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Literature Review of Research on Models in Mathematics Education

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There have been many discussions of models in mathematics education. Although there has been some agreement regarding the importance of clarifying perspectives on the concept and didactic significance of models, there is still no clear consensus on these issues. This study examines articles focused on models in mathematics education in order to clarify theoretical perspectives on models in the research community. The results of this study show that there are three perspectives on models in mathematics education and that these perspectives are closely related to researchers' ontological stances on mathematical knowledge and interpretations of the epistemological role of the model.

Ⅰ. Introduction

There have been various attempts to apply models in the learning of mathematics (e.g., English & Watters, 2004; Lesh & Doerr, 2003; Kim & Kim, 2004; Kim, 2005; Seo, Yeun, & Lee, 2013; Shin & Kwon, 2001; Son & Lew, 2007; Park & Lee, 2013). Although there is widespread agreement regarding the importance of clarifying perspectives on the concept and didactic significance of models, there is still no clear consensus on the meaning and role of models (Lesh & Fennewald, 2010). For example, some researchers investigated the mental model, which "describes the cognitive representations that individuals construct during various learning situations" (Chinnappan, 1998, p. 204) in students' learning of mathematics. On the other hand, some other researchers focused on models which are "objects, data, relations and conditions … translated into mathematics" (Blum et al., 2002, p. 153) to support learners' mathematical thinking and learning.

The diversity of arguments on the meaning and role of model use has contributed to the healthy growth of the research community (Lesh & Fennewald, 2010). On the other hand, the diversity can cause challenges for the research community since these various studies are based on indistinguishable stances (Arzarello, Bosch, Lenfant & Prediger, 2007). Kaiser & Sriraman (2006) argued that though various studies on models in mathematics education have been done based on

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different goals and backgrounds, there is a lack of discussion about these different perspectives.

With recognition of this ambiguity regarding the meaning and role of models in the mathematics education research community, we aim to review a selection of literature which focused on applying models in mathematics teaching and learning. The purpose of this study is to clarify the notion and role of a model, and to determine how these studies applied models in mathematics teaching and learning. To accomplish these purposes, we established the following research questions:

1. How do the studies published in mathematics education journals conceptualize the notion of a model?
2. How do these studies apply models in the teaching and learning of mathematics?

The major purpose of this study is to clarify and cluster theoretical perspectives of studies on models in mathematics education by answering the above research questions. By doing so, we also aim to highlight the significance and role of models in mathematics education.

Ⅱ. The notion and role of a model in mathematics education

In this section, we synthesize the work of researchers who examined the notion and role of models in mathematics teaching and learning. We first describe a method for our literature review for identifying how the notion of a model is conceptualized by researchers. We first present a method for this study since we need to establish a definition of a model based on our literature review results. Because of the absence of a comprehensive definition of a model which sheds light on central aspects of a model in mathematics education, we propose a provisional definition of a model, which emerged from the articles we reviewed. Second, we review and examine various ways of using the term "model" and aspects of a model, which we found in a selection of papers published in major journals in the mathematics education research community. Third, we propose an alternative provisional definition of a model which has the potential to shed light on central aspects of a model in mathematics education. This provisional definition is based on Sfard's work on the discursive perspective on mathematical knowledge and episteme (Sfard, 2007, 2008), since her perspective could highlight the significance and the role of models in mathematics education. Finally, we investigate the roles of a model chosen by researchers to identify and cluster ways of conceptualizing the notion of a model, and we highlight the significance and role of models in mathematics education. The rationale for the above procedures for literature review is elaborated in the following section.

1. Method for literature review for identifying how the notion of a model is conceptualized

Bryman (2008) argued that a literature review is "a means of reviewing the main ideas and researches relating to your chosen area of interest" (Bryman, 2008, p. 110). The systematic review is a way of reviewing literatures for generating “unbiased and comprehensive accounts of the literature” (Bryman, 2008, p. 85). Based on an investigation into the methods for literature review taken by prior studies, Bryman (2008) suggested five steps for systematic literature review: (1) identify the purpose of the review, (2) establish criteria for the selection of studies, (3) identify and review all articles which satisfy the criteria established by the reviewer, (4) identify the key features of each article, and (5) synthesize the results.

We followed the above five steps for systematic literature review. First, the purpose of literature review in this chapter is to identify how the studies published in mathematics education journals conceptualize the notion of a model. Second, we collected papers published in the following journals to conduct a literature review on studies focused on models in mathematics education: *Educational Studies in Mathematics* (Volumes 1-83), *Journal for Research in Mathematics Education* (Volumes 1-43), *Mathematical Thinking and Learning* (Volumes 1-14), *International Journal of Science and Mathematics Education* (Volumes 1-10), and *ZDM* (Volumes 29-44; from the first online version in 1997). These journals were peer-reviewed journals which were chosen with consideration given to the geographical spread of editorial influence (Ryve, 2011).

Third, we collected all papers published in the above journals which included the words model, modeling, or modelling in the title, abstract, or keywords in order to focus on models in mathematics education (Ryve, 2011). This search yielded 620 papers, and from those we excluded papers on models developed by researchers (e.g., learning model, developmental model, Toulmin’s model, and structural equation model), book reviews, introductions to special issues, and studies which include our keywords only in the abstract or keywords and do not deal with models in the manuscript. After these eliminations, we collected and analyzed 221 papers. Table 1 shows the overall distribution of these 221 papers.

Fourth, we first investigated researchers’ ways of using the term model to identify how the studies conceptualized the notion of a model, since identifying ways of using a key term is important for characterizing how the term is conceptualized (Ryve, 2011). Clarifying the usage of the term model is also one of the most important tasks of this study. As a result, we found three ways of

Table 1. Distribution of Articles by Time Period and Journal

Journal	1968-1989	1990-1994	1995-1999	2000-2004	2005-2009	2010-2012	Sum
ESM	23	15	14	14	12	13	91
JRME	11	4	8	4	6	4	37
MTL	-	-	4	10	5	4	23
IJSME	-	-	-	1	5	3	9
ZDM	-	-	5	5	36	15	61
Sum	34	19	31	34	64	39	221

using the term model in the articles which we reviewed. However, some researchers used the term model in a similar way but they interpreted a model's role in a totally different way. This is the main reason it is challenging to support only one of these ways of using the term model in this study, and each way of using the term model is also incommensurable. Bryman (2008) argued that it is possible to propose "an alternative perspective that is superior to the literatures as it stands (p. 84)" for these incommensurable literature review results. Given that, we synthesize prior studies by proposing an alternative definition of a model in the next step to shed light on central aspects of a model and to support communication among different perspectives.

