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ON THE EXTENDED HAAGERUP TENSOR PRODUCT IN OPERATOR SPACES

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Abstract. We describe the Haagerup tensor product $\ell^\infty \otimes_h \ell^\infty$ and the extended Haagerup tensor product $\ell^\infty \otimes_{eh} \ell^\infty$ in terms of Schur product maps, and show that $\ell^\infty \otimes_h \ell^\infty \cap \mathbb{B}(\ell^2)$ (resp. $\ell^\infty \otimes_{eh} \ell^\infty \cap \mathbb{B}(\ell^2)$) coincides with $c_0 \otimes_h c_0 \cap \mathbb{B}(\ell^2)$ (resp. $c_0 \otimes_{eh} c_0 \cap \mathbb{B}(\ell^2)$). For C*-algebras A, B, it is shown that $A \otimes_h B = A \otimes_{eh} B$ if and only if A or B is finite-dimensional.

1. Introduction

For Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, we let $\mathbb{B}(\mathcal{H},\mathcal{K})$ and $\mathbb{K}(\mathcal{H},\mathcal{K})$ denote the bounded operators and the compact operators of $\mathcal{H}$ to $\mathcal{K}$. An operator space X on $\mathcal{H}$ is a subspace of $\mathbb{B}(\mathcal{H}) = \mathbb{B}(\mathcal{H},\mathcal{H})$ which is endowed with norms to each $n \times m$ matrices $\mathbb{M}_{n,m}(X)$ over X as a subspace of $\mathbb{M}_{n,m}(\mathbb{B}(\mathcal{H})) \cong \mathbb{B}(\mathcal{H}^m, \mathcal{H}^n)$. We allow to use the notation $\mathbb{M}_{I,J}(\mathbb{B}(\mathcal{H})) \cong \mathbb{B}(\mathcal{H}^J, \mathcal{H}^I)$ for arbitrary index sets I and J. Let X and Y be operator spaces. The Haagerup tensor product of X and Y is the completion of the algebraic tensor product $X \otimes Y$ by the norm

$$\|u\|_h = \inf\{\|[a_1,\ldots,a_n]\|\|{}^t[b_1,\ldots,b_n]\| \mid u = \sum_{i=1}^n a_i \otimes b_i \in X \otimes Y,\ n \in \mathbb{N},\ a_i \in X,\ b_i \in Y\},$$

and is denoted by $X \otimes_h Y$ [4]. We also recall the extended Haagerup tensor product $X \otimes_{eh} Y$. An element u of $X \otimes_{eh} Y$ is represented by the following formal sum:

$$u = \sum_{i\in I} a_i \otimes b_i,$$

where $a = [a_i]_{i\in I} \in \mathbb{M}_{1,I}(X)$, $b = {}^t[b_i]_{i\in I} \in \mathbb{M}_{I,1}(Y)$ (in other words,

$$\|a\| = \|\sum_{i\in I} a_i a_i^*\|^{1/2} < \infty, \quad \|b\| = \|\sum_{i\in I} b_i^* b_i\|^{1/2} < \infty$$

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for $a_i \in X$ and $b_i \in Y$). We appreciate this formal sum as the bilinear form on $X^* \times Y^*$ as follows:

$$u(f,g) = \sum_{i\in I} f(a_i)g(b_i) \quad \text{for } f \in X^*, g \in Y^*.$$

For this element $u \in X \otimes_{eh} Y$, its norm is defined by

$$\|u\|_{eh} = \inf\{\|a\|\|b\| \mid u = \sum_{i\in I} a_i \otimes b_i,\ a \in \mathbb{M}_{1,I}(X),\ b \in \mathbb{M}_{I,1}(Y)\}.$$

Then we can realize $X \otimes_{eh} Y$ as a subspace of the dual operator space $(X^* \otimes_h Y^*)^*$ ([7], [8]).

In [9], the authors studied the Schur product on $\mathbb{B}(\mathcal{H})$ and used the extended Haagerup tensor product to describe the property of Schur product maps. Effros and Ruan has shown that $X \otimes_h Y$ is (completely isometrically) embedded to $X \otimes_{eh} Y$ [8]. We will be concerned with the difference between the Haagerup tensor product and the extended Haagerup tensor product, since it is essential to deal with Schur product maps derived from (possibly unbounded) operators. The Schur product map on $\mathbb{B}(\ell^2)$ is a normal ℓ^∞-bimodule map, where ℓ^∞ is a maximal abelian subalgebra of $\mathbb{B}(\ell^2)$ and is identified with the bounded sequences on $\mathbb{N}$ (c.f. [9]). As a deep result concerning (normal) bimodule maps, we often refer to the following theorem by Blecher and Smith in [3]: if M is a von Neumann algebra, then $M \otimes_{w^*h} M$ is completely isomorphic to the completely bounded M'-bimodule maps of $\mathbb{K}(\mathcal{H})$ to $\mathbb{B}(\mathcal{H})$ denoted by $CB_{M'}(\mathbb{K}(\mathcal{H}), \mathbb{B}(\mathcal{H}))$, where $\otimes_{w^*h}$ coincides with $\otimes_{eh}$ in this setting.

In Section 2, we study the difference between $\ell^\infty \otimes_h \ell^\infty$ and $\ell^\infty \otimes_{eh} \ell^\infty$ from the view point of Schur product and characterize them in terms of Schur product maps. Moreover we characterize $c_0 \otimes_h c_0$ and $c_0 \otimes_{eh} c_0$ in terms of Schur product maps, where c_0 is the complex sequences on $\mathbb{N}$ tends to 0. As a result for Schur product maps derived from bounded operators, we show that $\ell^\infty \otimes_h \ell^\infty \cap \mathbb{B}(\ell^2)$ (resp. $\ell^\infty \otimes_{eh} \ell^\infty \cap \mathbb{B}(\ell^2)$) coincides with $c_0 \otimes_h c_0 \cap \mathbb{B}(\ell^2)$ (resp. $c_0 \otimes_{eh} c_0 \cap \mathbb{B}(\ell^2)$).

In Section 3, we introduce some notions (right-compact, weakly right-compact, left-compact, weakly left-compact) for which distinguish the Haagerup tensor product from the extended Haagerup tensor product for operator spaces. As a main result in this section, for C*-algebras A, B, it is shown that $A \otimes_h B = A \otimes_{eh} B$ if and only if A or B is finite-dimensional.

2. $\ell^\infty \otimes_h \ell^\infty$ and $\ell^\infty \otimes_{eh} \ell^\infty$

Let X and Y be operator spaces and $X \otimes Y$ the algebraic tensor product of X and Y. For $a = [a_1, \ldots, a_n] \in \mathbb{M}_{1,n}(X)$, $b = {}^t[b_1, \ldots, b_n] \in \mathbb{M}_{n,1}(Y)$ and $\alpha = [\alpha_{ij}] \in \mathbb{M}_n(\mathbb{C})$, we denote $\sum_{i=1}^n a_i \otimes b_i \in X \otimes Y$ by $a \odot b$, and $\sum_{i,j=1}^n \alpha_{ij} a_i \otimes b_j$ by $a\alpha \odot b$.

