TWO-SCALE PRODUCT APPROXIMATION FOR SEMILINEAR PARABOLIC PROBLEMS IN MIXED METHODS

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ABSTRACT. We propose and analyze two-scale product approximation for semilinear heat equations in the mixed finite element method. In order to efficiently resolve nonlinear algebraic equations resulting from the mixed method for semilinear parabolic problems, we treat the nonlinear terms using some interpolation operator and exploit a two-scale grid algorithm. With this scheme, the nonlinear problem is reduced to a linear problem on a fine scale mesh without losing overall accuracy of the final system. We derive optimal order $L^\infty((0,T); L^2(\Omega))$-error estimates for the relevant variables. Numerical results are presented to support the theory developed in this paper.

1. Introduction

In order to efficiently compute a number of integrals involving the coefficients or nonlinear terms arising in the variational formulation of the differential equations, one uses approximation of the coefficients, e.g., the interpolation, the projection or the quadrature formulas of the coefficients. The use of product approximation of the coefficients in Galerkin finite element methods for nonlinear problems has been analyzed by several authors (e.g., [8, 28, 7, 12, 17]). They applied the interpolation operator to the nonlinear coefficients and obtained error estimates of optimal convergence order with much less computational effort.

Chen and Douglas [5] analyzed the approximation of coefficients in mixed finite element method for nonlinear parabolic problems in the continuous-in-time setting. There, the authors suggested two examples for the approximation of coefficients, namely, projection of the nonlinear coefficients into finite element

spaces and evaluation of the integral involving nonlinear term by a quadrature rule. However, in computing the discrete solution for the semilinear parabolic model problem under consideration, their schemes can be complicated from the implementation point of view.

In our paper, we propose elementwise interpolation of the nonlinear coefficients in mixed finite element methods for semilinear parabolic equations. This leads to great computational savings in obtaining matrix systems at every time step. Since the reduced system obtained by interpolation of the nonlinear term is yet nonlinear, it still requires an efficient solver for the resulting nonlinear algebraic equations on fine-scale mesh. For this, we introduce a two-scale method based on finite element spaces with grids of two different scales.

The two-scale method was first introduced by Xu [31, 32] with the Galerkin method for semilinear and nonlinear elliptic boundary value problems. A new discretization technique, based on finite element spaces defined on two scales of different sizes, is presented. The author suggests an efficient algorithm and gives an error analysis. In the linear but not symmetric positive-definite (SPD) case, the idea of two scales is used to reduce the problem to an SPD. This is done by solving a non-SPD problem on a much smaller space. For nonlinear problems, the standard finite element method is used on the coarsest mesh. This solution is used as a starting value for the iterative solution on the finer mesh. Optimal accuracy is obtained even if the ratio of the coarse to the fine mesh is large.

Since the pioneering work of Xu, the two-scale method was further investigated by many authors. For instance, Bi and Ginting [1] have applied this method to finite volume element method for linear and nonlinear elliptic problems. Layton and Lenferink [18] have also studied this method for Navier-Stokes equations. Recently, Jin, Shu and Xu [16] have applied the two-grid idea to decouple systems of partial differential equations.

On the other hand, mixed finite element methods have been successfully applied to several areas of interest, in particular, fluid flows in porous media [14, 27, 29, 22]. Linear and nonlinear second order elliptic problems are studied in [25, 11, 3, 26, 23, 24, 19, 20]. Two-scale mixed methods for parabolic problems are studied in references [10, 30, 6]. In particular, Dawson and Wheeler [10] have applied two-grid method combined with mixed finite element method to parabolic problems with nonlinear diffusivity. There, the fully nonlinear system is solved on a coarse scale of size H. The nonlinearities are expanded about the coarse scale solution, and the resulting linear but nonsymmetric system is solved on a finescale of size h. Error estimates are derived to demonstrate $\mathcal{O}(\Delta t + h^{k+1} + H^{2k+1})$ convergence with standard mixed finite element spaces of order k. Later, Wu and Allen [30] have applied two-grid scheme to mixed finite element method for reaction-diffusion equations with nonlinear reaction and obtained the error bound $\mathcal{O}(\Delta t + h^{k+1} + H^{2k+2})$. But a gap is found in the proof of Theorem 3.2 in the paper [30]. The proof requires $L^4(\Omega)$ error

estimate in order to bound quadratic terms arising from linearization. More-
over, the authors of the paper [6] essentially fixed this gap using the properties
of superconvergence. However, Lemma 4.1 in their paper should be modified
due to the use of the inverse estimate and Lemma 4.2 is valid under the space
and time mesh restriction $\Delta t \cdot H^{-1} = \mathcal{O}(1)$. Therefore, their main theorem
(Theorem 4.1) should be modified accordingly.

Aforementioned issues are incorporated into our paper. Moreover, from
the implementation point of view, our scheme is much simpler than that of
[6] mainly due to the elementwise interpolation of the nonlinear coefficients.
Consequently, we derive error estimates bounded by $\mathcal{O}(\Delta t + h^{k+1} + H^{2k+1})$
for $k \geq 1$ without losing overall accuracy of the final system on fine scale
mesh. Note that we show the error bound $\mathcal{O}(\Delta t + h + H^2)$ for the lowest index
$k = 0$. In order to improve coarse scale convergence for higher k, we propose an
interpolation operator based on Gaussian points. Then, we demonstrate error
estimates $\mathcal{O}(\Delta t + h^{k+1} + H^{2k+2})$, $k = 1, 2$, using the interpolation operator
based on some Gaussian nodal points on triangular mesh.

Now, consider the following semilinear parabolic model problem:

$$
(1) \qquad
\begin{cases}
u_t - \operatorname{div}(a(x)\nabla u) = f(u) & \text{in } \Omega \times J, \\
\qquad\qquad\qquad u = 0 & \text{on } \partial\Omega \times J, \\
\qquad\quad u(\cdot, 0) = u^0 & \text{on } \Omega,
\end{cases}
$$

where $\Omega \subset \mathbb{R}^2$ is a convex planar domain with smooth boundary $\partial\Omega$, $J := (0, T]$
and $0 < \beta_0^{-1} \leq a(x) \leq \alpha_0^{-1}$.

Let $V := L^2(\Omega)$ and $\boldsymbol{M} := H(\operatorname{div}; \Omega)$, where the space

$$
H(\operatorname{div}; \Omega) := \{\boldsymbol{\sigma} \in [L^2(\Omega)]^2 : \operatorname{div}\boldsymbol{\sigma} \in L^2(\Omega)\}.
$$

Let $(\cdot, \cdot)$ denote the $L^2(\Omega)$-inner product and given an integer $m \geq 0$, $W^{m,p}(\Omega)$
denote the usual Sobolev space provided the norm

$$
\|v\|_{m,p} := \left\{ \sum_{|\beta| \leq m} \|D^\beta v\|_p^p \right\}^{1/p}
$$

and semi-norm

$$
|v|_{m,p} := \left\{ \sum_{|\beta| = m} \|D^\beta v\|_p^p \right\}^{1/p}.
$$

When $p = 2$, we denote $W^{m,p}(\Omega)$ by $H^m(\Omega)$ and write $\|v\|_m := \|v\|_{m,2}$. If
$m = 0$ we usually write $\|v\| := \|v\|_0$. Furthermore, for finite element K we use
the notation $\|v\|_{m,p,K}$ and $|v|_{m,p,K}$.

We present the regularity assumptions which are sufficient to obtain required
estimates.

(A1) The $f(u)$ is a smooth function of $u \in \mathbb{R}$, i.e., there exists a bound B_1 such that
$$|f^{(i)}| \le B_1, \quad i = 0, 1, 2, \ldots.$$

(A2) If u is the solution of (1), there exists a constant B_2 such that
$$\|u\|_{L^2(J;H^{k+1}(\Omega))} + \left\|\frac{\partial u}{\partial t}\right\|_{L^2(J;H^{k+1}(\Omega))} + \left\|\frac{\partial^2 u}{\partial t^2}\right\|_{L^2(J;L^2(\Omega))} \le B_2,$$

where
$$\|w\|_{L^2(J;X)} := \left(\int_0^T \|w(\cdot, t)\|_X^2 \, dt\right)^{1/2}.$$

We note that under the hypotheses of the theorems to follow, our approximations converge uniformly to u; thus (A1) actually needs hold only in a neighborhood of the solution (c.f., [13]).

The rest of this paper is organized as follows. In Section 2, we describe our two-scale schemes combined with interpolation of coefficients for mixed finite element method. In Section 3, we derive $L^\infty(J; L^2(\Omega))$-error estimates for the scheme which show the optimal order of convergence on fine scale mesh. Finally, in Section 4, we present numerical results which confirm the theory developed in this paper.