Fifth, we adopted the grounded theory (Strauss & Corbin, 1998) to synthesize prior studies. Frejd (2013) and Wolfswinkel, Furtmuller, & Wilderom (2013) pointed out that the grounded theory is used to validate by means of literature review. Frejd (2013) reviewed literatures investigating assessment of mathematical modeling using *open coding* and *axial coding* from the grounded theory. Given that, we first organized collected articles into open categories based on the content of each article (*open coding*). We iteratively elicited categorized perspectives, and then revised them (*axial coding*) (Frejd, 2013). In this study, we applied investigator triangulations to establish validity and reliability (Creswell, 2009; Stake, 1995). Triangulation proceeded by checking inter-rater agreement. That is, several researchers participated in a peer-review of the interpretations of studies to confirm the results of analysis and obtain additional interpretations.

As a result, we propose an alternative definition of a model which sheds light on central aspects of a model. We then categorize prior studies on models in accordance with the roles of models that emerged from the studies to identify how these studies are conceptualizing the notion of a model in mathematics education. In the following section, we address the results of the review on the key term model, and we propose a provisional definition of a model from the discursive perspective.

2. Ways of using the term model

Although it is difficult to find general agreement on the notion and role of a model in mathematics education (Lesh & Fennewald, 2010), we identified three ways of using the term model in mathematics education. First, a model is "objects, data, relations and conditions … translated into mathematics" (Blum et al., 2002, p. 153). This way of using the term model is originated from discussions on the developments of scientific theory, and models are considered to be relevant to scientific discoveries (Freudenthal, 1991; Niaz, 1999; Clement, 2009). For example, Rutherford-Bohr's atom model (see Freudenthal, 1991) and James Maxwell's model of electromagnetic induction (see Clement, 2009) are models which correspond to the above definition.

Second, a model is a mental model which "describes the cognitive representations that individuals construct during various learning situations" (Chinnappan, 1998, p. 204). That is, a mental model represents the organization of a learner's mental activity. This way of using the term model is originated from discussions on psychological studies, and "[the term mental model] has been quite successful in cognitive psychology" (Van Dijk, 1998, p. 189). For example, *selection* *models, partition models,* and *distribution models* with respect to problem-solving strategy in combinatorial problems (Batanero, Navaroo-Pelayo & Godino, 1997) are mental models.

Third, a model is either a concrete, manipulable, or visual object which embodies mathematical concepts (Bartolini Bussi, Taimina & Isoda, 2010). This way of using the term model differs from the first way of using the term model (Freudenthal, 1991). That is, these concrete models are relevant to making implicit abstract mathematical constructs more explicit (Freudenthal, 1991). For example, a particular group is a model of the general group concept (Freudenthal, 1991) and a plaster model of the Steiner surface (Bartolini Bussi, Taimina & Isoda, 2010) is also a model which corresponds to this definition. Table 2 summarizes the above three ways of using the term model, and the number of studies using the term model in each way.

As we can see from Table 2, the percentage of studies that used the term model for signifying "objects, data, relations and conditions ⋯ translated into mathematics" was about 46.2%, and the percentage of studies that used the term model for "describing the cognitive representations that individuals construct during various learning situations" was about 39.8%. Finally, the percentage of studies that used the term model for signifying "a concrete, manipulable, or visual object which embodies mathematical concepts" was 13.6%.

With the above diverse ways of using the term model, we also identified three central aspects of a model which make it difficult to both combine the works of researchers on models in mathematics education and build on their works. First, a model encompasses not only *abstraction* and *idealization* but also *particularization* and *exemplification* (Dapueto & Parenti, 1999). The term "idealization" is quite similarly used as abstraction or generalization, and the term "particularization" is similar to exemplification (Dapueto & Parenti, 1999). For example, the theory of groups is an idealization model of the structure of whole numbers, and the structure of whole numbers is a particularization model of the theory of groups (Dapueto & Parenti, 1999). However, each way of using the term model (Table 2) focuses on only one aspect of models. That is, the first two ways focus on the idealization model, and the third one focuses on the particularization model (Table 2).

Second, a model can signify a real-world situation, a human subject's activity, and even mathematical objects. For example, Maxwell's model of electromagnetic induction signifies the real-world, the selection model of combinatorial reasoning signifies students' problem-solving activities, and a

Table 2. Ways of using the term model

Ways of using the term model	Percentage of Articles
objects, data, relations and conditions … translated into mathematics	46.2% (102 of 221)
describes the cognitive representations that individuals construct during various learning situations	39.8% (88 of 221)
concrete, manipulable, or visual object which embodies mathematical concepts	13.6% (30 of 221)
Not explicitly situated (meta-analysis article)	0.4% (1 of 221)

plaster model of the Steiner surface signifies a mathematical object.

Third, models have many different manifestations (Van den Heuvel-Panhuizen, 2003). According to Van den Heuvel-Panhuizen (2003):

> "[Models] can have different manifestations. This means that the term 'model' is not taken in a very literal way. Materials, visual sketches, paradigmatic situations, schemes, diagrams and even symbols can serve as models (p. 13)."

Because of the absence of a comprehensive definition of a model to highlight and shed light on the above three central aspects of a model, the need for an alternative definition of a model has arisen. It might be possible to categorize researches according to one of the above aspects of models focused on by each research. However, we observed that some researchers focused on the same aspects of models, but each of them interpreted the role of a model in mathematics teaching and learning differently, or vice versa. Also, if we categorize researches according to the role of a model which researchers are focusing on, the way of using the term model and aspect of a model focused on by researchers varies, which makes things complicated and unclear. With recognition of these difficulties in categorizing literatures on models, we propose an alternative definition of a model in a broad sense to encompass and shed light on the above three central aspects of models. For this alternative definition of a model, we adapt Sfard's discursive perspective on mathematics. In the following section, we first briefly address Sfard's discursive perspective, and propose an alternative definition of a model from her perspective. We then evaluate this alternative definition.