Proposition 2.1. *If $u \in X \otimes Y$, then*

$$\|u\|_h = \inf\{\|a\|\|\alpha\|\|b\| \mid u = a\alpha \odot b \in X \otimes Y,\\ n \in \mathbb{N},\ \alpha \in \mathbb{M}_n(\mathbb{C}),\ a \in \mathbb{M}_{1,n}(X),\ b \in \mathbb{M}_{n,1}(Y)\}.$$

Proof. It follows from

$$\begin{aligned}\|u\|_h &= \inf\{\|a\|\|1_n\|\|b\| \mid u = a1_n \odot b\}\\ &\geq \inf\{\|a\|\|\alpha\|\|b\| \mid u = a\alpha \odot b\}\\ &\geq \inf\{\|a\alpha\|\|b\| \mid u = a\alpha \odot b\}\\ &\geq \inf\{\|a\|\|b\| \mid u = a \odot b\} = \|u\|_h.\end{aligned}$$

□

For $\alpha = [\alpha_{ij}]_{i,j=1}^{\infty} \in \mathbb{B}(\ell^2)$, $a = [a_1, a_2, \ldots] \in \mathbb{M}_{1,\infty}(X)$ and $b = {}^t[b_1, b_2, \ldots] \in \mathbb{M}_{\infty,1}(Y)$, we set

$$\alpha(k) = [\alpha_{ij}]_{i,j=1}^{k} \in \mathbb{M}_k(\mathbb{C}), \quad a(k) = [a_1, a_2, \ldots, a_k] \in \mathbb{M}_{1,k}(X)\\ \text{and} \quad b(k) = {}^t[b_1, b_2, \ldots, b_k] \in \mathbb{M}_{k,1}(Y)$$

for $k = 1, 2, 3, \ldots$. If the sequence $\{\alpha(k)a(k) \odot b(k)\}_{k=1}^{\infty}$ becomes a Cauchy sequence in $X \otimes_h Y$, then we denote this limit by

$$a\alpha \otimes b \in X \otimes_h Y.$$

When α belongs to $\mathbb{K}(\ell^2)$, we can see $a\alpha \otimes b$ as the usual limit of convergent sequences

$$\lim_{k\to\infty} \sum_{i,j=1}^{k} \alpha_{i,j} a_i \otimes b_j = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij} a_i \otimes b_j$$

in $X \otimes_h Y$ by the following reason. We choose a finite subset $J(k)$ of $\{(i,j) \mid \max\{i,j\} > k\}$, then we have

$$\|\sum_{(i,j)\in J(k)} \alpha_{ij} a_i \otimes b_j\|_h = \|a(N)\beta \odot b(N)\|_h \leq \|\beta\|\|a\|\|b\|,$$

where $N = \max\{i, j \mid (i,j) \in J(k)\}$, $\beta = [\beta_{ij}]_{i,j=1}^{N} \in \mathbb{M}_N(\mathbb{C})$ and

$$\beta_{ij} = \begin{cases} 0 & (i,j) \notin J(k)\\ \alpha_{ij} & (i,j) \in J(k). \end{cases}$$

If we choose a sufficiently large k, then we can make $\|\sum_{(i,j)\in J(k)} \alpha_{ij} a_i \otimes b_j\|_h$ sufficiently small because of the compactness of α.

Proposition 2.2. *If $u \in X \otimes_h Y$, then*

$$\begin{aligned}\|u\|_h = \inf\{&\|a\|\|\alpha\|\|b\| \mid \alpha \in \mathbb{K}(\ell^2),\ a \in \mathbb{M}_{1,\infty}(X),\ b \in \mathbb{M}_{\infty,1}(Y),\\ &u = a\alpha \otimes b = \sum_{i,j=1}^{\infty} \alpha_{ij} a_i \otimes b_j\}\\ = \inf\{&\max_i |\lambda_i|\|a\|\|b\| \mid (\lambda_i) \in c_0, a \in \mathbb{M}_{1,\infty}(X),\ b \in \mathbb{M}_{\infty,1}(Y),\end{aligned}$$

$$u = \sum_{i=1}^{\infty} \lambda_i a_i \otimes b_i\}$$

Proof. Suppose that $u \in X \otimes_h Y$ with $\|u\|_h < 1$. To prove the first equality, it suffices to show that there exist $a = [a_1, a_2, \ldots] \in \mathbb{M}_{1,\infty}(X)$ with $\|a\| < 1$, $\alpha = [\alpha_{ij}] \in \mathbb{K}(\ell^2)$ with $\|\alpha\| < 1$ and $b =^t [b_1, b_2, \ldots] \in \mathbb{M}_{\infty,1}(Y)$ with $\|b\| < 1$ such that

$$\sum_{i,j=1}^{k} \alpha_{ij} a_i \otimes b_j$$

converges to u in $X \otimes_h Y$ when k tends to ∞.

Given $\varepsilon = 1 - \|u\|_h > 0$. Then we can choose a sequence $\{u_n\} \subset X \otimes Y$, which converges to u, satisfying that $\|u_n\|_h < 1 - \varepsilon$ and $\|u_{n+1} - u_n\|_h < 2^{-n}\varepsilon$ $(n \geq 1)$, $u_0 = 0$. If we put $t_n = u_{n+1} - u_n$, then it turns out

$$\| \sum_{n=0}^{k} t_n - u\|_h = \|u_{k+1} - u\|_h \to 0 \ (k \to \infty).$$

For $t_n \in X \otimes Y$, there exist $v_n \in \mathbb{M}_{1,\ell(n)}(X)$, $\beta_n \in \mathbb{M}_{\ell(n)}$ and $w_n \in \mathbb{M}_{\ell(n),1}$ such that $t_n = v_n\beta_n \otimes w_n$ with $\|\beta_n\| = 1 (n \geq 0)$, $\|v_n\|\|w_n\| < 2^{-n}\varepsilon(n \geq 1)$, $\|v_0\|\|w_0\| < 1 - \varepsilon$ and $\|v_n\| = \|w_n\|$. It follows that

$$\sum_{n=0}^{\infty} \|t_n\|_h \leq \sum_{n=0}^{\infty} \|v_n\|\|w_n\| < 1.$$

Then we can choose an increasing sequence $\{c_n\} \subset \mathbb{R}$ such that

$$c_n > 1, \ \lim_{n\to\infty} c_n = \infty, \ \sum_{n=0}^{\infty} c_n\|v_n\|\|w_n\| < 1.$$

Now we put $a(i) = \sqrt{c_i}v_i$, $\alpha_i = \beta_i/c_i$ and $b(i) = \sqrt{c_i}w_i$. Then we have

$$\begin{aligned} u_{k+1} &= \sum_{n=0}^{k} v_n\beta_n \odot w_n = \sum_{n=0}^{k} a(n)\alpha_n \odot b(n) \\ &= \begin{bmatrix} a(0) & a(1) & \ldots & a(k) \end{bmatrix} \begin{bmatrix} \alpha_0 & & & \\ & \alpha_1 & & \\ & & \ddots & \\ & & & \alpha_k \end{bmatrix} \odot \begin{bmatrix} b(0) \\ b(1) \\ \vdots \\ b(k) \end{bmatrix}, \end{aligned}$$

and $\|[a(0), a(1), \ldots, a(k)]\|, \|^t[b(0), b(1), \ldots, b(k)]\| < 1$, $\|\alpha_k\| \to 0$ $(k \to \infty)$. If we define $a_n \in X$, $b_n \in Y$ and $\alpha \in \mathbb{K}(\ell^2)$ by the following relation:

$$[a(0), a(1), \ldots, a(k)] = [a_1, a_2, \ldots, a_{\ell(0)+\ell(1)+\cdots+\ell(k)}]$$
$$[b(0), b(1), \ldots, b(k)] = [b_1, b_2, \ldots, b_{\ell(0)+\ell(1)+\cdots+\ell(k)}]$$
$$\alpha = \bigoplus_{k=0}^{\infty} \alpha_k,$$

then we can get the first equality.