2. Description of the methods

Let $\boldsymbol{\sigma} := -a(x)\nabla u$ and $\alpha := a(x)^{-1}$, then problem (1) has the following variational form: find $(u, \boldsymbol{\sigma}) \in V \times \boldsymbol{M}$ such that

(2) $(\alpha\boldsymbol{\sigma}, \boldsymbol{\tau}) - (u, \operatorname{div}\boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{\tau} \in \boldsymbol{M}, \quad \forall t \in J,$

(3) $\left(\dfrac{\partial u}{\partial t}, v\right) + (\operatorname{div}\boldsymbol{\sigma}, v) = (f(u), v), \quad \forall v \in V, \quad \forall t \in J,$

$$u(\cdot, 0) = u^0, \quad \text{in } \Omega.$$

Let $\mathcal{T}_h$ be a family of quasi-uniform partition of Ω into triangles or rectangles of diameter not greater than h. Boundary finite elements are allowed to have one curvilinear edge or side. Associated with $\mathcal{T}_h$, we take standard mixed finite dimensional subspaces $\boldsymbol{M}_h \subset \boldsymbol{M}$ and $V_h \subset L^2(\Omega)$, for example, the Raviart-Thomas space RT_k of order k.

If we use the backward Euler scheme for discretization with respect to the time variable it follows from (3) and (2) that for $n = 1, \ldots, N$,

(4) $(\alpha\boldsymbol{\sigma}^n, \boldsymbol{\tau}) - (u^n, \operatorname{div}\boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{\tau} \in \boldsymbol{M},$

(5) $\left(\dfrac{u^n - u^{n-1}}{\Delta t}, v\right) + (\operatorname{div}\boldsymbol{\sigma}^n, v) = (f(u^n), v) + (e^n, v), \quad \forall v \in V,$

where Δt is time step size, $t_n := n\Delta t$, $u^n(\cdot) := u(\cdot, t_n)$, $\boldsymbol{\sigma}^n(\cdot) := \boldsymbol{\sigma}(\cdot, t_n)$, and $e^n := \frac{u^n - u^{n-1}}{\Delta t} - \frac{\partial u^n}{\partial t}$ is the truncation error associated with the backward Euler

difference approximation. The space-time fully discrete problem is defined as follows: for $n = 1, \ldots, N$, find $(u_h^n, \boldsymbol{\sigma}_h^n) \in V_h \times \boldsymbol{M}_h$ such that

$$(\alpha \boldsymbol{\sigma}_h^n, \boldsymbol{\tau}) - (u_h^n, \text{div}\boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{\tau} \in \boldsymbol{M}_h,$$

$$\left(\frac{u_h^n - u_h^{n-1}}{\Delta t}, v \right) + (\text{div}\boldsymbol{\sigma}_h^n, v) = (f(u_h^n), v), \quad \forall v \in V_h.$$

The following estimates are well known results (c.f., [21]): for each time level n, there exists positive constant C, independent of h but dependent on function u such that

(6) $$\|u^n - u_h^n\| \leq C(\Delta t + h^r), \quad 1 \leq r \leq k + 1, \quad k \geq 0,$$

(7) $$\|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}_h^n\| \leq C(\Delta t + h^r), \quad 1 \leq r \leq k + 1, \quad k \geq 0.$$

Now, we define an operator Q_h to interpolate the nonlinear term $f(u_h)$. For any element-wise continuous function v, i.e., $v|_K \in C(\overline{K})$ for each element $K \in \mathcal{T}_h$, let Q_h be the operator which maps $f(v)$ to $Q_h f(v) \in V_h$ defined by

$$Q_h f(v)(\xi) := f(v(\xi))$$

at each nodal point ξ. Then we have well-known approximation property of Q_h:

(8) $$\|f(v) - Q_h f(v)\|_{0,2,k} \leq C h^{k+1} |f(v)|_{k+1,2,K} .$$

For example, if $f(u_h) = u_h^2$ and $u_h = \sum_{i=1}^{N_h} c_i \phi_i$, where $\phi_i \in V_h$ is the basis satisfying $\phi_i(\xi_j) = \delta_{ij}$, where N_h is the number of interior nodes, δ_{ij} is the symbol of Kronecker delta, for a nodal point ξ_j, then since $Q_h f(u_h)(\xi_j) = u_h(\xi_j)^2 = c_j^2$ we have $Q_h f(u_h) = \sum_{i=1}^{N_h} c_i^2 \phi_i$. Generally,

$$Q_h f(u_h) = \sum_{i=1}^{N_h} f(c_i) \phi_i$$

for any function f. Note that if V_h is the space of the lowest order, i.e., the space of order $k = 0$, then $Q_h f(v_h) = f(v_h)$ for any $v_h \in V_h$. So from now on we consider $k \geq 1$ only. The main algorithm of this paper is as follows:

Algorithm :

- **Step 1**. For $n = 1, \ldots, N$, on the coarse grid $\mathcal{T}_H$, find $(\hat{u}_H^n, \hat{\boldsymbol{\sigma}}_H^n) \in V_H \times \boldsymbol{M}_H$ satisfying the original nonlinear system:

(9) $$(\alpha \hat{\boldsymbol{\sigma}}_H^n, \boldsymbol{\tau}) - (\hat{u}_H^n, \text{div}\boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{\tau} \in \boldsymbol{M}_H,$$

(10) $$\left(\frac{\hat{u}_H^n - \hat{u}_H^{n-1}}{\Delta t}, v \right) + (\text{div}\hat{\boldsymbol{\sigma}}_H^n, v) = (Q_H f(\hat{u}_H^n), v), \quad \forall v \in V_H.$$

- **Step 2**. For $n = 1, \ldots, N$, on the fine grid $\mathcal{T}_h$, find $(\hat{u}_h^n, \hat{\boldsymbol{\sigma}}_h^n) \in V_h \times \boldsymbol{M}_h$ satisfying the linearized system:

(11) $$(\alpha \hat{\boldsymbol{\sigma}}_h^n, \boldsymbol{w}) - (\hat{u}_h^n, \text{div}\boldsymbol{w}) = 0, \quad \forall \boldsymbol{\tau} \in \boldsymbol{M}_h,$$

$$(12) \quad \left(\frac{\hat{u}_h^n - \hat{u}_h^{n-1}}{\Delta t}, v\right) + (\mathrm{div}\hat{\boldsymbol{\sigma}}_h^n, v) = (f(\hat{u}_H^n) + f_u(\hat{u}_H^n)(\hat{u}_h^n - \hat{u}_H^n), v), \ \forall v \in V_h.$$

3. Error analysis

We define the "elliptic projection" of the solution $(u^n, \boldsymbol{\sigma}^n)$ of the problem (4)-(5) as the function $(\tilde{u}_h^n, \tilde{\boldsymbol{\sigma}}_h^n) \in V_h \times \boldsymbol{M}_h$ at each time level n that satisfies the following equations:

$$(13) \qquad (\alpha\tilde{\boldsymbol{\sigma}}_h^n, \boldsymbol{\tau}) - (\tilde{u}_h^n, \mathrm{div}\boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{\tau} \in \boldsymbol{M}_h,$$

$$(14) \qquad (\mathrm{div}\tilde{\boldsymbol{\sigma}}_h^n, v) = \left(f(u^n) - \frac{\partial u^n}{\partial t}, v\right), \quad \forall v \in V_h.$$

The following estimates are well known results (see [4]): for each time level n, $1 \le p < \infty$,

$$(15) \qquad \|u^n - \tilde{u}_h^n\|_{0,p} \le Ch^r\|u^n\|_{r,p}, \quad 2 \le r \le k+1,$$

$$(16) \qquad \|\boldsymbol{\sigma}^n - \tilde{\boldsymbol{\sigma}}_h^n\|_{0,p} \le Ch^r\|\boldsymbol{\sigma}^n\|_{r,p}, \quad 1 \le r \le k+1,$$

$$(17) \qquad \|\mathrm{div}(\boldsymbol{\sigma}^n - \tilde{\boldsymbol{\sigma}}_h^n)\|_{0,p} \le Ch^r\|\mathrm{div}\boldsymbol{\sigma}^n\|_{r,p}, \quad 1 \le r \le k+1.$$

First, we consider **Step 1** (coarse grid). Let $\varepsilon^n := u^n - \hat{u}_H^n$ and $\boldsymbol{\theta}^n := \boldsymbol{\sigma}^n - \hat{\boldsymbol{\sigma}}_H^n$. Subtracting (9)-(10) from (4)-(5) respectively, we arrive at the following error equations:

$$(18) \qquad (\alpha\boldsymbol{\theta}^n, \boldsymbol{\tau}) - (\varepsilon^n, \mathrm{div}\boldsymbol{\tau}) = 0, \ \forall \boldsymbol{\tau} \in \boldsymbol{M}_H,$$

$$(19) \quad \left(\frac{\varepsilon^n - \varepsilon^{n-1}}{\Delta t}, v\right) + (\mathrm{div}\boldsymbol{\theta}^n, v) = (e^n, v) + (f(u^n) - Q_H f(\hat{u}_H^n), v), \ \forall v \in V_H.$$

For $\eta(\cdot, t_n) := \eta^n := u^n - \tilde{u}_H^n$, $\xi_H^n := \tilde{u}_H^n - \hat{u}_H^n$, $\boldsymbol{\tau}^n := \boldsymbol{\sigma}^n - \tilde{\boldsymbol{\sigma}}_H^n$, and $\boldsymbol{\rho}_H^n := \tilde{\boldsymbol{\sigma}}_H^n - \hat{\boldsymbol{\sigma}}_H^n$, we have $\varepsilon^n = \eta^n + \xi_H^n$ and $\boldsymbol{\theta}^n = \boldsymbol{\tau}^n + \boldsymbol{\rho}_H^n$. If we insert these decompositions into (18) and (19), then we get from (13) and (14), for $n = 1, \ldots, N$,

$$(20) \qquad (\alpha\boldsymbol{\rho}_H^n, \boldsymbol{\tau}) - (\xi_H^n, \mathrm{div}\boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{\tau} \in \boldsymbol{M}_H,$$

$$(21) \qquad \left(\frac{\xi_H^n - \xi_H^{n-1}}{\Delta t}, v\right) + (\mathrm{div}\boldsymbol{\rho}_H^n, v) = (e^n, v) + (f(u^n) - Q_H f(\hat{u}_H^n), v)$$
$$\qquad\qquad\qquad\qquad\qquad - \left(\frac{\eta^n - \eta^{n-1}}{\Delta t}, v\right), \quad \forall v \in V_H.$$

We introduce three lemmas before considering main theorems.