3. Proposing an alternative definition of a model from the discursive perspective

Cobb (2009) and Stahl (2009) argued that Sfard's discursive perspective on existence and episteme of mathematical knowledge provides some insight that can resolve quandaries in mathematics education. Sfard provides a coherent, comprehensive, and clear perspective on existence and episteme of mathematical knowledge (Stahl, 2009).

Sfard (2007) defines discourse as "the different types of communication that bring some people together while excluding some others" (p. 573), and regarding mathematical discourse, states that "discourse counts as mathematical if it features mathematical words" (p. 573) with historically established certain rules. Using these definitions of discourse and mathematical discourse, Sfard (2008) discusses the historical development of mathematics and individual subjects' learning processes from the discursive perspective.

Sfard (1991, 2008) investigates the emerging process of self-generative mathematical discourse. This is a process where a human subject's actions on concrete objects are condensed and alienated to become mathematical objects, and these objects are modified and transplanted within their relational net in order to build mathematical discourse. She views the objectification of human actions into mathematical objects as development from ground-level discourse into meta-level discourse. For example, *natural numbers* emerge by the condensation, alienation, and reification of a subject's act of counting (Sfard, 1991). This reification is interpreted as an *ontological shift*, since this seeing of human actions as objects with static structures in meta-view signifies a leap of level (Sfard, 2008). It is important that new operations, for example, *dividing* can be performed on these new objects, natural numbers (Sfard, 1991). Then, positive rational numbers also emerge as a result of the condensation, alienation, and reification of a subject's act of dividing natural numbers by other natural numbers. Thus we can say that positive rational numbers are locating relatively meta-level discourse than natural numbers (Sfard, 1991). From her investigation on the historical development of mathematical discourse, Sfard (2008) states that "[mathematical discourse] is decomposable into relatively neatly delineated, *hierarchically organized layers* that allow for many different levels of engagement and performance (p. xviii)."

We now propose an alternative provisional definition of a model from Sfard's discursive perspective: a model is *anything which signifies anything else in a different discursive layer*. Although we are proposing an alternative definition of a model, this definition is broad and potentially revisable. This definition of a model reflects our intentions of theorizing the notion and role of models in mathematics education. Our purpose in attempting to define what we mean by the term model is not to shut down current various discussions on models but to encompass and cluster various researches to support communication and theorizing based on the various perspectives.

We first emphasize that a model is *anything* which signifies *anything else*. Models not only have many different manifestations (Van den Heuvel-Panhuizen, 2003), but also can signify a real-world situation, a human subject's activity, and even mathematical objects. Thus we considered that it is reasonable to define a model as *anything* which signifies *anything else* to encompass both of these two aspects of a model.

Second, we defined a model as signifying anything else that *lies in a different discursive layer* to distinguish and highlight the notion of a model from other notions of external representations, symbols, or signs, although manifestations of models can be in the form of symbols or signs. That is, this definition of a model encompasses two directions of signifying: idealization and particularization (Dapueto & Parenti, 1999). From the discursive perspective, the theory of group is located in meta-level discourse via the structure of whole numbers, which is located in ground-level discourse. It is important to note that a model is relevant to the relationship between ground-level discourse and meta-level discourse, whether a model is either an idealization model or a particularization model. Gravemeijer & Stephan (2002) also highlight the role of models as a mediator of ground-level discourse and meta-level discourse. That is, they emphasize the shift from a *model of* a context situation to a *model for* mathematical reasoning, and they focus on how models can bridge the gap between learners' informal reasoning (or ground-level discourse) and formal mathematics (or meta-level discourse) with inscriptions on the chain of signification (Gravemeijer & Stephan, 2002). The chain of signification (Gravemeijer & Stephan, 2002) seizes on the central aspect of ideal models. But this chain is only bottom up and deeply rooted in one specific perspective on models which we will review in the next section.

Third, it is possible to highlight the ontological value of models in mathematics and provide a theoretical background for investigating why and how models can support mathematics learning through a re-interpretation of a model from the discursive perspective. Sfard (2008) argued that “the vision of mathematics as a discourse, and thus as a form of human activity, makes it possible to identify mechanisms that are common to the historical development of mathematics and to its individual learning (p. xviii).” From her argumentation, we defined a model from the discursive perspective since this makes it possible to interpret students’ mathematical development with consideration of the entire historical development of mathematics, individuals’ progress and the development of models. Like many other technical terms in mathematics education, the term model is borrowed from other disciplines such as science or psychology. Given that, we defined the term model from Sfard’s discursive perspective on the development of mathematics and individual subjects to bring the term model into our discourse of mathematics education. In the following section, we review perspectives on models in mathematics education, and Sfard’s perspective shines through in the interpreting and highlighting of these perspectives. We believe that her perspective sheds light on theorizing teaching and learning mathematics with models.

Fourth, our provisional definition encompasses four roles of models which we categorized in researches in mathematics education, which are presented in the following section. We especially focused on the roles of models since we identified that the role of a model chosen by a researcher was closely related to the researcher’s ontological stance on mathematics. Given that, we identified the role of a model chosen by researchers to clarify how studies are conceptualizing the notion of a model, in the following section.

4. Roles of models in the learning of mathematics

In this section we describe four roles of models that emerged from researches which focus on models in mathematics education. First, a model is a mediator for students’ participation in metal-level discourse from their informal activities, and this kind of model supports students’ reflection upon their own informal activities. The researchers who pursue this role of models focus on the fact that mathematics is an organization of human actions and mathematics has a multilayered structure. That is, these researchers emphasize both the triggering of students’ informal activities located in ground-level discourse and students’ reflections upon these activities to foster their participation in meta-level discourse. From this perspective, these researchers consider that models are either signifying the condensed or reified actions of learners, or that models are concrete, manipulable, or visual objects which embody mathematical concepts.