For the above $\alpha \in \mathbb{K}(\ell^2)$, we can take unitaries u_k, $v_k \in \mathbb{M}_{\ell(k)}$ such that

$$\alpha_k = u_k \begin{bmatrix} \lambda_{\sum_{i=0}^{k-1} \ell(i)+1} & & & \\ & \lambda_{\sum_{i=0}^{k-1} \ell(i)+2} & & \\ & & \ddots & \\ & & & \lambda_{\sum_{i=0}^{k} \ell(i)} \end{bmatrix} v_k \quad (k = 0, 1, 2, \ldots).$$

If we put

$$U = \bigoplus_{k=0}^{\infty} u_k, \ V = \bigoplus_{k=0}^{\infty} v_k, \ \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \end{bmatrix},$$

then we can get

$$\|\Lambda\| = \max_i |\lambda_i|,$$
$$aU \in \mathbb{M}_{1,\infty}(X) \text{ and } \|a\| = \|aU\|,$$
$$Vb \in \mathbb{M}_{\infty,1}(Y) \text{ and } \|b\| = \|Vb\|,$$

for any $a \in \mathbb{M}_{1,\infty}(X)$ and $b \in \mathbb{M}_{\infty,1}(Y)$. By the fact

$$a\alpha \otimes b = aU\Lambda V \otimes b = (aU)\Lambda \otimes (Vb),$$

we can get the second equality. □

By the above proof, we also get the following fact:

$$\begin{aligned} X \otimes_h Y &= \{a\alpha \otimes b \mid \alpha \in \mathbb{K}(\ell^2), a \in \mathbb{M}_{1,\infty}(X), b \in \mathbb{M}_{\infty,1}(Y)\} \\ &= \{\sum_{i=1}^{\infty} \lambda_i a_i \otimes b_i \mid (\lambda_i) \in c_0, a \in \mathbb{M}_{1,\infty}(X), b \in \mathbb{M}_{\infty,1}(Y)\}. \end{aligned}$$

Let $\mathcal{H}$ be a separable Hilbert space, $\{f_i\}_{i=1}^{\infty}$ a completely orthonormal system of $\mathcal{H}$ and $\{e_{ij}\}_{i,j=1}^{\infty}$ a system of matrix units of $\mathbb{B}(\mathcal{H})$ defined by

$$e_{ij}\xi = (\xi|f_j)f_i, \qquad \xi \in \mathcal{H}.$$

We can naturally identify the bounded sequences ℓ^∞ on $\mathbb{N}$ with the maximal abelian subalgebra of $\mathbb{B}(\mathcal{H})$ generated by $\{e_{ii}\}_{i=1}^{\infty}$. We denote by

$$CB_{\ell^\infty}(\mathbb{K}(\mathcal{H}), \mathbb{B}(\mathcal{H}))$$

the ℓ^∞-bimodule completely bounded maps of $\mathbb{K}(\mathcal{H})$ to $\mathbb{B}(\mathcal{H})$. Then there exists completely isometric isomorphism between $\ell^\infty \otimes_{eh} \ell^\infty$ and $CB_{\ell^\infty}(\mathbb{K}(\mathcal{H}), \mathbb{B}(\mathcal{H}))$ by the following: for $\sum_i a_i \otimes b_i \in \ell^\infty \otimes_{eh} \ell^\infty$, $\langle \sum_i a_i \otimes b_i \rangle \in CB_{\ell^\infty}(\mathbb{K}(\mathcal{H}), \mathbb{B}(\mathcal{H}))$ is defined by

$$\langle \sum_i a_i \otimes b_i \rangle(k) = \sum_i a_i k b_i$$

for $k \in \mathbb{K}(\mathcal{H})$ [3]. By the ℓ^∞-bimodularity of $\langle x\rangle$ for $x \in \ell^\infty \otimes_{eh} \ell^\infty$, there exists a scalar x_{ij} satisfying that

$$\langle x\rangle(e_{ij}) = x_{ij}e_{ij}, \qquad i, j = 1, 2, \ldots.$$

Then we can define an infinite dimensional matrix

$$[x] = [x_{ij}]_{i,j=1}^\infty,$$

and also identify $[x]$ with a linear map from $c_c(\mathbb{N})$ to ℓ^∞ as follows: for $\xi = [\xi_1, \xi_2, \ldots] \in c_c(\mathbb{N})$,

$$[x]\xi = [\sum_{j=1}^\infty x_{1j}\xi_j, \sum_{j=1}^\infty x_{2j}\xi_j, \ldots],$$

where $\xi = [\xi_1, \xi_2, \ldots] \in c_c(\mathbb{N})$ means that $\xi_n = 0$ for sufficiently large n. Clearly $c_c(\mathbb{N})$ is contained in ℓ^2 and the image of $c_c(\mathbb{N})$ by $[x]$ is not necessarily contained in ℓ^2. If $[x]$ can be extended to $\mathbb{B}(\ell^2)$ (resp. $\mathbb{K}(\ell^2)$), then we write $x \in (\ell^\infty \otimes_{eh} \ell^\infty) \cap \mathbb{B}$ (resp. $x \in (\ell^\infty \otimes_{eh} \ell^\infty) \cap \mathbb{K}$). We also use the following notation: for any subspace S of $\ell^\infty \otimes_{eh} \ell^\infty$,

$$S \cap \mathbb{B} = (\ell^\infty \otimes_{eh} \ell^\infty) \cap \mathbb{B} \cap S,$$
$$S \cap \mathbb{K} = (\ell^\infty \otimes_{eh} \ell^\infty) \cap \mathbb{K} \cap S.$$

Lemma 2.3. *$x \in \ell^\infty \otimes_h \ell^\infty$ if and only if there exist $\beta \in \mathbb{K}(\ell^2)$, ξ_i, $\eta_i \in \ell^2$ $(i = 1, 2, \ldots)$ such that*

$$\sup_i\{\|\xi_i\|, \|\eta_i\|\} < \infty \text{ and } x_{ij} = (\beta\xi_i|\eta_j).$$

Proof. By Proposition 2.2, for given $\varepsilon > 0$ and $x \in \ell^\infty \otimes_h \ell^\infty$, there exist $[a_1, a_2, \ldots] \in \mathbb{M}_{1,\infty}(\ell^\infty)$, ${}^t[b_1, b_2, \ldots] \in \mathbb{M}_{\infty,1}(\ell^\infty)$ and $[\alpha_{ij}] \in \mathbb{K}(\mathcal{H})$ satisfying $\|[a_1, a_2, \ldots]\|\ \|{}^t[b_1, b_2, \ldots]\| < 1$ and $\|[\alpha_{ij}]\| < \|x\|_h + \varepsilon$ such that $x = \sum_{i,j=1}^\infty \alpha_{ij}a_i \otimes b_j$. If $\xi_i = [a_1(i), a_2(i), \ldots]$, $\eta_i = [\overline{b_1(i)}, \overline{b_2(i)}, \ldots]$ and $\beta = [\beta_{ij}]$ where $\beta_{ij} = \alpha_{ji}$, then it is clear that $\sup_i\{\|\xi_i\|, \|\eta_i\|\} < \infty$. Thus we have

$$\begin{aligned}\langle x\rangle(e_{ij}) &= \sum_{s,t} \alpha_{st}a_s e_{ij} b_t \\ &= \sum_{s,t} \alpha_{st}a_s(i)b_t(j)e_{ij} \\ &= (\beta\xi_i|\eta_j)e_{ij}.\end{aligned}$$