Lemma 3.1. *If u is the solution of* (1), *then for $e^n = \frac{u^n - u^{n-1}}{\Delta t} - \frac{\partial u^n}{\partial t}$,*

$$\|e^n\|^2 = \frac{\Delta t}{3}\int_{(n-1)\Delta t}^{n\Delta t}\left\|\frac{\partial^2 u}{\partial t^2}\right\|^2 dt.$$

Proof. By the Cauchy-Schwarz inequality, we see that

$$\|e^n\|^2 = \left\|\frac{\partial u^n}{\partial t} - \frac{u^n - u^{n-1}}{\Delta t}\right\|^2 = \left\|\frac{1}{\Delta t}\int_{(n-1)\Delta t}^{n\Delta t}[t - (n-1)\Delta t]\frac{\partial^2 u}{\partial t^2}dt\right\|^2.$$

So,

$$\|e^n\|^2 \leq \frac{1}{\Delta t^2}\left\|\sqrt{\int_{(n-1)\Delta t}^{n\Delta t}\left[t-(n-1)\Delta t\right]^2 dt}\sqrt{\int_{(n-1)\Delta t}^{n\Delta t}\left(\frac{\partial^2 u}{\partial t^2}\right)^2 dt}\right\|^2$$

$$= \frac{\Delta t}{3}\int_\Omega \int_{(n-1)\Delta t}^{n\Delta t}\left(\frac{\partial^2 u}{\partial t^2}\right)^2 dt dx$$

$$= \frac{\Delta t}{3}\int_{(n-1)\Delta t}^{n\Delta t}\left\|\frac{\partial^2 u}{\partial t^2}\right\|^2 dt. \qquad \square$$

Lemma 3.2. *If u is the solution of (4)-(5), $\tilde{u}_H^n$ is the solution of (13)-(14) on the coarse grid $\mathcal{T}_H$, and $\eta(\cdot, t_n) := \eta^n := u^n - \tilde{u}_H^n$, then for any $\epsilon > 0$ and for $k \geq 1$,*

$$|(\eta^n - \eta^{n-1}, v)| \leq \frac{CH^{2k+2}}{2\epsilon^2}\int_{(n-1)\Delta t}^{n\Delta t}\left\|\frac{\partial u}{\partial t}\right\|_{k+1}^2 dt + \frac{\Delta t\epsilon^2}{2}\|v\|^2, \quad \forall v \in V_H.$$

Proof. By the Cauchy-Schwarz inequality,

$$|(\eta^n - \eta^{n-1}, v)| = \left|\left(\int_{(n-1)\Delta t}^{n\Delta t}\frac{\partial\eta}{\partial t}dt, v\right)\right|$$

$$\leq \left(\sqrt{\int_{(n-1)\Delta t}^{n\Delta t}dt}\sqrt{\int_{(n-1)\Delta t}^{n\Delta t}\left(\frac{\partial\eta}{\partial t}\right)^2 dt}, |v|\right)$$

$$= \left(\sqrt{\int_{(n-1)\Delta t}^{n\Delta t}\left(\frac{\partial\eta}{\partial t}\right)^2 dt}, \sqrt{\Delta t}\,|v|\right)$$

$$\leq \frac{1}{2\epsilon^2}\int_\Omega \int_{(n-1)\Delta t}^{n\Delta t}\left(\frac{\partial\eta}{\partial t}\right)^2 dt dx + \frac{\Delta t\epsilon^2}{2}(v, v).$$

Interchanging the order of integration and applying (15) imply for $k \geq 1$,

$$|(\eta^n - \eta^{n-1}, v)| \leq \frac{1}{2\epsilon^2}\int_{(n-1)\Delta t}^{n\Delta t}\left\|\frac{\partial\eta}{\partial t}\right\|^2 dt + \frac{\Delta t\epsilon^2}{2}(v, v)$$

$$\leq \frac{CH^{2k+2}}{2\epsilon^2}\int_{(n-1)\Delta t}^{n\Delta t}\left\|\frac{\partial u}{\partial t}\right\|_{k+1}^2 dt + \frac{\Delta t\epsilon^2}{2}\|v\|^2. \qquad \square$$

The following lemma is proved easily.

Lemma 3.3. *Let g be a piecewise smooth on the partition $\mathcal{T}_h$. If $\bar{g}$ is the average value of $g(u)$ on each element $K \in \mathcal{T}_h$ and $\|\nabla g\|_{0,\infty} \leq B$, then*

$$|(g(u)\theta, \psi) - (\bar{g}\theta, \psi)| \leq CBh\|\theta\|_0\|\psi\|_0.$$

Theorem 3.4. *For the solution $(\hat{u}_H^n, \hat{\boldsymbol{\sigma}}_H^n) \in V_H \times \boldsymbol{M}_H$ of (9)-(10) and the solution $(u^n, \boldsymbol{\sigma}^n)$ of the equations (4)-(5), there exists a positive constant C independent of Δt and H such that*

(22) $$\|u^n - \hat{u}_H^n\| \leq C(\Delta t + H^{k+1}),$$

(23) $$\|\boldsymbol{\sigma}^n - \hat{\boldsymbol{\sigma}}_H^n\| \leq C(\Delta t + H^{k+1}).$$

Proof. For $n = 1, 2, \ldots, \ell \leq N$, choosing $v = \xi_H^n$ and $\boldsymbol{\tau} = \boldsymbol{\rho}_H^n$ in (20) and (21) respectively, adding two equations, and multiplying by Δt, we obtain

$$(\xi_H^n - \xi_H^{n-1}, \xi_H^n) + \Delta t(\alpha \boldsymbol{\rho}_H^n, \boldsymbol{\rho}_H^n)$$
$$= \Delta t(e^n, \xi_H^n) + \Delta t(f(u^n) - Q_H f(\hat{u}_H^n), \xi_H^n) - (\eta^n - \eta^{n-1}, \xi_H^n).$$

Applying Cauchy-Schwarz inequality to the first term in left hand side and using the results of Lemma 3.1-3.2 in the right hand side, we get with $\epsilon = 1$

(24) $$\frac{1}{2}(\xi_H^n, \xi_H^n) - \frac{1}{2}(\xi_H^{n-1}, \xi_H^{n-1}) + \Delta t(\alpha \boldsymbol{\rho}_H^n, \boldsymbol{\rho}_H^n)$$

$$\leq \frac{1}{6}\Delta t^2 \int_{(n-1)\Delta t}^{n\Delta t} \left\|\frac{\partial^2 u}{\partial t^2}\right\|^2 dt + \frac{1}{2}CH^{2k+2} \int_{(n-1)\Delta t}^{n\Delta t} \left\|\frac{\partial u}{\partial t}\right\|_{k+1}^2 dt$$

$$+ \Delta t\|\xi_H^n\|^2 + \Delta t|(f(u^n) - Q_H f(\hat{u}_H^n), \xi_H^n)|.$$

In order to bound the last term of the right-hand side of (24), we consider smoothness of $f(u)$ (i.e., assumption (A1)) and the standard approximation property of interpolation (8). Then we see that

$$|(f(u^n) - Q_H f(\hat{u}_H^n), \xi_H^n)|$$
$$\leq |(f(u^n) - f(\tilde{u}_H^n), \xi_H^n)| + |(f(\tilde{u}_H^n) - Q_H f(\tilde{u}_H^n), \xi_H^n)|$$
$$\quad + |(Q_H f(\tilde{u}_H^n) - Q_H f(\hat{u}_H^n), \xi_H^n)|$$
$$\leq \frac{B_1^2}{2}\|\eta^n\|^2 + \frac{1}{2}\|\xi_H^n\|^2 + CH^{2k+2}\sum_K |f(\tilde{u}_H^n)|_{k+1,2,K}^2$$
$$\quad + \frac{1}{2}\|\xi_H^n\|^2 + \frac{B_1^2}{2}\|\xi_H^n\|^2 + \frac{1}{2}\|\xi_H^n\|^2.$$

The derivative $D^\beta f(\tilde{u}_H^n)$ with $|\beta| = k + 1$, when expanded, yields nonlinear terms involving the derivative of f up to order $k + 1$ and the derivatives of $\tilde{u}_H^n$ up to order $k + 1$. On the other hand from inverse inequality and the estimate (15), we have for $r = 0, 1, \ldots, k$ and $1 \leq p \leq \infty$,

$$\|P_H u^n - \tilde{u}_H^n\|_{r,p} \leq CH^{-r-1+2/p}\|P_H u^n - \tilde{u}_H^n\|$$
$$\leq CH^{-r-1+2/p}(\|P_H u^n - u^n\| + \|u^n - \tilde{u}_H^n\|)$$
$$\leq CH^{k-r+2/p},$$

where P_H is L^2-projection operator onto V_H with well known approximation property(e.g., [2, 9]), for $r = 0, 1, \ldots, k$, $1 \leq p \leq \infty$ and $K \in \mathcal{T}_h$

$$\|v - P_H v\|_{r,p,K} \leq CH^{k-r+2/p}\|v\|_{k+1,2,K}, \quad \forall v \in H^{k+1}(K).$$