Researchers who consider models as signifying the condensed or reified actions of learners emphasize the transition from the model of a problem situation to the model for students’ mathematical reasoning (Gravemeijer & Stephan, 2002). From this perspective, a cascade of models emerged through a designed series of (word) problems, and these models support students’ participation in formal mathematics (Gravemeijer & Stephan, 2002; Van den Heuvel-

Panhuizen, 2003). For example, the double number line model is emerged and developed in many different manifestations while students are passing through their learning trajectory and developing reasoning on percentages (Van den Heuvel-Panhuizen, 2003). Gravemeijer & Stephan (2002) also illustrates the emergence of the empty number line with the development of students' reasoning. This model supports the development of students' informal reasoning up to formal mathematics since a model is, on the one hand, a result of the organization of students' adding and subtracting activities, and on the other hand, signifies the real number system. We can view the double number line and empty number line as *pseudo-mathematical objects*, since they signify static mathematical objects but themselves are incomplete via a mathematical concept (Lee, 2011). On the other hand, models can signify the operational aspects of mathematics. Streefland (1993) described models which display the operational aspects of generating processes of formal mathematics through the organization of human activity using the term *operational model*. For example, selection models, partition models, and distribution models (Batanero, Navaroo-Pelayo & Godino, 1997), which we described earlier, signify the problem-solving processes of combinatorial problems, which are operational models.

In a similar way, concrete, manupulable, or visual models also signify mathematical objects but they are incomplete via mathematical objects, so these models can be interpreted as pseudo-mathematical objects. Researchers focusing on these models emphasize the significance of learning mathematics by students' manipulation of these models and students' reflection upon manipulation activities (Bartolini Bussi, Taimina & Isoda, 2010). Thus, a pseudo-mathematical object is either the result of the organization of students' activities (e.g., empty number line), or concrete, manipulable, or visual models (e.g., a plaster model of the Steiner surface) which is given by the instructor to trigger students' mathematical activities. Although both of them are pseudo-mathematical objects, while the former (organization of students' activities) is emerged by students' own activities, the latter (concrete, manipulable, or visual model) is provided by the instructor.

Although we categorized studies interpreting that a model is a mediator for students' participation in metal-level discourse from their informal activities, the ways of emphasizing their perspectives on models were various. That is, these studies are subcategorized into studies focused on learners' mental models (e.g., tacit model, intuitive model or implicit model), studies which investigated models signifying the condensed or reified actions of students, studies which investigated models signifying mathematical operations (e.g., model of addition, model of multiplication), studies on models signifying the condensed actions of students' problem-solving processes, and studies on concrete, manipulable or visual models (e.g., pictorial model, or geometrical model). These several terminologies were used by researchers considering that the role of a model is a mediator. Although these ways of describing the term model are various, these researchers commonly consider models as signifying either the condensed or reified actions of learners, or embodying mathematical objects. They commonly consider that the role of models is to act as a mediator between ground-level discourse and meta-level discourse. As we described in the previous section, researchers focusing on students' mental models regard models as representing the organization of students' mental activity. In other words, these models signify the condensed actions of students, and tacit, implicit models as well. On the one hand, the mental model is the term describing the condensed actions of students neutrally; on the other hand, the tacit and implicit models tend to describe students' errors or misinterpretations of mathematical ideas (see Fischbein, 1999). However, as Fischbein (1987) argued, these researchers focusing on the tacit and implicit models also consider that one of the major roles of models is "to constitute an intervening device between the intellectually inaccessible and the intellectually acceptable and manipulable" (Fischbein, 1987, p. 123).

In summary, the first role of models emerged by researchers focusing on models in mathematics education is a mediator for students' participation in meta-level discourse. In this perspective, models are either operational (e.g., selection models, partition models, distribution models) or pseudo-mathematical objects (e.g., empty number line, a plaster model of the Steiner surface). Operational models signify the condensed actions of learners, and pseudo-mathematical objects are either signifier of the reified actions of learners, or concrete, manipulable or visual objects embodying mathematical objects. Although there are three ways of using the term model, these models commonly support students' participation in meta-level discourse by triggering learners' actions in ground-level discourse and fostering students' reflections upon their own activities to participate in meta-level discourse. Also, we identified that researchers with this perspective on models commonly consider mathematics as the organization of human activities, so they considered models to be intermediate products between students' informal activities in ground-level discourse and formal mathematics in meta-level discourse.

Second, a model is a conceptual system, or a signifier of a local version of huge mathematical discourse. This kind of model supports learners' mathematical thinking and participation in wider and inclusive mathematical discourse through eliciting, revising, and generalizing models which represent authentic problem-solving situations. Researchers who pursue this role of models focus on the fact that knowledge is an accumulation of wisdom from active adaptation and problem-solving (Dewey & Bently, 1949; Westbrook, 1999). In this perspective, knowledge is interpreted as emerging from a problem-solving experience, and likened to an interrelated *living organism* (Lesh & Doerr, 2003). From this perspective on mathematical knowledge, models are interpreted as an accumulation of mathematical thinking from problem-solving, and students' participation in mathematical discourse is widening and deepening as models continue to grow, develop, and undergo revision. Researchers with this perspective on models argue that models of problem situations can also contribute to both the development of mathematical discourse and individuals' mathematical thinking. For example, Rutherford-Bohr's atom model and James Maxwell's model contribute to the development of scientific discourse. On the other hand, these models contribute to individuals' understanding of problematic situations in the real-world (atoms and electromagnetics) and scientific discourse.

According to Fischbein (1987), a model is "well structured, internally consistent, [and] governed by its own laws" (p. 123). Given that, researchers from this perspective argue that learners' participation in mathematical discourse can be expanded by modeling-eliciting and its iterative revisions to ought be "concrete to abstract, particular to general, situated to de-contextualized, intuitive to analytic to axiomatic, undifferentiated to refined, and fragmented to integrated" (Lesh & Doerr, p. 32). They regard models as conceptual systems since a model goes further from signifying a single-problem situation to signifying "autonomy in respect to the original" (Fischbein, 1987) and interprets the major role of a model in its "heuristic capacity" (Fischbein, 1987, p. 123). For example, English (2006) shows students' conceptual development, mathematization, and critical reflections through developing a generalizable, reusable model of deciding on the best snack chip. Lesh & Harel (2003) also illustrates students' conceptual development while constructing and revising a model for determining the height or weight of a person from his footprints. From this perspective, models are more likely to be models of challenging problem-solving situations which contain great potential for students' development, while the models of researchers holding the first perspective (models are mediators) are emerged and transited through several problems with the learning trajectory.

Therefore we can consider these models as signifying local versions of huge mathematical discourse, and eliciting and revising models can be interpreted as experiencing development and revision of local mathematical discourse. As a result, researchers with this view emphasize not only model-eliciting for authentic problem-solving, but also mathematical thinking and reasoning which can foster the development of students' participation in wider mathematical discourse (Lesh & Doerr, 2003).