Conversely, for given $\xi_i = [\xi_i(1), \xi_i(2), \ldots]$, $\eta_i = [\eta_i(1), \eta_i(2), \ldots] \in \ell^2$ and $\beta = [\beta_{ij}] \in \mathbb{K}(\ell^2)$, we put $a_i = [\xi_1(i), \xi_2(i), \ldots]$, $b_i = [\overline{\eta_1(i)}, \overline{\eta_2(i)}, \ldots] \in \ell^\infty$ and $\alpha = [\alpha_{ij}] \in \mathbb{K}(\ell^2)$ where $\alpha_{ij} = \beta_{ji}$. Then we have, for any positive integer N,

$$\|[a_1, a_2, \ldots, a_N]\|, \|{}^t[b_1, b_2, \ldots, b_N]\| \le \sup\{\|\xi_i\|, \|\eta_i\|\} < \infty.$$

For an element

$$x_n = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix} \odot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \ell^\infty \otimes \ell^\infty,$$

we have

$$x_{n+k} - x_n = \begin{bmatrix} a_1 & \dots & a_{n+k} \end{bmatrix} \times \begin{bmatrix} 0 & \cdots & 0 & \alpha_{1,n+1} & \cdots & \alpha_{1,n+k} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \vdots & & \vdots \\ \alpha_{n+1,1} & \cdots & \cdots & \alpha_{n+1,n+1} & & \vdots \\ \vdots & & & & \ddots & \vdots \\ \alpha_{n+k,1} & \cdots & \cdots & \cdots & \cdots & \alpha_{n+k,n+k} \end{bmatrix} \odot \begin{bmatrix} b_1 \\ \vdots \\ b_{n+k} \end{bmatrix}.$$

By the compactness of α and Proposition 2.1,

$$\lim_{n\to\infty} \|x_{n+k} - x_n\|_h = 0$$

for any positive integer k. Thus we have that the sequence $\{x_n\}$ converges to x in $\ell^\infty \otimes_h \ell^\infty$. □

Since c_0 is a C*-subalgebra of ℓ^∞, we can see $c_0 \otimes_h c_0$ as a subspace of $\ell^\infty \otimes_h \ell^\infty$.

Lemma 2.4. *$x \in c_0 \otimes_h c_0$ if and only if there exist $\xi_i, \eta_i \in \ell^2 (i = 1, 2, \dots)$ such that*

$$\lim_i \|\xi_i\| = \lim_i \|\eta_i\| = 0 \text{ and } x_{ij} = (\xi_i \mid \eta_j).$$

Proof. By Proposition 2.2, for given $\varepsilon > 0$ and $x \in c_0 \otimes_h c_0$, there exist $[a_1, a_2, \dots] \in \mathbb{M}_{1,\infty}(c_0)$, ${}^t[b_1, b_2, \dots] \in \mathbb{M}_{\infty,1}(c_0)$ and $[\alpha_{ij}] \in \mathbb{K}(\mathcal{H})$ satisfying $\|[a_1, a_2, \dots]\| \ \|{}^t[b_1, b_2, \dots]\| < 1$ and $\|[\alpha_{ij}]\| < \|x\|_h + \varepsilon$ such that $x = \sum_{i,j=1}^\infty \alpha_{ij} a_i \otimes b_j$. We put $\xi_i = [a_1(i), a_2(i), \dots]$, $\eta_i = [\overline{b_1(i)}, \overline{b_2(i)}, \dots]$ and $\beta = [\beta_{ij}]$ where $\beta_{ij} = \alpha_{ji}$. Then we have

$$x_{ij} = (\beta\xi_i | \eta_j)$$

and, by the fact $a_i, b_i \in c_0$,

$$\lim_{i\to\infty} \xi_i(j) = \lim_{i\to\infty} \eta_i(j) = 0 \text{ for any } j \in \mathbb{N}.$$

This means that $\{\xi_i\}$, $\{\eta_i\} \subset \ell^2$ weakly converge to 0. We can choose β_1, β_2 $\in \mathbb{K}(\ell^2)$ such that $\beta = \beta_2^*\beta_1$. Then we have

$$\lim_i \|\beta_1\xi_i\| = \lim_i \|\beta_2\eta_i\| = 0$$

and

$$x_{ij} = (\beta\xi_i|\eta_j) = (\beta_1\xi_i|\beta_2\eta_j).$$

Conversely suppose that $\lim_{i\to\infty}\|\xi_i\| = \lim_{i\to\infty}\|\eta_i\| = 0$. We may assume that $\|\xi_i\| < c$ for all $i \in \mathbb{N}$. Then, for any $\varepsilon > 0$, we can choose a number N such that

$$\|\xi_i'\| < c \text{ for all } i \quad \text{and} \quad \|\xi_i'\| < \varepsilon \text{ if } i > N,$$

where

$$\xi_i'(j) = \begin{cases} \xi_i(j) & \text{if } j < N \\ 2\xi_i(j) & \text{otherwise.} \end{cases}$$

Clearly we have $\lim_{i\to\infty}\|\xi_i'\| = 0$. Applying this argument to $\{\xi_i\}$ repeatedly, we can choose $1 = n(0) < n(1) < n(2) < \cdots$ and $\{\zeta_i\} \subset \ell^2$ such that

$$\zeta_i(j) = 2^k\xi_i(j) \quad \text{if } n(k) \le j < n(k+1),$$

$$\|\zeta_i\| < c \text{ for all } i \quad \text{and} \quad \|\zeta_i\| < 2^{-k} \text{ if } i > n(k).$$

We put $a_i = [\zeta_1(i), \zeta_2(i), \ldots]$, $b_i = [\overline{\eta_1(i)}, \overline{\eta_2(i)}, \ldots]$ and

$$\lambda_i = 2^{-k} \quad \text{if } n(k) \le i < n(k+1).$$

Then we have $a_i, b_i, (\lambda_i) \in c_0$, and

$$\|[a_1, a_2, \ldots]\|, \|{}^t[b_1, b_2, \ldots]\| \le \sup\{\|\zeta_i\|, \|\eta_i\|\} < \infty.$$

Thus we have

$$x = \sum_{i=1}^{\infty} \lambda_i a_i \otimes b_i \in c_0 \otimes_h c_0.$$

□

Combining these lemmas, we can get the following fact:

Theorem 2.5. (1) *For $x \in \ell^\infty \otimes_h \ell^\infty$,*

$$\|x\|_h = \inf\{\sup_{ij}\|\xi_i\|\|\eta_j\|\|\beta\| \mid x_{ij} = (\beta\xi_i|\eta_j),\ \xi_i,\ \eta_j \in \ell^2, \beta \in \mathbb{K}(\ell^2)\}.$$

(2) *For $x \in c_0 \otimes_h c_0$,*

$$\|x\|_h = \inf\{\sup_{ij}\|\xi_i\|\|\eta_j\| \mid x_{ij} = (\xi_i|\eta_j),\ \xi_i,\ \eta_j \in \ell^2, \xi_i \to 0,\ \eta_j \to 0 \ \textit{strongly}\}.$$