In particular, when $r = k + 1$ and $p = 2$, the following holds

$$\|v - P_H v\|_{k+1,2,K} \leq C\|v\|_{k+1,2,K}, \quad \forall v \in H^{k+1}(K).$$

Thus, for $H \leq 1$, the norm $\|\tilde{u}_H^n\|_{r,p}$ is bounded by a constant C which depends on only $\|u^n\|_{k+1}$. Hence $\sum_K |f(\tilde{u}_H^n)|_{k+1,2,K}^2$ is bounded by a constant C which is independent of H. Since $\alpha(x) > \alpha_0 > 0$, (24) becomes

$$
\begin{aligned}
\|\xi_H^n\|^2 - \|\xi_H^{n-1}\|^2 \leq & \frac{\Delta t^2}{3} \int_{(n-1)\Delta t}^{n\Delta t} \left\| \frac{\partial^2 u}{\partial t^2} \right\|^2 dt \\
& + CH^{2k+2} \int_{(n-1)\Delta t}^{n\Delta t} \left\| \frac{\partial u}{\partial t} \right\|_{k+1}^2 dt + B_1^2 \Delta t \|\eta^n\|^2 \\
& + C\Delta t H^{2k+2} + (5 + B_1^2)\Delta t \|\xi_H^n\|^2.
\end{aligned}
$$

Summing above equations from $n = 1, \ldots, \ell$ ($1 \leq \ell \leq N$), we obtain

$$
\begin{aligned}
\|\xi_H^l\|^2 \leq & \|\xi_H^0\|^2 + \frac{\Delta t^2}{3} \int_0^{\ell \Delta t} \left\| \frac{\partial^2 u}{\partial t^2} \right\|^2 dt + CH^{2k+2} \int_0^{\ell \Delta t} \left\| \frac{\partial u}{\partial t} \right\|_{k+1}^2 dt \\
& + B_1^2 \Delta t \sum_{n=1}^{\ell} \|\eta^n\|^2 + Cl\Delta t H^{2k+2} + (5 + B_1^2)\Delta t \sum_{n=1}^{\ell} \|\xi_H^n\|^2 \\
\leq & (1 - (5 + B_1^2)\Delta t)\|\xi_H^0\|^2 + \frac{\Delta t^2}{3} \int_0^{\ell \Delta t} \left\| \frac{\partial^2 u}{\partial t^2} \right\|^2 dt \\
& + CH^{2k+2} \int_0^{\ell \Delta t} \left\| \frac{\partial u}{\partial t} \right\|_{k+1}^2 dt + B_1^2 \Delta t \sum_{n=1}^{\ell} \|\eta^n\|^2 \\
& + CTH^{2k+2} + (5 + B_1^2)\Delta t \sum_{n=0}^{\ell} \|\xi_H^n\|^2.
\end{aligned}
$$

When $1 - (5 + B_1^2)\Delta t > 0$, by discrete Gronwall's lemma it follows that

$$
\begin{aligned}
\|\xi_H^\ell\|^2 \leq & C\left(\|\xi_H^0\|^2 + \Delta t^2 \int_0^{\ell \Delta t} \left\| \frac{\partial^2 u}{\partial t^2} \right\|^2 dt + H^{2k+2} \int_0^{\ell \Delta t} \left\| \frac{\partial u}{\partial t} \right\|_{k+1}^2 dt \right. \\
& \left. + B_1^2 T \max_{1 \leq n \leq \ell} \|\eta^n\|^2 + H^{2k+2} \right).
\end{aligned}
$$

In particular, let us choose the initial function $\hat{u}_H^0, \hat{\boldsymbol{\sigma}}_H^0$ to be the elliptic projection $\tilde{u}_H^0, \tilde{\boldsymbol{\sigma}}_H^0$ of the exact solution then ξ_H^0 vanish. Hence, the estimate $\|\eta^n\| \leq CH^{k+1}\|u^n\|_{k+1}$ for $k \geq 1$ and the assumptions (A2) on u imply

$$
(25) \qquad \|\varepsilon^n\| \leq \|\eta^n\| + \|\xi_H^n\| \leq C(\Delta t + H^{k+1}).
$$

To estimate $\boldsymbol{\theta}^n$, it suffices to bound $\boldsymbol{\rho}_H^n$, since $\boldsymbol{\theta}^n = \boldsymbol{\tau}^n + \boldsymbol{\rho}_H^n$ and $\boldsymbol{\tau}^n$ can be bounded using (16). For time level n and $n-1$ in (20), we obtain the following equation:

$$
(26) \qquad \left(\frac{\alpha(\boldsymbol{\rho}_H^n - \boldsymbol{\rho}_H^{n-1})}{\Delta t}, \boldsymbol{\tau} \right) - \left(\text{div}\boldsymbol{\tau}, \frac{\xi_H^n - \xi_H^{n-1}}{\Delta t} \right) = 0, \quad \forall \boldsymbol{\tau} \in \boldsymbol{M}_H.
$$

Taking $\boldsymbol{\tau} = \boldsymbol{\rho}_H^n$ in (26) and $v = \frac{\xi_H^n - \xi_H^{n-1}}{\Delta t}$ in (21) and then adding the resulting equations, we have

$$\left(\frac{\alpha(\boldsymbol{\rho}_H^n - \boldsymbol{\rho}_H^{n-1})}{\Delta t}, \boldsymbol{\rho}_H^n\right) + \left(\frac{\xi_H^n - \xi_H^{n-1}}{\Delta t}, \frac{\xi_H^n - \xi_H^{n-1}}{\Delta t}\right)$$

$$= \left(e^n, \frac{\xi_H^n - \xi_H^{n-1}}{\Delta t}\right) + \left(f(u^n) - Q_H f(\hat{u}_H^n), \frac{\xi_H^n - \xi_H^{n-1}}{\Delta t}\right)$$

$$- \left(\frac{\eta^n - \eta^{n-1}}{\Delta t}, \frac{\xi_H^n - \xi_H^{n-1}}{\Delta t}\right).$$

Let $\lambda^n = \frac{\xi_H^n - \xi_H^{n-1}}{\Delta t}$. Then multiplying above equation by Δt, we have from Cauchy-Schwarz inequality and Lemma 3.1-3.2, for $\epsilon > 0$

$$(27) \qquad \frac{1}{2}(\alpha\boldsymbol{\rho}_H^n, \boldsymbol{\rho}_H^n) + \Delta t \|\lambda^n\|^2$$

$$\leq \frac{1}{2}(\alpha\boldsymbol{\rho}_H^{n-1}, \boldsymbol{\rho}_H^{n-1}) + \frac{\Delta t^2}{6\epsilon^2} \int_{(n-1)\Delta t}^{n\Delta t} \left\|\frac{\partial^2 u}{\partial t^2}\right\|^2 dt$$

$$+ \frac{\Delta t \epsilon^2}{2}\|\lambda^n\|^2 + \frac{\Delta t}{2}\|f(u^n) - Q_H f(\hat{u}_H^n)\|^2 + \frac{\Delta t}{2}\|\lambda^n\|^2$$

$$+ \frac{CH^{2k+2}}{2\epsilon^2} \int_{(n-1)\Delta t}^{n\Delta t} \left\|\frac{\partial u}{\partial t}\right\|_{k+1}^2 dt + \frac{\Delta t \epsilon^2}{2}\|\lambda^n\|^2.$$

The estimate (22) proved above, the boundedness of f (i.e., assumption (A1)) and the standard approximation property of interpolation (8) imply that there exists a positive constant C independent of H such that

$$\|f(u^n) - Q_H f(\hat{u}_H^n)\|^2 \leq 2\|f(u^n) - f(\hat{u}_H^n)\|^2 + 2\|f(\hat{u}_H^n) - Q_H f(\hat{u}_H^n)\|^2$$

$$\leq C\left(\Delta t + H^{k+1}\right)^2.$$

Thus, for $\epsilon = \frac{1}{\sqrt{2}}$, (27) becomes

$$\frac{1}{2}(\alpha\boldsymbol{\rho}_H^n, \boldsymbol{\rho}_H^n) \leq \frac{1}{2}(\alpha\boldsymbol{\rho}_H^{n-1}, \boldsymbol{\rho}_H^{n-1}) + \frac{\Delta t^2}{3} \int_{(n-1)\Delta t}^{n\Delta t} \left\|\frac{\partial^2 u}{\partial t^2}\right\|^2 dt$$

$$+ CH^{2k+2} \int_{(n-1)\Delta t}^{n\Delta t} \left\|\frac{\partial u}{\partial t}\right\|_{k+1}^2 dt + C\Delta t^3 + C\Delta t H^{2k+2}.$$