Although we categorized studies interpreting models as signifying a local version of mathematical discourse, the ways of emphasizing their perspectives on models were various. That is, these studies are subcategorized into studies focusing on conceptual systems, model-eliciting, data modeling, and heuristic models. These several terminologies were commonly used by researchers focusing on models which signify a local version of mathematical discourse.

Third, another role of a model is the application of mathematics to real-world problem situations, and researchers with this perspective consider the building and application of models to solve real-world problems as a goal of mathematics teaching and learning rather than as a vehicle (Galbraith & Stillman, 2006). Researchers with this perspective use the term model similar to researchers with the second perspective (models as signifying the local version of mathematical discourse), and both of them consider models as representing real-world problem situations. However, the role of models pursued by researchers with each perspective is different. On the one hand, researchers with the second perspective focus on students' mathematical thinking and participation in wider mathematical discourse; on the other hand, researchers with this third perspective focus on students' application of mathematical discourse. Whereas researchers with the previous two perspectives on models (mediator, conceptual system) aim at learners' participation in wider mathematical discourse, researchers with this perspective aim at searching for and applying appropriate mathematical discourse in which students are participating.

Researchers with this perspective are strongly influenced by Pollak's (1968) emphasis on the application of mathematics to real-world situations in school mathematics. From his emphasis on the application of mathematics, researchers with this perspective focus on students' experience of the entire modeling process and expect the growth of learners' modeling competency (Blomhøj & Jensen, 2003; Galbraith & Stillman, 2006). Modeling competence encompasses "being able to analyse the foundations and properties of existing models and being able to assess their range and validity" (Niss & Højgaard, 2011, p. 99). For example, Galbraith & Stillman (2006) is a generic research that was conducted based on this perspective, and in this study, students' blockages are identified while the building and revising of models of real situations occur.

On the other hand, there are arguments that teaching and learning for the development of modeling competency can also support the learning of mathematics (Blum & Borromeo Ferri, 2009). That is, since modeling iterates the processes of building, revising, and validating (Blum & Borromeo Ferri, 2009), these processes can be interpreted as bridging real-world phenomena and mathematics, which could support mathematics learning. Although these researchers point out the potential of models to support the learning of mathematics, their major concern is the development of students' modeling competency.

Although we categorized studies interpreting the role of a model as the application of mathematics to real-world situations, the ways of emphasizing their perspectives on models were various. That is, these studies are subcategorized into studies focusing on applied mathematics (as opposed to pure mathematics), applications, optimization, industry or engineering mathematics, modeling competency, and modeling course design. These several terminologies were commonly used by researchers focusing on the role of models as an application of mathematics to real-world situations.

Fourth, the role of a model is a vehicle for social critics (Barbosa, 2006), and researchers with this perspective emphasize that both mathematical discourse and models are ideological products (Skovsmose, 1990a, 1990b). Gee (2007) argues that ideology is inherent in our discourse and the term ideology is crucial in education.

"'Ideology'" is an 'upside-down' version of reality. Things are not really the way the elite and powerful believe them to be; rather their beliefs invert reality to make it appear the way they would like it to be, the way it 'needs' to be if their power is to be enhanced and sustained." (Gee, 2007, p. 28)

This perspective is closely related to D'Ambrosio's (1990) view of regarding mathematics as an intellectual instrument for seeing reality. Skovsmose (1990b) argued that the building of models should be based on critical investigation in the real world and that models enable the critical understanding of real-world phenomena. Researchers with this perspective interpret modeling competency and mathematical discourse as vehicles for the criticism of models and realities (Barbosa, 2006). From this perspective, researchers focus on social critics with model construction and interpretation. For example, while building and interpreting models on the distribution of bean and corn seeds donated by the government to farmers, students can raise questions like "What criteria would be *fairer* [for this distribution]?" (Barbosa, 2006, p. 295).

Although we categorized studies interpreting models as ideological products which are relevant to reality, the ways of emphasizing their perspectives on models were various. That is, these studies are subcategorized into studies focusing on social criticism, ideological product, emancipation, and emphasis on critical reflections on building and revising models. These several terminologies were commonly used by researchers focusing on the role of models as ideological products which are relevant to reality.

Table 3 summarizes the above four roles of models, and the number of studies interpreting the role of a model in each way.

5. Summary

We reviewed articles focusing on models in mathematics teaching and learning, and we attempted to identify how these studies are conceptualizing the notion of a model. To accomplish this purpose, we identified three aspects of models, three ways of using the term model, and four roles of models. We found that each study conceptualized the notion of a model by focusing on one or several aspects of a model, the roles of a model, and ways of using the term model.

Although it seems to be that the notion of a model is complex, we regarded a model as anything signifying anything else in a different discursive layer. Sometimes a model sheds light on the mechanism of real-world phenomena (e.g., Rutherford model); sometimes a model supports the posing of a problem illustrating social inequality (e.g., models on the distribution of bean and corn). A model located in a relatively lower-level discourse helps us to access perceptually inaccessible abstract objects (e.g., a Steiner model), and a model located in a relatively meta-level discourse which encompasses particulars enables us to see their underlying structure (e.g., the group theory). A mental model enables us to see and guide students' activities from ground-level discourse to meta-level discourse (e.g., addition model, selection model).

Although there are many aspects of a model, roles of a model, and ways of using the term model, a model is commonly a lens for seeing other original things in a clearer and better way. It

Table 3. Roles of model in mathematics learning

Role of model in mathematics education		Percentage of Articles
Mediator	Operational model	33.0% (73 of 221)
	Pseudo-mathematical object	20.4% (45 of 221)
Conceptual system		24.0% (53 of 221)
Application		18.6% (41 of 221)
Social critic		3.6% (8 of 221)
Mixed (meta-analysis article)		0.4% (1 of 221)

can give us insight into a real-world situation, a human subject's activity, and even mathematical objects. From these perspectives on models, researchers applied models in accord with their own ontological stance (e.g., pragmatism) and goals to achieve through mathematics teaching and learning (e.g., social critic).

We also found that Sfard's discursive perspective is quite successful for proposing a definition of a model encompassing and highlighting three central aspects of models, and clarifying and highlighting each of the four roles of models chosen by researchers. As we expected, her discursive perspective was effective for bridging researchers' perspectives on the development of mathematics and individuals' learning mathematics with models. We expect that the above results of this study can increase understanding and communication among different perspectives on models.