The correspondence between Schur multiplier and the tensor product with a suitable norm is known as in [12], and the result is obtained by Spronk in [15]. By using the extended Haagerup norm on $\ell^\infty \otimes_{eh} \ell^\infty$ and $c_0 \otimes_{eh} c_0$, we can rewrite them as follows:

Proposition 2.6 ([15], Corollary 3.2)**.** (1) *For $x \in \ell^\infty \otimes_{eh} \ell^\infty$,*

$$\|x\|_{eh} = \inf\{\sup_{ij}\|\xi_i\|\|\eta_j\| \mid x_{ij} = (\xi_i|\eta_j),\ \xi_i,\ \eta_j \in \ell^2\}.$$

(2) *For $x \in c_0 \otimes_{eh} c_0$,*

$$\|x\|_{eh} = \inf\{\sup_{ij}\|\xi_i\|\|\eta_j\| \mid x_{ij} = (\xi_i|\eta_j),\ \xi_i,\ \eta_j \in \ell^2, \xi_i \to 0,\ \eta_j \to 0 \ \textit{weakly}\}.$$

Theorem 2.7. (1) $(c_0 \otimes_h c_0) \cap \mathbb{B} = (\ell^\infty \otimes_h \ell^\infty) \cap \mathbb{B}$.
(2) $(c_0 \otimes_{eh} c_0) \cap \mathbb{B} = (\ell^\infty \otimes_{eh} \ell^\infty) \cap \mathbb{B}$.

Proof. (1) It is clear that $(c_0 \otimes_h c_0) \cap \mathbb{B} \subset (\ell^\infty \otimes_h \ell^\infty) \cap \mathbb{B}$.

Let $x \in (\ell^\infty \otimes_h \ell^\infty) \cap \mathbb{B}$. By Lemma 2.3, there exist $\beta \in \mathbb{K}(\ell^2)$, ξ_i, $\eta_i \in \ell^2$ $(i = 1, 2, \ldots)$ such that

$$\sup_i \{\|\xi_i\|, \|\eta_i\|\} < \infty \text{ and } x_{ij} = (\beta\xi_i|\eta_j).$$

We choose β_1, $\beta_2 \in \mathbb{K}(\ell^2)$ such that $\beta = \beta_2^*\beta_1$, that is,

$$x_{ij} = (\beta_1\xi_i|\beta_2\eta_j),$$

and we may assume that

$$\text{Range}(\beta_1) \subset \overline{\text{span}\{\beta_2\eta_j \mid j \in \mathbb{N}\}}$$

and

$$\text{Range}(\beta_2) \subset \overline{\text{span}\{\beta_1\xi_j \mid j \in \mathbb{N}\}}.$$

It is sufficient to show that

$$\lim_i \|\beta_1\xi_i\| = \lim_i \|\beta_2\eta_i\| = 0.$$

Assume that

$$\limsup_i \|\beta_1\xi_i\| > 0.$$

Then there exist $\delta > 0$ and a subsequence $\{n(k)\}$ such that $\|\beta_1\xi_{n(k)}\| > \delta$ for $k = 1, 2, \ldots$. Since $\sup\|\xi_i\| < \infty$, we may also assume that $\{\xi_{n(k)}\}$ weakly converges to some $\xi_0 \in \ell^2$. By the compactness of β_1, we have

$$\lim_i \|\beta_1\xi_{n(k)} - \beta_1\xi_0\| = 0.$$

Thus it turns out $\beta_1\xi_0 \neq 0$. We can choose j_0 such that

$$(\beta_1\xi_0|\beta_2\eta_{j_0}) \neq 0.$$

Then there exists $K \in \mathbb{N}$ such that

$$|x_{n(k),j_0}| = |(\beta_1\xi_{n(k)}|\beta_2\eta_{j_0})| > \frac{|(\beta_1\xi_0|\beta_2\eta_{j_0})|}{2} \quad \text{for } k > K.$$

This contradicts to $[x] = [x_{ij}] \in \mathbb{B}(\ell^2)$.

(2) For $x \in (\ell^\infty \otimes_{eh} \ell^\infty) \cap \mathbb{B}$, that is, $[x] \in \mathbb{B}(\ell^2)$, we can choose $\alpha = [\xi_{ij}]$ and $\beta = [\eta_{ij}]$ in $\mathbb{B}(\ell^2)$ such that

$$[x] = \alpha\beta \text{ and } \|\alpha\| = \|\beta\| = \|[x]\|^{1/2}.$$

Remarking the fact

$$x_{ij} = \sum_k \xi_{ik}\eta_{kj},$$

we define

$$a_i = [\xi_{1i}, \xi_{2i}, \ldots],\ b_i = [\eta_{i1}, \eta_{i2}, \ldots] \in \ell^2 \subset c_0$$

for all i. Then we have

$$\|[a_1, a_2, \ldots]\|, \|^t[b_1, b_2, \ldots]\| \le \|[x]\|^{1/2} < \infty$$

and $x = \sum_i a_i \otimes b_i \in c_0 \otimes_{eh} c_0$. □

Corollary 2.8. *Let* $x \in \ell^\infty \otimes_{eh} \ell^\infty$ *and*

$$\limsup_k |x_{i(k),j(k)}| > 0$$

for some injection $\mathbb{N} \ni k \to (i(k), j(k)) \in \mathbb{N} \times \mathbb{N}$. *Then* x *does not belong to* $c_0 \otimes_h c_0$.

Moreover, if x *satisfies an additional condition* $[x] \in \mathbb{B}(\ell^2)$, *then* x *does not belong to* $\ell^\infty \otimes_h \ell^\infty$.

Example 2.9. (1) Let $x = \sum_{i,j=1}^\infty (\frac{\lambda_i}{\lambda_j})^{\sqrt{-1}t} e_i \otimes e_j \in \ell^\infty \otimes_{eh} \ell^\infty$, where λ_i's are positive real and t is real. Then we have

$$[x] = \begin{bmatrix} (\frac{\lambda_1}{\lambda_1})^{\sqrt{-1}t} & (\frac{\lambda_1}{\lambda_2})^{\sqrt{-1}t} & \cdots \\ (\frac{\lambda_2}{\lambda_1})^{\sqrt{-1}t} & (\frac{\lambda_2}{\lambda_2})^{\sqrt{-1}t} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \notin \mathbb{B}(\ell^2),$$

$$x_{ij} = (\frac{\lambda_i}{\lambda_j})^{\sqrt{-1}t} = (\begin{bmatrix} 1 & & \\ & 0 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \lambda_i^{\sqrt{-1}t} \\ 0 \\ \vdots \end{bmatrix} | \begin{bmatrix} \lambda_j^{\sqrt{-1}t} \\ 0 \\ \vdots \end{bmatrix})$$

and $|x_{ij}| = 1$. This means $x \notin (\ell^\infty \otimes_{eh} \ell^\infty) \cap \mathbb{B}$, $x \notin c_0 \otimes_{eh} c_0$ (by Proposition 2.6) and $x \in \ell^\infty \otimes_h \ell^\infty$ (by Lemma 2.3).

(2) Let $x = \sum_{k=1}^\infty e_k \otimes e_k \in c_0 \otimes_{eh} c_0$. Since

$$[x] = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \in \mathbb{B}(\ell^2)$$

then we have $x \notin \ell^\infty \otimes_h \ell^\infty$ (by Corollary 2.8).

(3) $(\ell^\infty \otimes_{eh} \ell^\infty) \cap \mathbb{K} \subsetneqq (\ell^\infty \otimes_h \ell^\infty) \cap \mathbb{B}$.