Multiplying by 2 and summing from $n = 1$ to ℓ ($1 \leq \ell \leq N$), we get

$$(\alpha\boldsymbol{\rho}_H^\ell, \boldsymbol{\rho}_H^\ell) \leq (\alpha\boldsymbol{\rho}_H^0, \boldsymbol{\rho}_H^0) + \frac{2\Delta t^2}{3} \int_0^{\ell\Delta t} \left\|\frac{\partial^2 u}{\partial t^2}\right\|^2 dt$$

$$+ CH^{2k+2} \int_0^{\ell\Delta t} \left\|\frac{\partial u}{\partial t}\right\|_{k+1}^2 dt + C \sum_{1 \leq n \leq \ell} \left(\Delta t^3 + \Delta t H^{2k+2}\right).$$

So, for $n = 1, \ldots, N$,

$$\alpha_0 \|\boldsymbol{\rho}_H^n\|^2 \leq \beta_0 \|\boldsymbol{\rho}_H^0\|^2 + \frac{2\Delta t^2}{3} \int_0^T \left\| \frac{\partial^2 u}{\partial t^2} \right\|^2 dt$$

$$+ CH^{2k+2} \int_0^T \left\| \frac{\partial u}{\partial t} \right\|_{k+1}^2 dt + CT\left(\Delta t^2 + H^{2k+2}\right).$$

Since $\boldsymbol{\rho}_H^0$ vanishes, we obtain from (16), (25) and the assumption $(A2)$ on u

$$\|\boldsymbol{\theta}^n\| \leq \|\boldsymbol{\tau}^n\| + \|\boldsymbol{\rho}_H^n\| \leq C(\Delta t + H^{k+1}). \qquad \square$$

We now derive an error estimate for the approximate solution of the linearized system (11)-(12). Subtracting (11)-(12) from (13)-(14), we obtain the following error equations

(28) $$\qquad\qquad (\alpha \boldsymbol{\gamma}_h^n, \boldsymbol{\tau}) - (\mu_h^n, \text{div}\boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{\tau} \in \boldsymbol{M}_h,$$

(29)
$$\left(\frac{\mu_h^n - \mu_h^{n-1}}{\Delta t}, v \right) + (\text{div}\boldsymbol{\gamma}_h^n, v)$$
$$= \left(f(u^n) - f(\hat{u}_H^n) - f_u(\hat{u}_H^n)(\hat{u}_h^n - \hat{u}_H^n), v \right)$$
$$+ \left(\frac{\tilde{u}_h^n - \tilde{u}_h^{n-1}}{\Delta t} - \frac{u^n - u^{n-1}}{\Delta t}, v \right) + (e^n, v), \quad \forall v \in V_h,$$

where $\mu_h^n := \tilde{u}_h^n - \hat{u}_h^n$, $\boldsymbol{\gamma}_h^n := \tilde{\boldsymbol{\sigma}}_h^n - \hat{\boldsymbol{\sigma}}_h^n$.

Lemma 3.5. *If $\tilde{u}_H^n$ is the solution of (13)-(14) and $\hat{u}_H^n$ is the solution of (9)-(10), then there exists a positive constant C independent of Δt and H such that*

$$\|\tilde{u}_H^n - \hat{u}_H^n\|_{0,4} \leq C\left(H^{-1/2}\Delta t + H^{k+1/2} \right).$$

Proof. Subtracting (9)-(10) from (13)-(14), we have

(30) $$\qquad\qquad (\alpha \boldsymbol{\rho}_H^n, \boldsymbol{\tau}) - (\xi_H^n, \text{div}\boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{\tau} \in \boldsymbol{M}_H,$$

(31) $$\qquad \left(\frac{\xi_H^n - \xi_H^{n-1}}{\Delta t}, v \right) + (\text{div}\boldsymbol{\rho}_H^n, v) = (T_0, v), \quad \forall v \in V_H,$$

where

$$T_0 := \left\{ \frac{\tilde{u}_H^n - \tilde{u}_H^{n-1}}{\Delta t} - \frac{\partial u^n}{\partial t} \right\} + \{ f(u^n) - Q_H f(\hat{u}_H^n) \} := T_1 + T_2.$$

Putting $v = \xi_H^n$ in (31) and $\boldsymbol{\tau} = \boldsymbol{\rho}_H^n$ in (30) and adding the resulting equations, we get

(32) $$\qquad \left(\frac{\xi_H^n - \xi_H^{n-1}}{\Delta t}, \xi_H^n \right) + (\alpha \boldsymbol{\rho}_H^n, \boldsymbol{\rho}_H^n) = (T_1, \xi_H^n) + (T_2, \xi_H^n).$$

To bound the first term of right-side of (32), we split T_1 in two terms:

$$T_1 = \left\{ \frac{\tilde{u}_H^n - \tilde{u}_H^{n-1}}{\Delta t} - \frac{P_H u^n - P_H u^{n-1}}{\Delta t} \right\} + \left\{ \frac{P_H u^n - P_H u^{n-1}}{\Delta t} - \frac{\partial u^n}{\partial t} \right\}$$

$$:= d_1 + d_2.$$

From the well-known superconvergnce result (e.g., [11]) between the L^2-projection and the elliptic projection of u^n, we see that

$$(33) \qquad |(d_1, \xi_H^n)| = \left\| (\Delta t)^{-1} \int_{t_{n-1}}^{t_n} \frac{\partial \omega_H}{\partial t}(\cdot, s) ds \right\| \, \|\xi_H^n\|$$

$$\leq (\Delta t)^{-1} \int_{t_{n-1}}^{t_n} \left\| \frac{\partial \omega_H}{\partial t} \right\| ds \, \|\xi_H^n\|$$

$$\leq CH^{2k+4} + \frac{1}{2}\|\xi_H^n\|^2,$$

where $\omega_H := \tilde{u}_H^n - P_H u^n$.

From the definition of L^2-projection and Lemma 3.1, we get

$$|(d_2, \xi_H^n)| = \left| \left(\frac{P_H u^n - P_H u^{n-1}}{\Delta t} - \frac{u^n - u^{n-1}}{\Delta t}, \xi_H^n \right) \right.$$

$$\left. + \left(\frac{u^n - u^{n-1}}{\Delta t} - \frac{\partial u^n}{\partial t}, \xi_H^n \right) \right|$$

$$\leq \frac{1}{2}\|e^n\|^2 + \frac{1}{2}\|\xi_H^n\|^2$$

$$\leq \frac{1}{6} \Delta t \int_{(n-1)\Delta t}^{n\Delta t} \left\| \frac{\partial^2 u}{\partial t^2} \right\|^2 dt + \frac{1}{2}\|\xi_H^n\|^2.$$

Next, to estimate for T_2, we consider the following relation

$$(34) \qquad T_2 = f(u^n) - Q_H f(\hat{u}_H^n)$$

$$= (f(u^n) - f(P_H u^n)) + (f(P_H u^n) - f(\tilde{u}_H^n))$$

$$+ (f(\tilde{u}_H^n) - f(\hat{u}_H^n)) + (f(\hat{u}_H^n) - Q_H f(\hat{u}_H^n)).$$

From the assumption (A1) on f, the well-known superconvergence result (e.g., [11]) between the L^2-projection and the elliptic projection of u^n and the approximation property of interpolation (8), we see that for some $u^{**} \in V$

$$(35) \qquad (f(u^n) - f(P_H u^n), \xi_H^n) = (f_u(P_H u^n)(u^n - P_H u^n), \xi_H^n)$$

$$+ \left(\frac{f_{uu}(u^{**})}{2}(u^n - P_H u^n)^2, \xi_H^n \right),$$

$$(36) \qquad\qquad = ((f_u(P_H u^n) - \bar{f}_u(P_H u^n))(u^n - P_H u^n), \xi_H^n)$$

$$+ \left(\frac{f_{uu}(u^{**})}{2}(u^n - P_H u^n)^2, \xi_H^n \right),$$

$$(37) \quad |(f(P_H u^n) - f(\tilde{u}_H^n), \xi_H^n)| \leq CH^{2k+4} + \frac{1}{2}\|\xi_H^n\|^2,$$

$$(38) \qquad |(f(\tilde{u}_H^n) - f(\hat{u}_H^n), \xi_H^n)| \leq B_1\|\xi_H^n\|^2,$$

(39) $|(f(\hat{u}_H^n) - Q_H f(\hat{u}_H^n), \xi_H^n)| \leq CH^{2k+2} \sum_K |f(\hat{u}_H^n)|_{k+1,2,K}^2 + \frac{1}{2}\|\xi_H^n\|^2.$

As in the proof of Theorem 3.4, we can show with (22) that $\sum_K |f(\hat{u}_H^n)|_{k+1,2,K}^2$ is bounded by a constant C which is independent of H. Applying Lemma 3.3 for $g(u^n) = f_u(P_H u^n)$, with the standard approximation property of L^2-projection P_H and the assumption on f, we can bound each term on the right hand side of (35):

(40)
$$|(f(u^n) - f(P_H u^n), \xi_H^n)|$$
$$\leq CB_1\big(H\|u^n - P_H u^n\|\|\xi_H^n\| + \|u^n - P_H u^n\|_{0,4}^2 \|\xi_H^n\|\big)$$
$$\leq CH^{2k+4} + \|\xi_H^n\|^2.$$