In the following section, we focus on the second research question: How do these studies apply models in the teaching and learning of mathematics? After a brief description of the method used for literature review, we present results for the second research question.

Ⅲ. Applying models in the teaching and learning of mathematics

In this section, we synthesize the work of researchers who attempted to apply models in the teaching and learning of mathematics. We first describe the method used for our literature review for identifying how studies apply models in the teaching and learning of mathematics. Second, we set up three analytic questions to guide the answering of this research question, and evaluate these analytic questions. Finally, we present the results for these analytic questions and synthesize the results with our framework on the roles of models in mathematics teaching and learning (Table 3).

1. Method for literature review for identifying how studies applied models in the teaching and learning of mathematics

We followed five steps for systematic literature review which we addressed in the previous chapter: (1) identify the purpose of the review, (2) establish the criteria for the selection of studies, (3) identify and review all articles which satisfy the criteria established by the reviewer, (4) identify the key features of each article, and (5) synthesize the results.

First, the purpose of literature review in this chapter is to identify how the studies published in mathematics education journals applied models in the teaching and learning of mathematics. Second and third, we collected and analyzed 221 papers published in major mathematics educational research journals to conduct a literature review on studies focused on models in mathematics education as described in the previous chapter (Table 1).

Fourth, we set up three analytical questions to investigate how the studies published in mathematics education journals apply models in the teaching and learning of mathematics, and we present results for these analytic questions. We especially investigated the types of tasks, employment of technologies, and ages of participants, which were focused on by the studies on models selected in the third step. Evaluation of these variables (types of tasks, employment of technologies and ages of participants) for identifying ways of applying models in the teaching and learning of mathematics is presented in the next section.

Fifth, we synthesized literature review results with our framework on the roles of models in mathematics teaching and learning (Table 3). Theoretical grounds for difference in the ways of applying models in accordance with the perspective on the roles of a model in mathematics education are presented in the next section while establishing analytic questions. Given that, we investigated the types of tasks, employment of technology, and ages of participants according to the perspectives on the roles of a model in mathematics education.

2. Establishing analytic questions

Ryve (2011) argued that setting and using analytic questions can guide literature review and organizing information. Given that, we established the following three analytic questions to investigate how the studies published in mathematics education journals applied models in the teaching and learning of mathematics.

1. What kind of task is used in the study?
2. Was technology employed or not?
3. What were the ages of the participants?

We established analytical questions especially focusing on the types of tasks, the employment of technology and the ages of the participants. Although there are many factors for investigating the teaching and learning of mathematics attempted by prior studies, we focused on the above three factors for identifying how models are applied.

We first investigated the types of tasks (analytic question 1) since tasks play a key role in mathematics teaching and learning (Krainer, 1993; Kaur & Yeap, 2009; Lee, Lee, & Park, 2013), and the type of task is closely related to students' mathematics learning (Lee, Lee, & Park, 2013). On the other hand, for applying models in the teaching and learning of mathematics, "there are a huge variety of tasks to choose from; consequently, selecting the appropriate task according to specific aims and target groups can be difficult" (Maaß, 2010, p. 286). In her argument, it is also important to note that selecting tasks for teaching mathematics is closely related to 'specific aims' and 'target groups.' Given that, we considered that types of tasks might be correlated with perspectives on the role of a model, or the ages of the participants. So we investigated whether there is a tendency of types of tasks in accordance with the ages of the participants, or perspective on the role of a model which is pursued by each study.

As Maaß (2010) indicates, students' school level is also known to be a key variable in the teaching and learning of mathematics using models (Blum et al, 2002; Niss, Blum & Galbraith, 2007). Stillman, Kaiser, Blum, & Brown (2013) also point out that it is difficult for young students to solve modeling tasks, and note the potential of simple word problems for these young learners. Given that, we investigated the ages of the participants in each study (analytic question 3) to identify whether there are correlations among the types of tasks, the ages of the participants and the perspectives on the roles of models in mathematics teaching and learning.

We also investigated the employment of technology in each research in response to widespread attention from researchers to the employment of technology in the teaching and learning of mathematics using models (Burkhardt, 2006; Henn, 2007; Stillman, Kaiser, Blum, & Brown, 2013). Henn (2007) argued that a technological environment is helpful for accomplishing various goals with models. These are: (1) "to understand phenomena in the world around us," (2) "to learn about and to understand mathematical issues," and (3) "to acquire problem-solving (heuristic) skills" (p. 321). Although the goals are not exactly the same as those of our framework, his argument is important for us in considering the various roles of technology in accordance to the perspectives on the roles of models in the teaching and learning of mathematics. Although it is interesting to investigate the roles of technology which are pursued by each researcher, this goes beyond the scope of this study. Given that, we investigate the employment of technology according to the perspectives on the roles of models pursued by each study to identify whether there is a correlation between them.

3. Types of modeling tasks

Maaß (2010) categorized modeling tasks by their "nature of relationship to reality." Nevertheless, as she acknowledged in her paper, we found that problems with an "inner-mathematical context (entirely within mathematics and with no connection to reality)" (Maaß, 2010, p. 287) were actively applied to studies on teaching and learning with models. Given that, we investigated types of modeling tasks employed by studies with the categorization of *reality-related tasks* and *inner problem-solving tasks*.

Also, we further subcategorized reality-related tasks into *authentic tasks* and *word problems*. This further subcategorization is aimed at identifying the degree of implementation of *authenticity* that was expected by learning through these types of modeling activities (Niss, Blum & Galbraith, 2007). Niss(1991) argued that a situation is authentic when it is relevant by experts in related fields. Based on Niss's (1992) perspective on authenticity, Maaß (2010) categorized modeling tasks as authentic "when the data itself, the way in which the data is represented or the question itself is authentic" (p. 287). Given that, we used Maaß's (2010) perspective to categorize tasks as authentic. We also categorized reality-related tasks which were not authentic as world problems and other inner-mathematical tasks as inner problem-solving tasks.