By Lemma 2.3, it is clear that $(\ell^\infty \otimes_{eh} \ell^\infty) \cap \mathbb{K} \subset (\ell^\infty \otimes_h \ell^\infty) \cap \mathbb{B}$.

We consider the following infinite dimensional matrix:

$$p = \begin{bmatrix} 1 & & & & & & & \\ & \frac{1}{2} & \frac{1}{2} & & & & & \\ & \frac{1}{2} & \frac{1}{2} & & & & & \\ & & & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & & \\ & & & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & & \\ & & & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & & \\ & & & & & & \ddots & \\ & & & & & & & \ddots \end{bmatrix}.$$

Since p is an infinite dimensional projection, p does not belong to $\mathbb{K}(\ell^2)$. If we put

$$\xi_1 = [1, 0, 0, 0, \ldots]$$
$$\xi_2 = \xi_3 = [0, \frac{1}{\sqrt{2}}, 0, 0 \ldots]$$
$$\xi_4 = \xi_5 = \xi_6 = [0, 0, 0, \frac{1}{\sqrt{3}}, 0, \ldots]$$
$$\cdots$$

and $\xi_n = \eta_n$ $(n = 1, 2, \ldots)$, then ξ_n, $\eta_n \in \ell^2$ satisfy

$$\lim_n \|\xi_n\| = \lim_n \|\eta_n\| = 0 \text{ and } p = [(\xi_i|\eta_j)].$$

This means that

$$((\ell^\infty \otimes_h \ell^\infty) \cap \mathbb{B}) \cap ((\ell^\infty \otimes_{eh} \ell^\infty) \cap \mathbb{K})^c \neq \phi.$$

(4) Let $a = b = [1, \frac{1}{\sqrt{2}}, \ldots, \frac{1}{\sqrt{n}}, \ldots] \in c_0$. Then $x = a \otimes b \in c_0 \otimes_h c_0$ and

$$[x_{ij}] = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & \cdots \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \notin \mathbb{B}(\ell^2).$$

By the above argument, we can get the following diagram of inclusions:

$$\begin{array}{ccccc} (c_0 \otimes_h c_0) \cap \mathbb{K} & \subsetneqq & (c_0 \otimes_h c_0) \cap \mathbb{B} & \subsetneqq & (c_0 \otimes_{eh} c_0) \cap \mathbb{B} \\ & & \| & & \| \\ & & (\ell^\infty \otimes_h \ell^\infty) \cap \mathbb{B} & \subsetneqq & (\ell^\infty \otimes_{eh} \ell^\infty) \cap \mathbb{B} \\ & & \not\shortparallel\cap & & \not\shortparallel\cap \\ & & c_0 \otimes_h c_0 & \subsetneqq & c_0 \otimes_{eh} c_0 \\ & & \not\shortparallel\cap & & \not\shortparallel\cap \\ & & \ell^\infty \otimes_h \ell^\infty & \subsetneqq & \ell^\infty \otimes_{eh} \ell^\infty \end{array}.$$

3. $X \otimes_h Y$ and $X \otimes_{eh} Y$

Let X be an operator space. We call X is right-compact (resp. left-compact) if $\mathbb{M}_{1,I}(X) = \mathbb{M}_{1,I}(X)\mathbb{K}(\ell^2(I))$ (resp. $\mathbb{M}_{I,1}(X) = \mathbb{K}(\ell^2(I))\ \mathbb{M}_{I,1}(X)$). If X is right-compact, then, for any $a = [a_i]_{i \in I} \in \mathbb{M}_{1,I}(X)$, there exist $b = [b_i]_{i \in I} \in \mathbb{M}_{1,I}(X)$ and $\alpha = [\alpha_{ij}]_{i,j \in I} \in \mathbb{K}(\ell^2(I))$ such that

$$a = b\alpha \quad (a_j = \sum_{i \in I} b_i \alpha_{ij}).$$

We also call X is weakly right-compact (resp. weakly left-compact) if we have, for any $a = [a_i] \in \mathbb{M}_{1,I}(X)$ (resp. $a = [a_i] \in \mathbb{M}_{I,1}(X)$), that $\{i \in I \mid a_i \neq 0\}$ is countable and $\lim_{i \to \infty} \|a_i\| = 0$.

Lemma 3.1. *If X is a right-compact (resp. left-compact) operator space, then X is weakly right-compact (resp. weakly left-compact).*

Proof. Since $\alpha = [\alpha_{ij}]_{i,j\in I} \in \mathbb{K}(\ell^2(I))$, we have that $\{(i,j) \in I \times I \mid \alpha_{ij} \neq 0\}$ is countable and $\{i \in I \mid a_i \neq 0\}$ is also countable. So we may assume that $I = \mathbb{N}$. Then we have $a_i = \sum_{j=1}^{\infty} b_j \alpha_{ji}$ and

$$\begin{aligned}\|a_i\| &= \| \begin{bmatrix} b_1 & b_2 & \cdots & \end{bmatrix} \begin{bmatrix} \alpha_{1i} \\ \alpha_{2i} \\ \vdots \end{bmatrix} \| \leq \| \begin{bmatrix} b_1 & b_2 & \cdots & \end{bmatrix} \| \| \begin{bmatrix} \alpha_{1i} \\ \alpha_{2i} \\ \vdots \end{bmatrix} \| \\ &= \| \sum_{j=1}^{\infty} |\alpha_{ji}|^2 \|^{1/2} \| \| \sum_{i=1}^{\infty} b_i b_i^* \|^{1/2}.\end{aligned}$$

By the assumtion, we have

$$\| \sum_{i=1}^{\infty} b_i b_i^* \| < \infty \text{ and } \lim_{i\to\infty} \sum_{j=1}^{\infty} |\alpha_{ji}|^2 = 0.$$

This means that

$$\lim_{i\to\infty} \|a_i\| = 0. \qquad \square$$

As a typical example of right-compact operator spaces, we can get the following:

Lemma 3.2. *Let X be an operator space on a Hilbert space $\mathcal{H}$. If $X \subset p\mathbb{B}(\mathcal{H})$ for some finite-dimensional projection $p \in \mathbb{B}(\mathcal{H})$, then X is right-compact.*

In particular, any finite-dimensional C^*-algebra is left- and right-compact.

Proof. We assume that $\dim p\mathcal{H} = n < \infty$. Let $a = [a_i]_{i\in I} \in \mathbb{M}_{1,I}(X)$, i.e.,

$$\| \sum_{i\in I} a_i a_i^* \| < \infty.$$

We can consider $a_i a_i^*$ as an element of $\mathbb{M}_n(\mathbb{C})$, so we put $a_i a_i^* = (\alpha^i_{jk})$ $(j,k = 1,2,\ldots,n)$. By the positivity of $a_i a_i^*$, we have

$$0 \leq \max_{1\leq j\leq n} \sum_{i\in I} \alpha^i_{jj} \leq \| \sum_{i\in I} a_i a_i^* \|.$$