From (33)-(40), (32) becomes

$$\|\xi_H^n\|^2 - \|\xi_H^{n-1}\|^2 \leq \frac{\Delta t^2}{3} \int_{(n-1)\Delta t}^{n\Delta t} \left\|\frac{\partial^2 u}{\partial t^2}\right\|^2 dt + (6 + 2B_1)\Delta t\|\xi_H^n\|^2 + C\Delta t H^{2k+2}.$$

Summing from $n = 1$ to ℓ $(1 \leq \ell \leq N)$, we have

$$\|\xi_H^\ell\|^2 - \|\xi_H^0\|^2 \leq \frac{\Delta t^2}{3} \int_0^{\ell\Delta t} \left\|\frac{\partial^2 u}{\partial t^2}\right\|^2 dt + (6+2B_1)\Delta t \sum_{1 \leq n \leq \ell} \|\xi_H^n\|^2 + C\ell\Delta t H^{2k+2}.$$

Hence,

$$\|\xi_H^\ell\|^2 \leq (1 - (6 + 2B_1)\Delta t)\|\xi_H^0\|^2 + \frac{\Delta t^2}{3} \int_0^T \left\|\frac{\partial^2 u}{\partial t^2}\right\|^2 dt$$
$$+ (6 + 2B_1)\Delta t \sum_{0 \leq n \leq \ell} \|\xi_H^n\|^2 + CTH^{2k+2}.$$

When $1 - (6 + 2B_1)\Delta t > 0$, by Gronwall's lemma it follows that

$$\|\xi_H^\ell\|^2 \leq C\left(\|\xi_H^0\|^2 + \Delta t^2 \int_0^T \left\|\frac{\partial^2 u}{\partial t^2}\right\|^2 dt + H^{2k+2}\right).$$

We choose the initial function $\hat{u}_H^0$ to be elliptic projection $\tilde{u}_H^0$ of the exact solution. Then $\xi_H^0 = 0$ and we have

$$\|\xi_H^n\| \leq C\big(\Delta t + H^{k+1}\big).$$

Thus, we conclude with the inverse inequality

$$\|\xi_H^n\|_{0,4} \leq CH^{-1/2}\|\xi_H^n\| \leq C\big(H^{-1/2}\Delta t + H^{k+1/2}\big). \qquad \square$$

Theorem 3.6. *Let $(\hat{u}_h^n, \hat{\boldsymbol{\sigma}}_h^n) \in V_h \times \boldsymbol{M}_h$ be the solution of (11)-(12). Assume that the relation between size of time step Δt and mesh size H of coarse grid satisfies $\Delta t \cdot H^{-1} = \mathcal{O}(1)$. Then there exists a positive constant C independent of Δt, H and h such that*

$$\|u^n - \hat{u}_h^n\| \leq C\big(\Delta t + h^{k+1} + H^{2k+1}\big),$$

$$\|\boldsymbol{\sigma}^n - \hat{\boldsymbol{\sigma}}_h^n\| \le C\Big(\Delta t + h^{k+1} + H^{2k+1}\Big).$$

Proof. By Taylor expansion, there exists $u^* \in V$ such that

$$f(u^n) = f(\hat{u}_H^n) + f_u(\hat{u}_H^n)(u^n - \hat{u}_H^n) + \frac{1}{2} f_{uu}(u^*)(u^n - \hat{u}_H^n)^2.$$

Hence, for any $v \in V$

$$(41) \qquad \big(f(u^n) - f(\hat{u}_H^n) - f_u(\hat{u}_H^n)(\hat{u}_h^n - \hat{u}_H^n), v\big)$$
$$= \big(f_u(\hat{u}_H^n)(u^n - \hat{u}_h^n) + \frac{1}{2} f_{uu}(u^*)(u^n - \hat{u}_H^n)^2, v\big).$$

Taking $\boldsymbol{\tau} = \boldsymbol{\gamma}_h^n$ in (28) and $v = \mu_h^n$ in (29) and adding two resulting equations, we have from (41)

$$\Big(\frac{\mu_h^n - \mu_h^{n-1}}{\Delta t}, \mu_h^n\Big) + \Big(\alpha \boldsymbol{\gamma}_h^n, \boldsymbol{\gamma}_h^n\Big) = \Big(f_u(\hat{u}_H^n)(u^n - \hat{u}_h^n) + \frac{1}{2} f_{uu}(u^*)(u^n - \hat{u}_H^n)^2, \mu_h^n\Big)$$
$$+ \Big(\frac{\tilde{u}_h^n - \tilde{u}_h^{n-1}}{\Delta t} - \frac{u^n - u^{n-1}}{\Delta t}, \mu_h^n\Big) + (e^n, \mu_h^n).$$

Applying the assumption on f_u and f_{uu} and using the result of Lemma 3.2 and the following relationship

$$u^n - \hat{u}_h^n = u^n - \tilde{u}_h^n + \tilde{u}_h^n - \hat{u}_h^n = u^n - \tilde{u}_h^n + \mu_h^n,$$

we get with the fact $\alpha > \alpha_0 > 0$

$$\frac{1}{2\Delta t}\big(\|\mu_h^n\|^2 - \|\mu_h^{n-1}\|^2\big)$$
$$\le B_1^2 \|u^n - \tilde{u}_h^n\|^2 + \frac{1}{8} B_1^2 \|(u^n - \hat{u}_H^n)^2\|^2$$
$$+ \frac{1}{2\Delta t} C h^{2k+2} \int_{(n-1)\Delta t}^{n\Delta t} \Big\|\frac{\partial u}{\partial t}\Big\|_{k+1}^2 dt + \frac{1}{2}\|e^n\|^2 + (2 + B_1^2)\|\mu_h^n\|^2.$$

Summing over $n = 1$ to ℓ ($1 \le \ell \le N$) and multiplying by $2\Delta t$, we have from Lemma 3.1

$$\|\mu_h^\ell\|^2 - \|\mu_h^0\|^2 \le (4 + 2B_1^2)\Delta t \sum_{n=1}^{\ell} \|\mu_h^n\|^2$$
$$+ \Delta t B_1^2 \sum_{n=1}^{\ell} \Big(\|u^n - \tilde{u}_h^n\|^2 + \frac{1}{4}\|(u^n - \hat{u}_H^n)^2\|^2\Big)$$
$$+ C h^{2k+2} \int_0^{\ell \Delta t} \Big\|\frac{\partial u}{\partial t}\Big\|_{k+1}^2 dt + C \Delta t^2 \int_0^{\ell \Delta t} \Big\|\frac{\partial^2 u}{\partial t^2}\Big\|^2 dt.$$

When $1 - (4 + 2B_1^2)\Delta t > 0$, by Gronwall's lemma

$$\|\mu_h^\ell\|^2 \le C\Delta t \sum_{n=1}^{\ell} \Big(\|u^n - \tilde{u}_h^n\|^2 + \|(u^n - \hat{u}_H^n)^2\|^2\Big)$$

$$+ C\left(\Delta t^2 + h^{2k+2} + \|\mu_h^0\|^2\right).$$

If we choose the initial function $\hat{u}_h^0 = \tilde{u}_h^0$, $\hat{\boldsymbol{\sigma}}_h^0 = \tilde{\boldsymbol{\sigma}}_h^0$ then μ_h^0 and $\boldsymbol{\gamma}_h^0$ vanish. It follows from (6), (15) and Lemma 3.5 that

$$\|\mu_h^\ell\|^2 \leq C\ell\Delta t \max_{1\leq n\leq l}\left(\|u^n - \tilde{u}_h^n\|^2 + \|u^n - \tilde{u}_H^n\|_{0,4}^4 + \|\tilde{u}_H^n - \hat{u}_H^n\|_{0,4}^4\right)$$
$$+ C\Delta t^2 + Ch^{2k+2}$$
$$\leq C\left(\Delta t^2 + h^{2k+2} + H^{4(k+1)} + H^{-2}\Delta t^4 + H^{4(k+1/2)}\right).$$

Hence, we have from the assumption $\Delta t \cdot H^{-1} = \mathcal{O}(1)$

$$(42) \qquad \|u^n - \hat{u}_h^n\| \leq \|u^n - \tilde{u}_h^n\| + \|\mu_h^n\| \leq C\left(\Delta t + h^{k+1} + H^{2k+1}\right).$$

Now, in order to estimate $\|\boldsymbol{\sigma}^n - \hat{\boldsymbol{\sigma}}_h^n\|$, we consider the equation (28) for time level $n-1$ and n. Then we have from (29) and (41)

$$\left(\frac{\alpha(\boldsymbol{\gamma}_h^n - \boldsymbol{\gamma}_h^{n-1})}{\Delta t}, \boldsymbol{\tau}\right) - \left(\text{div}\boldsymbol{\tau}, \frac{\mu_h^n - \mu_h^{n-1}}{\Delta t}\right) = 0, \quad \forall \boldsymbol{\tau} \in \boldsymbol{M}_h,$$

$$\left(\frac{\mu_h^n - \mu_h^{n-1}}{\Delta t}, v\right) + \left(\text{div}\boldsymbol{\gamma}_h^n, v\right)$$
$$= \left(f_u(\hat{u}_H^n)(u^n - \hat{u}_h^n) + \frac{1}{2}f_{uu}(u^*)(u^n - \hat{u}_H^n)^2, v\right)$$
$$+ \left(\frac{\tilde{u}_h^n - \tilde{u}_h^{n-1}}{\Delta t} - \frac{u^n - u^{n-1}}{\Delta t}, v\right) + (e^n, v), \quad \forall v \in V_h.$$