For example, we consider modeling tasks on local heating (see Lingerfjard, 2006) to be authentic tasks. A local heating task is building a mathematical model of the total annual heating cost of a house, taking into consideration of each square meter of the exterior wall with northern, eastern, western, and southern exposures of the house (Lingerfjard, 2006). Fan & Zhu (2007) poses a word-problem task that asks students to solve for the number of defective diskettes by setting up linear equations based on the sum of all diskettes, using the ratios of defective diskettes of each type of diskette provided by the text. We consider a task posed by Chinnappan (1998) which asks students to solve a geometric problem using the provided circles and triangles to be an inner problem-solving task. That is, "AE is a tangent to the circle, centre C. AC is perpendicular to CE, and the angle DCE has a measure of 30 degrees. The radius of the circle is equal to 5cm. Find AB (p. 206)." Table 4 shows the results of the literature review on the studies showing the above task categories.

Table 4. Types of tasks

Type of task	Percentage of Articles
authentic tasks	33.0%(73 of 221)
word problem	35.3%(78 of 221)
inner problem-solving	19.0%(42 of 221)
articles with no tasks	12.7%(28 of 221)
Sum	221

As we can see in Table 4, 33.0% of the studies dealt with authentic tasks, 35.3% dealt with word problems, 19.0% dealt with inner problem-solving tasks, and 12.7% did not provide any tasks.

Also, we determined whether there was a tendency in the types of tasks used depending on the researcher's perspective on the role of models (Table 5).

There is a noticeable tendency in the types of tasks employed by each study depending on the researcher's perspective on the role of models. The group of studies with the perspective that a model has the role of a mediator mainly employed word problems and inner problem-solving tasks. The group of studies with the perspective that a model has the role of a conceptual system mainly employed authentic tasks and word problems. The group of studies with the perspective that a model has the role of either an application or a critic employed only authentic tasks. We believe that since the studies focusing on applicative aspects of models emphasize the whole process, from simplifying real-world phenomena to building, revising, and validating models (Blomhøj & Jensen, 2003), they applied only authentic tasks.

4. Employment of technology

We identified studies which employed technological environments for teaching and learning with models. A limitation of this investigation was that we could only identify technological environments if authors explicitly described technological environments or showed data which is related to students' use of technology in their exploration. Table 6 shows the results of this survey.

Table 5. Types of tasks according to perspectives on the roles of models

Roles of model	Authentic	Word problem	Problem-solving	No task
Mediator	6.0% (7 of 118)	54.2% (64 of 118)	35.6% (42 of 118)	4.2% (5 of 118)
Conceptual system	60.4% (32 of 53)	26.4% (14 of 53)	-	13.2% (7 of 53)
Application	78.0% (32 of 41)	-	-	22.0% (9 of 41)
Critic	25.0% (2 of 8)			75.0% (6 of 8)

Table 6. Employment of technological environments

Technology	Percentage of Articles
Technology	24.9%(55 of 221)
Not employed	75.1%(166 of 221)

The survey showed that 24.9% of studies employed technological environments. We also determined whether there was a tendency for the employment of a technological environment depending on the researcher's perspective on the role of models (Table 7).

Table 7. Employment of technology according to perspectives on the roles of models

Roles of model	Technology	Not employed
Mediator	11.0% (13 of 118)	89.0% (105 of 118)
Conceptual system	34.0% (18 of 53)	66.0% (35 of 53)
Application	53.7% (22 of 41)	46.3% (19 of 41)
Critic	25.0% (2 of 8)	75.0% (6 of 8)

There is a noticeable tendency for the employment of a technological environment depending on the researcher's perspective on the role of models (Table 7). We considered that the main reason for this difference in the employment of a technological environment according to a researcher's perspective on the role of models is that there was a tendency in the types of tasks used depending on the researcher's perspective on the role of models. Given that, we determined whether the employment of a technological environment is determined by the type of modeling task (Table 8).

Table 8. Employment of technology according to the implemented type of task

Types of tasks	Technology	Not employed
Authentic	47.9% (35 of 73)	52.1% (38 of 73)
Word problem	11.5% (9 of 78)	88.5% (69 of 78)
Inner problem-solving	11.9% (5 of 42)	88.1% (37 of 42)

As we can see in Table 8, about half of the studies with authentic tasks employed a technological environment. Since technology is widely known as "[removing] much of the drudgery from modelling reality – long calculations, collecting and handling data, etc." (Burkhardt, p. 187) and authentic tasks require "simplifying and structuring the information extracted from the situation, and … choosing a suitable mathematical description of the situation," (Schukajlow, Leiss, Pekrun, Blum, Müller & Messner, 2012, p. 220) many of the studies with authentic tasks or applicative perspective researchers seem to have employed a technological environment.

5. Participants

We investigated the school ages of the participants of each study since this was considered an important variable in teaching and learning with models (Blum et al, 2002; Niss, Blum & Galbraith, 2007). Table 9 shows participants' school ages in the study sample.

Table 9. School ages of participants

School age	Percentage of Articles
Elementary	25.8%(57 of 221)
Secondary	32.6%(72 of 221)
Undergraduate	14.5%(32 of 221)
Mathematics Teachers	6.3%(14 of 221)
General	20.8%(46 of 221)

As we can see in Table 9, about half of the studies are focused on elementary and secondary school students. Especially, the results show that there were few model-related studies aimed at teacher education. The category of 'general' means that these articles do not target the mathematics learning of a specific school age.

We also determined whether participants' school age was correlated to a study's perspective on the role of models (Table 10).

The school ages of the participants are closely related to the researcher's perspective on the roles of models (Table 10). As we can see in Table 10, studies with the perspective on the role of a model as being a mediator mainly focused on elementary and secondary school students, studies with the perspective on the role of a model as being a conceptual system mainly focused on secondary school students, and studies with the perspective on the role of a model as being an application mainly focused on secondary school and undergraduate students.

6. Summary

We reviewed articles focusing on models in mathematics teaching and learning, and we attempted to identify how these studies are applying models in teaching and learning mathematics. To accomplish this purpose, we established three analytic questions and presented the results for addressing these questions. We found that the ways of applying models in each study, especially the types of tasks, the employment of technology, and the ages of the participants were closely related to the researcher's perspective on the role of models.

Researchers who consider the role of a model as that of a mediator mainly employed word problems and inner problem-solving tasks for elementary and secondary school students (Table 5). Since they employed a relatively small number of authentic tasks, they usually did not employ a technological

Table 10. School ages of participants according to perspectives on the roles of models

Roles of models	Elementary	Secondary	Undergraduate	Teacher	General
Mediator	41.5% (49 of 118)	33.9% (40 of 118)	13.6% (16 of 118)	0.8% (1 of 118)	10.2% (12 of 118)
Conceptual system	15.1% (8 of 53)	32.1% (17 of 53)	15.1% (8 of 53)	17.0% (9 of 53)	20.7% (11 of 53)
Application	-	34.1% (14 of 41)	19.5% (8 of 41)	9.8% (4 of 41)	36.6% (15 of 41)
Critic	-	12.5% (1 of 8)			87.5% (7 of 8)

environment (Table 7).