This implies that

$$I_0 = \{i \in I \mid \alpha^i_{jj} > 0 \text{ for some } j\}$$

is countable, so we may assume $I = \mathbb{N}$. Remarking the fact

$$\|a_i a_i^*\| \leq \sqrt{n} \max_{1\leq j\leq n} \alpha^i_{jj},$$

we have

$$\| \sum_{i=1}^{\infty} a_i a_i^* \| \leq \sum_{i=1}^{\infty} \|a_i a_i^*\| \leq \sqrt{n} \sum_{i=1}^{\infty} \max_{1\leq j\leq n} \alpha^i_{jj} \leq \sqrt{n} \sum_{i=1}^{\infty} \sum_{j=1}^{n} \alpha^i_{jj}$$

$$= \sqrt{n}\sum_{j=1}^{n}\sum_{i=1}^{\infty}\alpha_{jj}^{i} \leq n\sqrt{n}\max_{1\leq j\leq n}\sum_{i=1}^{\infty}\alpha_{jj}^{i} < \infty.$$

We can choose numbers $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ such that

$$\|\sum_{i=\lambda_k+1}^{\infty} a_i a_i^*\| < \frac{1}{2^k}\|\sum_{i=1}^{\infty} a_i a_i^*\|$$

and set $b_i = \sqrt{k}a_i$ ($\lambda_{k-1} < i \leq \lambda_k$). Since

$$\|\sum_{i=1}^{\infty} b_i b_i^*\| = \|\sum_{k=0}^{\infty}\sum_{i=\lambda_k+1}^{\infty} a_i a_i^*\| < \sum_{k=0}^{\infty}\frac{1}{2^k}\|\sum_{i=1}^{\infty} a_i a_i^*\| = 2\|\sum_{i=1}^{\infty} a_i a_i^*\| < \infty,$$

we have

$$a = [a_i]_{i\in\mathbb{N}} = [b_i]_{i\in\mathbb{N}}[\frac{1}{\sqrt{k(i)}}\delta_{ij}]_{i,j\in\mathbb{N}} \in \mathbb{M}_{1\mathbb{N}}(X)\mathbb{K}(\ell^2),$$

where $k(i) = k$ if $\lambda_{k-1} < i \leq \lambda_k$ and δ_{ij} means Kronecker's symbol. □

Lemma 3.3. *Let $\{a_i\}_{i=1}^{\infty}$ be a sequence of bounded operators on a Hilbert space $\mathcal{H}$ with $\|\sum_{i=1}^{\infty} a_i a_i^*\| < \infty$. Suppose that there exist sequences $\{\xi_i\}_{i=1}^{\infty}$, $\{\eta_i\}_{i=1}^{\infty}$ of unit vectors in $\mathcal{H}$ such that*

$$|(a_i\xi_i|\eta_i)| > 1 \quad \textit{for } i \in \mathbb{N}.$$

Then, for any $\varepsilon > 0$, there exists a number i_0 such that

$$\{i \in \mathbb{N} \mid |(a_{i_0}\xi_i|\eta_i)| < \varepsilon\}$$

is infinite.

Proof. Let $\varepsilon > 0$. Suppose that $\{j \in \mathbb{N} \mid |(a_i\xi_j|\eta_j)| < \varepsilon\}$ is finite for all $i \in \mathbb{N}$. If we omit the finite set

$$\{i \in \mathbb{N} \mid |(a_1\xi_i|\eta_i)| < \varepsilon\},$$

we may assume that

$$|(a_1\xi_1|\eta_1)| > 1,$$
$$|(a_2\xi_2|\eta_2)| > 1, \quad |(a_1\xi_2|\eta_2)| \geq \varepsilon.$$

If we omit again the finite set

$$\{i \in \mathbb{N} \mid i > 2,\ |(a_2\xi_i|\eta_i)| < \varepsilon\},$$

we may assume that

$$|(a_1\xi_3|\eta_3)| \geq \varepsilon, \quad |(a_2\xi_3|\eta_3)| \geq \varepsilon.$$

Using this argument repeatedly, we may assume that, for any n,

$$|(a_1\xi_n|\eta_n)| \geq \varepsilon, |(a_2\xi_n|\eta_n)| \geq \varepsilon, \ldots, |(a_{n-1}\xi_n|\eta_n)| \geq \varepsilon.$$

Then we have

$$(n-1)\varepsilon^2 \leq \sum_{i=1}^{n-1} |(a_i\xi_n|\eta_n)|^2 \leq \sum_{i=1}^{n-1} \|\xi_n\|^2 \|a_i^*\eta_n\|^2$$
$$\leq \sum_{i=1}^{n-1} (a_i a_i^* \eta_n|\eta_n) \leq \|\sum_{i=1}^{n-1} a_i a_i^*\|.$$

This contradicts to the assumption $\|\sum_{i=1}^{\infty} a_i a_i^*\| < \infty$. Therefore we can get a number i_0 required in the statement. □

Lemma 3.4. *Let X be an operator space on a Hilbert space $\mathcal{H}$. If X is not weakly right-compact, then there exist a sequence $\{a_i\}$ of X, sequences $\{\xi_i\}$, $\{\eta_i\}$ of unit vectors in $\mathcal{H}$ and some constant K such that*

(1) $\|\sum_{i=1}^{\infty} a_i a_i^*\| < \infty$.
(2) $3 < \|a_i\| < K$.
(3) $3 < |(a_i\xi_i|\eta_i)| < K$.
(4) $|(a_k\xi_j|\eta_j)| \leq \dfrac{1}{K^k}$ *for* $k \neq j$.

Proof. Since X is not weakly right-compact, we can choose a sequence $\{a_i\}$ of X such that

$$\|\sum_{i=1}^{\infty} a_i a_i^*\| < \infty$$

and $\{\|a_i\|\}$ is not convergent to 0. Then we may assume that $3 < \|a_i\| < K$ for any i and some constant K. We choose sequences $\{\xi_i\}$, $\{\eta_i\}$ of unit vectors in $\mathcal{H}$ satisfying

$$|(a_i\xi_i|\eta_i)| > 3 \qquad \text{for all } i \in \mathbb{N}.$$

Using Lemma 3.3, we can choose a subsequence $\{n(k)\}_{k=1}^{\infty}$ such that

$$|(a_{n(k)}\xi_{n(j)}|\eta_{n(j)})| < \frac{1}{K^k} \quad \text{for } k < j.$$

If we replace $\{a_{n(k)}\}$ with $\{a_i\}$, then we can get the conditions (1)–(3) and (4) for $k < j$.

We consider a sequence $\{a_i, \xi_i, \eta_i\}$ of triplets. By the calculation

$$\sum_{i=1}^{\infty} |(a_i\xi_j|\eta_j)|^2 \leq \sum_{i=1}^{\infty} \|a_i^*\eta_j\|^2 = (\sum_{i=1}^{\infty} a_i a_i^* \eta_j|\eta_j)$$
$$\leq \|\sum_{i=1}^{\infty} a_i a_i^*\| < \infty,$$

we have

$$\lim_{i\to\infty} |(a_i\xi_j|\eta_j)| = 0 \quad \text{for any } j.$$

Choosing a subsequence of $\{a_i, \xi_i, \eta_i\}$, we may assume that

$$|(a_k \xi_j | \eta_j)| < \frac{1}{K^k} \quad \text{for } k > j.$$

Thus we can get the conditions (1)–(4). □

Lemma 3.5. *Let $\alpha > \beta > 0$. If sequences $\{a_k\}$, $\{b_k\}$ of vectors in $\mathbb{C}^m$ satisfy the following conditions*:

$$|(a_k|b_k)| > \alpha \text{ and } |(a_k|b_\ell)| < \beta \text{ for } k \neq \ell,$$

then

$$\sup\{|a_k(i)|, |b_k(i)| \mid i = 1, \dots, m,\ k = 1, 2, \dots\} = \infty.$$

Proof. We assume that $\sup\{|a_k(i)|, |b_k(i)| \mid i = 1, \dots, n,\ k = 1, 2, \dots\}$ is finite. By the compactness, we can choose a pair of convergent subsequences $\{a_{n(k)}\}, \{b_{n(k)}\}$. Then we have

$$\lim_k |(a_{n(k)}|b_{n(k+1)})| = \lim_k |(a_{n(k)}|b_{n(k)})| \geq \alpha.$$

But this contradicts to

$$\limsup_k |(a_{n(k)}|b_{n(k+1)})| \leq \beta.$$

□

Theorem 3.6. *Let X and Y be operator spaces. Then we have*

(1) *$X \otimes_h Y = X \otimes_{eh} Y$ if X is right-compact or Y is left-compact.*
(2) *X is weakly right-compact or Y is weakly left-compact if $X \otimes_h Y = X \otimes_{eh} Y$.*