Taking $\boldsymbol{\tau} = \boldsymbol{\gamma}_h^n$, $v = \dfrac{\mu_h^n - \mu_h^{n-1}}{\Delta t}$ and adding the resulting equations, we obtain

$$\left(\frac{\alpha(\boldsymbol{\gamma}_h^n - \boldsymbol{\gamma}_h^{n-1})}{\Delta t}, \boldsymbol{\gamma}_h^n\right) + \left(\frac{\mu_h^n - \mu_h^{n-1}}{\Delta t}, \frac{\mu_h^n - \mu_h^{n-1}}{\Delta t}\right)$$
$$= \left(f_u(\hat{u}_H^n)(u^n - \hat{u}_h^n) + \frac{1}{2}f_{uu}(u^*)(u^n - \hat{u}_H^n)^2, \frac{\mu_h^n - \mu_h^{n-1}}{\Delta t}\right)$$
$$+ \left(\frac{\tilde{u}_h^n - \tilde{u}_h^{n-1}}{\Delta t} - \frac{u^n - u^{n-1}}{\Delta t}, \frac{\mu_h^n - \mu_h^{n-1}}{\Delta t}\right) + \left(e^n, \frac{\mu_h^n - \mu_h^{n-1}}{\Delta t}\right).$$

Letting $\lambda^n = \frac{\mu_h^n - \mu_h^{n-1}}{\Delta t}$ and multiplying by Δt, we get

$$\frac{1}{2}(\alpha\boldsymbol{\gamma}_h^n, \boldsymbol{\gamma}_h^n) + \Delta t\|\lambda^n\|^2$$
$$\leq \frac{1}{2}(\alpha\boldsymbol{\gamma}_h^{n-1}, \boldsymbol{\gamma}_h^{n-1}) + \Delta t(e^n, \lambda^n)$$
$$+ \Delta t\left|\left(f_u(\hat{u}_H^n)(u^n - \hat{u}_h^n) + \frac{1}{2}f_{uu}(u^*)(u^n - \hat{u}_H^n)^2, \lambda^n\right)\right|$$
$$+ \left|\left((\tilde{u}_h^n - u^n) - (\tilde{u}_h^{n-1} - u^{n-1}), \lambda^n\right)\right|.$$

So that, Cauchy-Schwarz inequality and Lemma 3.2 imply

$$\frac{1}{2}(\alpha\boldsymbol{\gamma}_h^n, \boldsymbol{\gamma}_h^n) + \Delta t\|\lambda^n\|^2$$

$$\leq \frac{1}{2}(\alpha\boldsymbol{\gamma}_h^{n-1}, \boldsymbol{\gamma}_h^{n-1}) + \Delta t\left(\|e^n\|^2 + \frac{1}{4}\|\lambda^n\|^2\right)$$

$$+ \Delta t\left(B_1^2\|u^n - \hat{u}_h^n\|^2 + \frac{1}{4}B_1^2\|(u^n - \hat{u}_H^n)^2\|^2 + \frac{1}{2}\|\lambda^n\|^2\right)$$

$$+ Ch^{2k+2}\int_{(n-1)\Delta t}^{n\Delta t}\left\|\frac{\partial u}{\partial t}\right\|_{k+1}^2 dt + \frac{1}{4}\Delta t\|\lambda^n\|^2.$$

Multiplying by 2 and summing from $n = 1$ to $\ell(1 \leq \ell \leq N)$, we have from Lemma 3.1 and the fact that $\boldsymbol{\gamma}_h^0$ vanishes

$$\alpha_0\|\boldsymbol{\gamma}_h^\ell\|^2 \leq \beta_0\|\boldsymbol{\gamma}_h^0\|^2 + C\Delta t^2 + Ch^{2k+2}$$

$$+ \Delta t\sum_{n=1}^{\ell}\left(B_1^2\|u^n - \hat{u}_h^n\|^2 + \frac{1}{4}B_1^2\|(u^n - \hat{u}_H^n)^2\|^2\right)$$

$$\leq C\Delta t^2 + Ch^{2k+2} + C\max_{1\leq n\leq\ell}\left(\|u^n - \hat{u}_h^n\|^2 + \|u^n - \hat{u}_H^n\|_{0,4}^4\right).$$

We conclude with (16), (42) and (15) associated with mesh size H and Lemma 3.5

$$\|\boldsymbol{\sigma}^n - \hat{\boldsymbol{\sigma}}_h^n\| \leq \|\boldsymbol{\sigma}^n - \tilde{\boldsymbol{\sigma}}_h^n\| + \|\boldsymbol{\gamma}_h^n\| \leq C\left(\Delta t + h^{k+1} + H^{2k+1}\right). \qquad \square$$

As mentioned in introduction, we can get improved error estimates using the interpolation operator with the following special nodal points (Gaussian points) on triangle elements (see, for example, p. 360 in [15]):

TABLE 1. Gaussian points and weights for quadratures on a triangle of area $|K|$.

k	Degree of Exactness	Barycentric Coordinates	Multiplicity	Weights
1	2	$(\frac{1}{2}, \frac{1}{2}, 0)$	3	$\omega = \frac{1}{3}|K|$
2	4	$(a_i, a_i, 1 - 2a_i)$ for $i = 1, 2$ $a_1 = 0.445948$ $a_2 = 0.091576$	3	ω_i for $i = 1, 2$ $\omega_1 = |K| \times 0.223382$ $\omega_2 = |K| \times 0.109952$

Lemma 3.7. *Assume that $\mathcal{T}_H$ is a quasi-uniform partition of Ω into triangles. For each $K \in \mathcal{T}_H$ and $v \in P_k(K)$, there is a constant C, independent of H, such that for $k = 1, 2$,*

$$\left|\int_K \left(f(v)\xi_H^n - Q_H f(v)\xi_H^n\right) dx\right| \leq CH^{2k+2}|f(v)\xi_H^n|_{2k+1,2,K}.$$

Proof. Since the order of quadrature (degree of exactness) is $2k$, $k = 1, 2$ for Gaussian quadrature formula on triangle and the polynomial $Q_H f(v) \xi_H^n$ is of degree $\leq 2k$, by the definition of $Q_H f(v)$ we have

$$
\int_K Q_H f(v) \xi_H^n \, dx = \sum_{j=1}^{n_k} \omega_j Q_H f(v(g_j)) \xi_H^n(g_j)
$$

$$
= \sum_{j=1}^{n_k} \omega_j f(v(g_j)) \xi_H^n(g_j)
$$

$$
= \int_K Q_H \big(f(v) \xi_H^n \big) \, dx,
$$

where ω_j are Gaussian weights and g_j are Gaussian points in K and $n_k = \frac{(k+1)(k+2)}{2}$ is the total number of points. Thus, the proof is completed by the error estimates of Gaussian quadrature formula (Lemma 8.4 in [15]). $\qquad\square$

Lemma 3.8. *Under the assumption of Lemma 3.7, if $v \in V_H$ and $\|v\|_{r,p}$, $r = 0, 1, 2, \ldots, k$, $p \geq 1$ is bounded by a constant independent of H, then there exists a positive constant C independent of H such that*

$$
\left| \Big(f(v) - Q_H f(v), \xi_H^n \Big) \right| \leq C H^{2k+3} + \|\xi_H^n\|^2.
$$

Proof. By Lemma 3.7 we have

(43)

$$
\left| \Big(f(v) - Q_H f(v), \xi_H^n \Big) \right| \leq \sum_{K \in \mathcal{T}_H} \left| \int_K \Big(f(v) \xi_H^n - Q_H f(v) \xi_H^n \Big) \, dx \right|
$$

$$
\leq C \sum_{K \in \mathcal{T}_H} H^{2k+2} |f(v) \xi_H^n|_{2k+1,2,K}
$$

$$
\leq C \sum_{K \in \mathcal{T}_H} \left(\sum_{|\beta|=2k+1} H^{4k+4} \|D^\beta (f(v)\xi_H^n)\|_{0,2,K}^2 \right)^{1/2}.
$$

Note that the product rule gives

$$
D^\beta (f(v)\xi_H^n) = \sum_{\gamma+\delta=\beta} \frac{\beta!}{\gamma!\delta!} D^\gamma f(v) D^\delta \xi_H^n.
$$

So, by Hölder inequality, we get

$$
\|D^\beta (f(v)\xi_H^n)\|_{0,2,K}^2 = \int_K |D^\beta (f(v)\xi_H^n)|^2 \, dx
$$

$$
\leq C \int_K \left(\sum_{\gamma+\delta=\beta} (D^\gamma f(v))^2 \right) \left(\sum_{\gamma+\delta=\beta} \big(D^\delta \xi_H^n \big)^2 \right) \, dx
$$

$$\leq C \left\| \sum_{\gamma+\delta=\beta} (D^\gamma f(v))^2 \right\|_{0,2,K}^2 \left\| \sum_{\gamma+\delta=\beta} (D^\delta \xi_H^n)^2 \right\|_{0,2,K}^2,$$

where $|\gamma| + |\delta| = 2k + 1$. Hence, the equation (43) becomes

$$(44) \qquad \left| \left(f(v) - Q_H f(v), \xi_H^n \right) \right|$$

$$\leq C \sum_K \left(H^{2k+3} \|f(v)\|_{2k+1,4,K}^2 H^{2k+1} \|\xi_H^n\|_{2k+1,4,K}^2 \right)^{1/2}$$

$$\leq C \sum_K \left(\frac{1}{2\epsilon} H^{4k+6} \|f(v)\|_{2k+1,4,K}^4 + \frac{\epsilon}{2} H^{4k+2} \|\xi_H^n\|_{2k+1,4,K}^4 \right)^{1/2}.$$