Researchers who consider the role of a model to be a conceptual system mainly employed authentic tasks and word problems for secondary school and undergraduate students (Table 5). They employed technological environments more often relative to the researchers who consider the role of a model to be a mediator (Table 7). These studies did not employ inner problem-solving tasks, since they emphasize model eliciting from authentic problem-solving situations. Also, further research is needed on young learners (such as English, 2006) since there is a relatively small number of studies on elementary school students (Table 10).

Researchers who consider the role of a model as an application employed only authentic tasks, and so about half of these studies employed technological environments (Table 5 and Table 7). It is not a surprising result that they applied only authentic tasks, since they emphasize both the application of models to the real-world and modeling competency, which encompasses the simplification and validation of models via real phenomena (see Blum & Borromeo Ferri, 2006). Actually, only researchers with this perspective on models tried to segregate authentic tasks from word problems or inner problem-solving tasks, and they attempted to find the role and significance of authentic tasks (e.g., Schukailow, et al., 2012). Further research on young learners' activities utilizing learning with the applicative perspective is necessary. Also, more empirical trials based on the perspective that a model has the role of a social critic need to be performed.

Ⅳ. Discussion and Conclusion

In this study, we analyzed studies on models in mathematics education, identified how these studies conceptualized the notion of a model, and showed how they applied models in the teaching and learning of mathematics. This study highlights several features inherent in the notion of a model and various ways of applying models to encourage communication among various perspectives on models, and supports cumulative research results. From the results of this study, we draw the following conclusions.

First, the results of this study once more indicate the variety of perspectives on models in the mathematics education research community (Kaiser & Sriraman, 2006; Lesh & Fennewald, 2010; Park & Lee, 2013) and the absence of developing theoretical approaches for conceptualizing the notion of a model. These are the main reasons for difficulties in communication and cumulative work on models in mathematics teaching and learning. As Lesh & Fennewald (2010) argued, this diversity can contribute to the growth of researches on models, but the absence of a theoretical approach for conceptualizing the notion of models might cause difficulties in cumulating various research results. Given that, we proposed a provisional definition of a model from Sfard's discursive perspective to encompass and cluster various approaches, but it is still broad and potentially revisable. As Ryve (2011) emphasizes, it is important to raise conceptual clarity and cumulative work of developing theoretical approaches for the development of research on models in mathematics education.

It is also necessary to examine the rationale and didactic potential of connecting various perspectives which are identified in this study. Blomhøj & Kjeldsen (2006) examined the possibility of pursuing two goals of teaching mathematics with models in a simultaneous way. One of these two goals was exactly the same as that of the applicative perspective, which we identified as a category, and the other one goal was "motivating and supporting students' learning of mathematics (p. 176)." Although some studies are categorized into different perspectives in accordance with the researcher's perspective on the role of a model (Table 3), there are still many common points among them in accordance with the researcher's focus on the aspects of models, ways of using the term model and so on. Thus we believe that there is still potential for further theoretical and empirical approaches to link various perspectives on models in the teaching and learning of mathematics.

Second, we found that the role of a model which is chosen by each researcher is closely related to both the researcher's ontological stance and their goals to achieve through mathematics teaching and learning. As Arzarello, Bosch, Lenfant & Prediger (2007) emphasize, it is effective to clarify various perspectives on models by identifying the underlying assumptions of each study. It is also helpful for us to clarify each way of conceptualizing the notion of a model by identifying three aspects of a model, three ways of using the term model and four roles of models. We expect that the results of this study will contribute to increasing communication among different perspectives on models and theories based on them.

Third, we proposed an alternative definition of a model based on Sfard's discursive perspective to encompass and shed light on three central aspects of a model and four roles of models in mathematics teaching and learning. Sfard's discursive perspective can be interpreted as a result of the endeavor to establish a coherent perspective on the relationship between the historical development of mathematics and individuals' mathematical progress (Sfard, 2008). As we already mentioned, we defined what a model is from the discursive perspective to interpret students' mathematical development, taking in to consideration the entire historical development of mathematics, individuals' progress and the development of models. From her perspective, it was also possible to clarify and highlight both each researcher's view of mathematics and their focus on the role of a model. Also, in this study our provisional definition, which is based on the discursive perspective, encompasses and sheds light on three central aspects of a model. Although the term model is widely used to describe scientific inquiry and psychological construct, we expect that Sfard's discursive perspective can guide our discussion on models for mathematical thinking and learning.

Fourth, we identified several tendencies in the ways of applying models in the teaching and learning of mathematics in accordance with the researcher's perspective on the role of models. From these results, we found several research areas which need further research. Especially, further researches on teacher education relevant to models, empirical studies on young learners' exploration for authentic modeling tasks, and studies with a social critic perspective on models are encouraged.

The limitation of this research resides with the selection of journals, articles, and analytic criteria for the research questions. Further attempts for clarifying the notion of a model and ways of applying models are encouraged by selecting other journals, articles or analytic criteria. Although this study highlights several central features of models in mathematics education, further theoretical approaches to conceptualize the notion of a model and improve our provisional perspective are encouraged.

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수학교육에서 모델의 활용에 대한 국외 문헌 연구

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수학교육에서 논의되고 있는 모델의 의미와 역할에 대한 여러 관점들을 분명히 할 필요성이 제기되고 있음에도 불구하고, 여전히 이에 대한 합의는 찾아보기 어렵다. 이에 본 연구에서는 모델에 대한 선행 연구들을 검토하여 수학교육 연구공동체에서 논의되고 있는 모델에 대한 여러 관점들을 분명히 하는 데 목적을 둔다. 연구 결과, 모델에 대한 세 가지 관점이 도출되었으며, 모델에 대한 각 관점은 수학적 지식에 대한 존재론적 관점과 모델의 인식론적 역할에 대한 해석과 밀접하게 관련된 것으로 확인되었다.

키워드 : 모델(model), 모델링(modeling), 문헌연구(literature review)

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