431–2442.
- Park, H. I. (2011). A new nonparametric test for the location and scale parameters, *Journal of Korean Data Analysis Society*, **13**, 2987–2995.
- Park, H. I. (2013). On the study for the simultaneous test. *Communications for Statistical Applications and Methods*, **20**, 241–246.
- Pesarin, F. (2001). *Multivariate Permutation Tests with Applications in Biostatistics*, Wiley, New York.
- Randles, R. H. and Hogg, R. V. (1971). Certain uncorrelated and independent rank statistics, *Journal of American Statistical Society*, **66**, 569–574.
- Tippett, L. H. C. (1931). *The Methods of Statistics*, Williams and Norgate, London.

Received August 13, 2013; Revised December 15, 2013; Accepted January 7, 2014