To bound the first term of (44), we apply generalized Hölder inequality to the identity

$$D^\gamma \big(f(v) \big) = \sum_{i=1}^{|\gamma|} \left(\sum_{\substack{\sum_1^i \gamma_j = \gamma \\ \gamma_j \neq 0}} f^{(i)}(v) D^{\gamma_1} v \cdots D^{\gamma_i} v \right).$$

Then we get

$$\|D^\gamma \big(f(v) \big)\|_{0,4,K}$$

$$\leq \sum_{i=1}^{|\gamma|} \left(\sum_{\substack{\sum_1^i \gamma_j = \gamma \\ \beta_j \neq 0}} \|f^{(i)}(v)\|_{L^\infty} \|D^{\gamma_1} v\|_{0,q_1,K} \cdots \|D^{\gamma_i} v\|_{0,q_i,K} \right) \leq C_1,$$

with $\sum 1/q_j = 1/4$. Here, we have used the smoothness of f and the fact that v is a polynomial of degree k on each element and $\|v\|_{r,p}$, $r = 0, 1, \ldots, k$, $p \geq 1$, is bounded. Next, we consider the second term. Since $D^\delta \xi_H^n$ for $|\delta| > k$ is identically zero, from the inverse inequality

$$\|D^\delta \xi_H^n\|_{0,4,K}^4 \leq C_2 H^{-4k-2} \|\xi_H^n\|_{0,2,K}^4, \quad \forall\, |\delta| \leq k.$$

For the constant C in the equation (44), letting $\epsilon = \frac{2}{C^2 C_2}$ the proof is completed. $\qquad \square$

Theorem 3.9. *Assume that $\mathcal{T}_H$ is a quasi-uniform partition of Ω into triangles. Let $(\hat{u}_h^n, \hat{\boldsymbol{\sigma}}_h^n) \in V_h \times \boldsymbol{M}_h$ be the solution of (11)-(12). Assume that the relation between size of time step Δt and mesh size H of coarse grid satisfies $\Delta t \cdot H^{-1} = \mathcal{O}(1)$. Moreover, assume that f is sufficiently smooth. Then, for $k = 0, 1, 2$, there exists a positive constant C independent of Δt, H and h such that*

$$\|u^n - \hat{u}_h^n\| \leq C \Big(\Delta t + h^{k+1} + H^{2k+2} \Big),$$

$$\|\boldsymbol{\sigma}^n - \hat{\boldsymbol{\sigma}}_h^n\| \le C\Big(\Delta t + h^{k+1} + H^{2k+2}\Big).$$

Proof. In the case $k = 0$, the identity

$$Q_h f(v_h) = f(v_h), \quad \forall\, v_h \in V_h$$

holds. This says that all terms related to interpolation disappear in the proofs of Theorem 3.4, Lemma 3.8 and Theorem 3.6. So the proof is easily completed.

In the case $k = 1, 2$, from the result of Lemma 3.8 the equation (39) in the proof of Lemma 3.5 becomes

$$|(f(\hat{u}_H^n) - Q_H f(\hat{u}_H^n), \xi_H^n)| \le CH^{2k+3} + \|\xi_H^n\|^2.$$

So we get

$$\|\tilde{u}_H^n - \hat{u}_H^n\|_{0,4} \le C\Big(H^{-1/2}\Delta t + H^{k+1}\Big),$$

which is the improved result compared with Lemma 3.5. Using a similar argument of the proof of Theorem 3.6, we conclude the proof. $\square$

Remark 3.10. In the case $k = 0$, Theorem 3.9 holds for any mixed finite element space on rectangular mesh.

4. Numerical results

In this section, we apply the two scale backward Euler mixed finite element method with interpolated coefficients. We perform numerical experiments to demonstrate the effect of our two scale algorithm. For this, we consider a semilinear parabolic problem:

$$(45) \qquad \begin{cases} u_t - \mathrm{div}(\nabla u) + u^3 = g & \text{in } \Omega \times [0,1) \\ \qquad\qquad\qquad u = 0 & \text{on } \partial\Omega \times [0,1) \\ \qquad\qquad u(\cdot, 0) = 0 & \text{in } \Omega \times \{0\}, \end{cases}$$

where $\Omega = (0,1) \times (0,1)$ and g is computed from the exact solution $u(t,x,y) = txy(1-x)(1-y)$.

Let $\mathcal{T}_h$ be a uniform partition of Ω into triangles. Consider the Raviart-Thomas finite element space of index 1 (RT_1) given as

$$\boldsymbol{M}_h := \{\boldsymbol{\tau} \in H(\mathrm{div}; \Omega) : \boldsymbol{\tau}|_K \in RT_1(K), \; \forall K \in \mathcal{T}_h\},$$

$$V_h := \{v \in L^2(\Omega) : v|_K \in P_1(K), \; \forall K \in \mathcal{T}_h\}.$$

Thanks to Theorem 3.9, the bases of $P_1(K)$ on each triangle K consist of Lagrangian nodal functions based on barycentric coordinates $(0, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 0, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2}, 0)$. More precisely, the basis of $P_1(\hat{K})$ on reference triangle $\hat{K}$ is given by

$$\{\; 2\xi + 2\eta - 1, \; -2\xi + 1, \; -2\eta + 1 \;\},$$

where reference triangle $\hat{K}$ is a triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$. Now, we apply our numerical scheme to the problem (45). We equally divide

time interval $[0, 1]$ and set $t_n = n\Delta t, n = 1, 2, \ldots, N$ and $\Delta t = 1/N$. We seek solutions at final time $t = 1$.

On coarse-scale grid, our nonlinear system can be solved by an iteration method, for example:

$$(\hat{\boldsymbol{\sigma}}_H^{n,m}, \boldsymbol{\tau}) - (\hat{u}_H^{n,m}, \operatorname{div}\boldsymbol{\tau}) = 0, \ \forall \boldsymbol{\tau} \in \boldsymbol{M}_H,$$

$$\Delta t(\operatorname{div}\hat{\boldsymbol{\sigma}}_H^{n,m}, v) + (\hat{u}_H^{n,m}, v) = \Delta t(g(x) - Q_H(\hat{u}_H^{n,m-1})^3, v) + (\hat{u}_H^{n-1}, v), \ \forall v \in V_H,$$

where n denotes the time level and m the iteration index. On fine-scale grid, we solve the following linear system:

$$(\hat{\boldsymbol{\sigma}}_h^n, \boldsymbol{\tau}) - (\hat{u}_h^n, \operatorname{div}\boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{\tau} \in \boldsymbol{M}_h$$

$$\Delta t(\operatorname{div}\hat{\boldsymbol{\sigma}}_h^n, v) + ((1 + 3\Delta t(\hat{u}_H^n)^2)\hat{u}_h^n, v) = \Delta t(g(x) + 2(\hat{u}_H^n)^3, v)$$
$$+ (\hat{u}_h^{n-1}, v), \quad \forall v \in V_h.$$

For each time step, the computation of $(Q_H(\hat{u}_H^n)^3, \phi_j)$ for $\hat{u}_H^n = \sum c_i\phi_i$ is carried out as follows

$$(Q_H(\hat{u}_H^n)^3, \phi_j) = D(\boldsymbol{c}.{}^\wedge 3),$$

where $\boldsymbol{c}.{}^\wedge 3$ is a vector whose ith component is c_i^3 and D is a diagonal matrix with $D_{ij} = (\phi_i, \phi_j)$ which is exactly computed by Gaussian quadrature rule for $k = 1$ as in Table 1

$$(\phi_i, \phi_j) = \sum_{K \in \mathcal{T}_h} \int_K \phi_i\phi_j \, dx = \sum_{K \in \mathcal{T}_h} \left(\sum_{\ell=1}^3 \omega\phi_i(g_\ell)\phi_j(g_\ell)\right).$$

Thus, our two-scale product approximation procedure can lead to great computational advantages without sacrificing the overall accuracy of the scheme. Indeed, we present convergence orders for $\|u - \hat{u}_h\|$ and $\|\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h\|$ in the following table. The coarse scale mesh H and fine scale h are chosen so that $H = h^{1/2}$ according to Theorem 3.9. Table 2 shows optimal convergence results which are in good agreement with the theory developed in this paper.

Table 2. L^2-error norms and convergence orders ($RT_1 \times P_1$ with $\Delta t = 0.1$).

$1/h(1/H)$	$\|u - \hat{u}_h\|$	$\|\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h\|$	C.O. for u	C.O. for $\boldsymbol{\sigma}$
4(2)	$1.3689e - 003$	$5.7954e - 003$		
16(4)	$8.7236e - 005$	$3.7649e - 004$	1.99	1.97
36(6)	$1.7250e - 005$	$7.4775e - 005$	2.00	1.99
64(8)	$5.4596e - 006$	$2.3704e - 005$	2.00	2.00

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