Proof. (1) We assume that X is right-compact. For any $s \in X \otimes_{eh} Y$, there exist $a = [a_i] \in \mathbb{M}_{1,I}(X)$ and $b = {}^t[b_i] \in \mathbb{M}_{I,1}(Y)$ such that

$$s = a \otimes b = \sum_{i \in I} a_i \otimes b_i.$$

By the assumption, there exist $c \in \mathbb{M}_{1,I}(X)$ and $\alpha \in \mathbb{K}(\ell^2(I))$ such that $a = c\alpha$. So we have

$$s = a \otimes b = c\alpha \otimes b \in X \otimes_h Y.$$

This means that $X \otimes_h Y = X \otimes_{eh} Y$. When Y is left-compact, we can also have $X \otimes_h Y = X \otimes_{eh} Y$ by the same argument.

(2) Let X (resp. Y) be an operator space on $\mathcal{H}$ (resp. $\mathcal{K}$). We assume that X is not weakly right-compact and Y is not weakly left-compact. By Lemma 3.4, we can choose a sequence $\{a_i\}$ of X (resp. a sequence $\{b_i\}$ of Y), sequences $\{\xi_i\}$, $\{\eta_i\}$ of unit vectors in $\mathcal{H}$ (resp. sequences $\{\tilde{\xi_i}\}$, $\{\tilde{\eta_i}\}$ of unit vectors in $\mathcal{K}$) and some constant K satisfying that

$$\|\sum_{i=1}^{\infty} a_i a_i^*\| < \infty,\ \|\sum_{i=1}^{\infty} b_i^* b_i\| < \infty,$$
$$3 < \|a_i\|, \|b_i\| < K,$$

$$3 < |(a_i\xi_i|\eta_i)|, |(b_i\tilde{\eta_i}|\tilde{\xi_i})| < K$$

and

$$|(a_k\xi_j|\eta_j)|, |(b_k\tilde{\eta_j}|\tilde{\xi_j})| < \frac{1}{K^k} \quad \text{for } k \neq j.$$

We define $s \in X \otimes_{eh} Y$, $\varphi_k \in X^*$ and $\psi_k \in Y^*$ as follows:

$$s = \sum_{i=1}^{\infty} a_i \otimes b_i, \quad \varphi_k(\cdot) = (\cdot\xi_k|\eta_k), \quad \psi_k(\cdot) = (\cdot\tilde{\eta_k}|\tilde{\xi_k}).$$

Then we have

$$\begin{aligned}
|s(\varphi_k, \psi_k)| &= |\sum_{i=1}^{\infty} \varphi_k(a_i)\psi_k(b_i)| \\
&= |\sum_{i=1}^{\infty} (a_i\xi_k|\eta_k)(b_i\tilde{\eta_k}|\tilde{\xi_k})| \\
&\geq |(a_k\xi_k|\eta_k)(b_k\tilde{\eta_k}|\tilde{\xi_k})| - \sum_{i\neq k} |(a_i\xi_k|\eta_k)(b_i\tilde{\eta_k}|\tilde{\xi_k})| \\
&\geq 9 - \sum_{i=1}^{\infty} \frac{1}{K^{2i}} > 8,
\end{aligned}$$

and, for $j \neq k$,

$$\begin{aligned}
|s(\varphi_j, \psi_k)| &= |\sum_{i=1}^{\infty} \varphi_j(a_i)\psi_k(b_i)| \\
&= |\sum_{i=1}^{\infty} (a_i\xi_j|\eta_j)(b_i\tilde{\eta_k}|\tilde{\xi_k})| \\
&\leq \sum_{i=1}^{\infty} |(a_i\xi_j|\eta_j)(b_i\tilde{\eta_k}|\tilde{\xi_k})| \\
&\leq \frac{1}{K^{j-1}} + \frac{1}{K^{k-1}} + \sum_{i=1}^{\infty} \frac{1}{K^{2i}} < 3.
\end{aligned}$$

Suppose that $X \otimes_{eh} Y = X \otimes_h Y$, then s belongs to $X \otimes_h Y$. We can choose

$$t = \sum_{i=1}^{m} x_i \otimes y_i \in X \otimes_h Y \text{ and } \|s - t\|_h < 1.$$

Since $\|\varphi_j\| = \|\psi_k\| = 1$,

$$|s(\varphi_j, \psi_k) - t(\varphi_j, \psi_k)| < 1,$$

that is,

$$|s(\varphi_j, \psi_k) - \sum_{i=1}^{m} \varphi_j(x_i)\psi_k(y_i)| < 1.$$

Then we have

$$|\sum_{i=1}^{m} \varphi_k(x_i)\psi_k(y_i)| > 7$$

and, for $j \neq k$,

$$|\sum_{i=1}^{m} \varphi_j(x_i)\psi_k(y_i)| < 4.$$

This contradicts to the boundedness of $\{|\varphi_k(x_i)|, |\psi_k(y_i)| \mid 1 \leq i \leq m, k \in \mathbb{N}\}$ by Lemma 3.5. We are done. □

Remark 3.7. The row Hilbert space $\mathcal{H}_r$ is right-compact and is not weakly left-compact and the column Hilbert space $\mathcal{H}_c$ is left-compact and is not weakly right-compact. Then it is clear that

$$\mathcal{H}_r \otimes_h \mathcal{H}_c = \mathcal{H}_r \otimes_{eh} \mathcal{H}_c, \ \mathcal{H}_c \otimes_h \mathcal{H}_r \neq \mathcal{H}_c \otimes_{eh} \mathcal{H}_r$$

(c.f.[6]).

Corollary 3.8. *Let A and B be C*-algebras. Then the following assertions are equivalent*:

(1) $A \otimes_h B = A \otimes_{eh} B$,
(2) *A or B is finite dimensional.*

Proof. We have already shown that every finite-dimensinal C*-algebra is right-compact and left-compact in Lemma 3.2. It is sufficient to show that every infinite-dimensional C*-algebra is neither weakly right-compact nor weakly left-compact.

Suppose that A is infinite dimensional. Since the maximal abelian *-subalgebras in A is infinite dimensional, there exist self-adjoint elements $\{a_n\} \subset A$ such that $\|a_n\| = 1$ and $a_i a_j = 0$ if $i \neq j$. Then we have

$$\|\sum_{i=1}^{\infty} a_i^2\| = \|\sum_{i=1}^{\infty} a_i a_i^*\| = \|\sum_{i=1}^{\infty} a_i^* a_i\| < \infty$$

and $\{\|a_i\|\}$ does not converge to 0. This means that A is neither weakly right-compact nor weakly left-compact. □

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